Can. J. Math.Vol. **44** (1), 1992 pp. 119–134

HOMOLOGY OF BRANCHED COVERINGS OF 3-MANIFOLDS

JOHN HEMPEL

ABSTRACT. We give a relation between the homology groups $H_1(\tilde{M})$ and $H_1(M)$ for a branched cyclic cover $\tilde{M} \longrightarrow M$ of arbitrary closed, oriented 3-manifolds which generalizes a classical result of Plans on covers of S^3 branched over a knot and provides other quantitative information as well. We include a general "free calculus" procedure for computing homology groups of branched covers and reinterpret the results in this computational setting.

1. **Introduction.** It is a theorem of Plans [P] that the first homology of an odd degree cyclic branched cover $\tilde{M}$ of S^3, branched over a knot, is always a direct double: $H_1(\tilde{M}) = A \oplus A$. This extends [VW] to cyclic covers of homology 3-spheres branched over links—modulo p-torsion for primes p dividing the degree of the cover. Some independent information about the first betti number and torsion numbers prime to the degree for such covers (cyclic covers branched over links in homology 3-spheres) is given in [CM].

We have two goals in this paper. The first is to show that the arguments of [VW] and [CM] can easily be adapted to treat cyclic covers $\tilde{M} \longrightarrow M$ branched over a link in an arbitrary 3-manifold, M, and show that the change from $H_1(M)$ to $H_1(\tilde{M})$ follows the same pattern. This allows us to give qualitative information about the homology of branched coverings which can be factored as a composition of cyclic coverings. Such covers include all solvable covers (in particular all abelian covers) and many irregular covers as well.

The second is to describe a procedure for calculating the first homology of a general branched covering from a relative Jacobian matrix of free derivatives of a presentation of a certain group pair in much the same way (cf. [H_2]) as one calculates the first homology of an unbranched covering from a Jacobian of a presentation of the fundamental group of the base. This procedure is mentioned briefly in [H_3], but does not seem to be well known. We give a more complete description in Section 3. We then reinterpret the results on cyclic branched covers in terms of this computational procedure in a way that provides alternate arguments for the results, explains the significance of the steps in the procedure, and provides some additional qualitative information including some information about the p-torsion of the homology for primes p which do divide the degree of the cover.

We establish the following extensions of Plans' theorem. Definitions are given at the end of this section. The proofs are given in Section 2 and again in Section 4 in terms of

Supported in part by NSF DMS 9004018.
Received by the editors June 25, 1990.
AMS subject classification: 57N10, 57M05, 57M12.

the computational procedure given in Section 3. Some pertinent examples are given in Section 5.

THEOREM A. *Let $\rho: \tilde{M} \longrightarrow M$ be a cyclic branched covering of closed, orientable 3-manifolds branched over a link $L \subset M$ and of prime degree d.*

If d is odd then:

(1) $\beta_1(\tilde{M}) = \beta_1(M) + (d-1)r$ for some $r \geq 0$.

(2) For each prime $q \neq d$

$$q\text{-torsion}\left(H_1(\tilde{M})\right) = q\text{-torsion}\left(H_1(M)\right) \oplus A^{\operatorname{lcm}(2, g_d(q))}$$

for some group A; where $g_d(q)$ is the order of q in the multiplicative group $\mathbb{Z}_d{}^*$.

If $d = 2$ and the nontrivial covering transformation of ρ has an orientation preserving square root then:

(1′) $\beta_1(\tilde{M}) = \beta_1(M) + 2r$ for some $r \geq 0$.

(2′) For each prime $q \neq 2$

$$q\text{-torsion}(H_1(\tilde{M})) = q\text{-torsion}(H_1(M)) \oplus A^2$$

for some group A.

In all cases we have:

(3) Suppose $\beta_1(\tilde{M}) = 0$. If nontrivial branching occurs, then d-torsion $\left(H_1(M)\right)$ is a quotient group of d-torsion $\left(H_1(\tilde{M})\right)$. In any event the image of d-torsion $\left(H_1(\tilde{M})\right)$ in d-torsion $\left(H_1(M)\right)$ has index $\leq d$.

(4) If L is connected, $\beta_1(M) = 0$ and $\beta_1(\tilde{M}) > 0$ then d-torsion $\left(H_1(M)\right) \neq 0$

Throughout we will regard each (finitely generated) abelian group A as being decomposed as a direct sum of a free abelian group and cyclic groups of prime power order. Then for a prime q, q-torsion (A) is just the sum of those summands whose orders are powers of q. The result of dividing A by all the q-torsion summands for which q divides a fixed integer n is called the *n-reduction* of A and denoted n-red (A). A branched covering will be called *subsolvable* if it can be factored as a composition of cyclic branched covers—and so can further be factored as a composition of cyclic branched covers of prime degree. Any regular branched cover with solvable covering group is subsolvable; but there are lots of irregular subsolvable branched covers. Theorem A clearly yields

THEOREM B. *Let $\rho: \tilde{M} \longrightarrow M$ be a subsolvable branched cover of odd degree n of closed, orientable 3-manifolds branched over a link $L \subset M$. Then:*

(1) $\beta_1(\tilde{M}) = \beta_1(M) + 2r$, for some $r \geq 0$;

(2) n-red$\left(H_1(\tilde{M})\right) = n$-red $\left(H_1(M)\right) \oplus A \oplus A$, for some group A;

(3) If q^α is the highest power of the prime q which divides n, then $o(q\text{-torsion}\,(H_1(\tilde{M}))) \geq o(q\text{-torsion}\,(H_1(M)))/\,q^\alpha$.

Note that the inequality in (3) can be improved by a power of q for each degree q factor in some factorization of ρ which is actually branched. Also the conclusion also holds for even n if the covering transformation of the degree 2 factors all have orientation preserving square roots. However both of these conditions are hard to detect in general.

Throughout this paper we will consider orientable 3-manifolds. A map

$$\rho : \tilde{M} \longrightarrow M$$

of such 3-manifolds is a *branched covering* of degree d, branched over a link $L \subset \operatorname{Int}(M)$ provided that

$$\rho \,|\, \tilde{M} - \rho^{-1}(L) : \tilde{M} - \rho^{-1}(L) \longrightarrow M - L$$

is a d sheeted covering space and each component, J, of L has a closed neighborhood V such that for each component $\tilde{V}$ of $\rho^{-1}(V)$ there are homeomorphisms $h: B^2 \times S^1 \longrightarrow V$, $\tilde{h}: B^2 \times S^1 \longrightarrow \tilde{V}$ with $h(0 \times S^1) = J$ and

$$h^{-1} \circ \rho \circ \tilde{h}(z, e^{i\theta}) = (z^n, e^{im\theta})$$

for some integers $n, m > 0$. The integer n is called *the branching index* of ρ at the component $\tilde{J} = \tilde{h}(0 \times S^1)$ of $\rho^{-1}(J)$. We use the expression *true covering* to describe a branched covering with all branching indices equal to one.

The covering is called *cyclic* if it is regular and the group of covering transformations is cyclic.

The *monodromy* of the covering is the homomorphism

$$\varphi : \pi_1(M - L) \longrightarrow S_d$$

describing the action of $\pi_1(M-L)$ on a fiber $\rho^{-1}(x_0)$, $\quad x_0 \in M-L$. It completely determines the covering: equivalence classes of connected degree d coverings of M branched over L are in one to one correspondence with transitive representations of $\pi_1(M-L)$ to S_d. The *corresponding unbranched covering,*

$$\rho \,|\, \tilde{M} - \rho^{-1}(L) : \tilde{M} - \rho^{-1}(L) \longrightarrow M - L$$

is determined by the condition:

$$\rho_*\big(\pi_1(\tilde{M} - \rho^{-1}(L)\big) = \varphi^{-1}(\mathrm{Stab}(1)).$$

We will write permutations on the right, as exponents: $i \longrightarrow i^{\varphi(g)}$.

If $\mu \in \pi_1(M-L)$ corresponds to a meridian of a component J of L then the lengths of the cycles of $\varphi(\mu)$ are the branching indices of ρ at the components of $\rho^{-1}(J)$—counted with some repetitions since m cycles of $\varphi(\mu)$ (all of the same length) will correspond to a component, $\tilde{J}$, of $\rho^{-1}(J)$ if $\rho|\tilde{J}: \tilde{J} \longrightarrow J$ is an m sheeted covering. If the covering is regular, then all the branching indices over a fixed component of L are equal. The converse is false.

2. **Proof of Theorem A.** Conclusion 4 will be proved in Section 4.

We remove an open regular neighborhood of the branch set $L \subset M$ to obtain a submanifold N of M and let $\tilde{N} = \rho^{-1}(N)$. Consider the diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\tilde{N}) & \xrightarrow{\rho_\sharp} & \pi_1(N) & \longrightarrow & \langle t : t^d = 1\rangle & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & & & \\ & & H_1(\tilde{M}) & \xrightarrow{\rho_*} & H_1(M) & & & & \end{array}$$

with horizontal exactness and vertical epimorphisms. Now $[H_1(M) : \rho_*\big(H_1(\tilde{M})\big)]$ divides $[\pi_1(N) : \rho_\sharp\big(\pi_1(\tilde{N})\big)] = d$.

Now if actual branching occurs, then since d is prime, it follows that some meridian μ of L in ∂N has $\rho^{-1}(\mu)$ connected. Thus for any loop x in N based at a point of μ there is some k such that $x\mu^k$ lifts to a loop in $\tilde{N}$ and so lies in $\rho_\sharp\big(\pi_1(\tilde{N})\big)$. Since x and $x\mu^k$ have the same image in $H_1(M)$, this establishes that $\rho_*\big(H_1(\tilde{M})\big) = H_1(M)$. Conclusion 3 follows easily from these observations.

Now $H_1(\tilde{M})$ is a module over the integral group ring $\mathbb{Z}\,C$, $C = \langle t : t^d = 1\rangle$ and there is a transfer homomorphism $\sigma : H_1(M) \longrightarrow H_1(\tilde{M})$ (cf. [F]) satisfying:

(1) $\sigma \circ \rho_* = 1 + t + t^2 + \ldots + t^{d-1} \stackrel{\text{def}}{=\!=} \Phi(t)$;

(2) $\rho_* \circ \sigma =$ multiplication by d.

By (1) we see that multiplication by $\Phi(t)$ is zero on $\mathrm{Ker}(\rho_*)$ and so $t \longmapsto \xi = e^{2\pi i/d}$ induces a factorization of the action of $\mathbb{Z}\,C$ on $\mathrm{Ker}(\rho_*)$ through $\mathbb{Z}[\xi]$—that is $\mathrm{Ker}(\rho_*)$ is a $\mathbb{Z}[\xi]$-module. Now we quote [CM; Theorem 2.5] that for a prime d the $\mathbb{Z}$-rank of a finitely generated $\mathbb{Z}[\xi]$-module is divisible by $d-1$. This establishes conclusion (1).

Let $\bar{T}$ denote the d-reduction of the torsion subgroup of $H_1(\tilde{M})$ and put

$$\bar{K} = \{x \in \bar{T} : \Phi(t) \cdot x = 0\}, \quad \bar{L} = \{x \in \bar{T} : (1-t) \cdot x = 0\}.$$

Now multiplication by d is an isomorphism on $\bar{T}$. So given $x \in \bar{T}$, $x = d \cdot x_1$ for some $x_1 \in \bar{T}$. Then $x = x - \Phi(t) \cdot x_1 + \Phi(t) \cdot x_1$, and $x - \Phi(t) \cdot x_1 \in \bar{K}$—since $f(t) \cdot \Phi(t) = f(1) \cdot \Phi(t)$ for all $f(t) \in \mathbb{Z}\,C$, so $\Phi^2(t) = d \cdot \Phi(t)$. Thus $\bar{T} = \bar{K} + \bar{L}$. If $x \in \bar{K} \cap \bar{L}$ then $t \cdot x = x$ so $0 = \Phi(t) \cdot x = d \cdot x$; so $\bar{K} \cap \bar{L} = 0$. We have shown:

(3) $\bar{T} = \bar{K} \oplus \bar{L}$;

(4) $\bar{K}$ is the d-reduction of the torsion subgroup of $(\mathrm{Ker}(\rho_*))$.

(5) $\bar{L}$ is isomorphic to the d-reduction of the torsion subgroup of $H_1(M)$.

As above $\bar{K}$ is a $\mathbb{Z}[\xi]$-module. Again we quote [CM; Theorem 2.5] that for an odd prime d and a prime $q \neq d$ the q-torsion subgroup of a finitely generated $\mathbb{Z}[\xi]$-module has the form $A^{g_d(q)}$ for some q-torsion group, A.

To complete the proof of conclusion (2) we must show that for d odd $\bar{K}$ is a direct double. This follows as in [VW] with slight modifications. We outline the argument for completeness. So we consider the linking form

$$\mathcal{L} : T \otimes T \longrightarrow \mathbb{Q}/\mathbb{Z}$$

defined on the torsion subgroup T of $H_1(\tilde{M})$. This is a symmetric, $\mathbb{Z}$-bilinear, unimodular form which is invariant under the action of t and which decomposes orthogonally over the prime power decomposition of T. So $\mathcal{L}$ induces a unimodular form on the d-reduction $\bar{T}$ of T. We further need to show that $\mathcal{L}$ decomposes orthogonally over $\bar{T} = \bar{K} \oplus \bar{L}$. So take $x \in \bar{K}, y \in \bar{L}$. Multiplication by d is an isomorphism on $\bar{L}$; so $y = d \cdot y_1$ for some y_1. Then

$$\begin{aligned}\mathcal{L}(x,y) &= \mathcal{L}(x, d \cdot y_1)\\ &= \sum \mathcal{L}(x, t^i \cdot y_1)\\ &= \sum \mathcal{L}(t^{-i} \cdot x, y_1)\\ &= \mathcal{L}\big(\Phi(t) \cdot x, y_1\big)\\ &= 0.\end{aligned}$$

Thus $\mathcal{L}$ is unimodular on $\bar{K}$ and we define a second form $\langle\ ,\ \rangle$ on $\bar{K}$ by

$$\langle x, y\rangle = \mathcal{L}\big(x, (t - t^{-1}) \cdot y\big)$$

This form is alternate, i.e. $\langle x, x\rangle = 0$, and unimodular because $t - t^{-1} = t^{-1}(1-t)(1+t)$ is an isomorphism on $\bar{K}$—$\mathrm{Ker}(1-t) \in \bar{L}$ and $\Phi(t) \equiv 1 \mod(1+t)$ since d is odd. The proof of (2) is now finished by quoting the theorem of de Rham [dR] that a finite abelian group supporting such a form is a direct double.

Now consider the case $d = 2$. We are assuming that the nontrivial covering transformation, which we identify with t, has an orientation preserving square root $s : s^2 = t$. Then $H_1(\tilde{M})$ is a module over $\mathbb{Z}C_1$, $C_1 = \langle s : s^4 = 1\rangle$, and we have a sequence

$$\tilde{M} \xrightarrow{\rho} M \xrightarrow{\rho_1} M_1$$

of covering maps where the composition $\rho_1 \circ \rho$ is a degree 4 cyclic covering with covering group C_1.

Then $\mathrm{Ker}(\rho_*)$ is annihilated by $1 + s^2$ and so is a module over $\mathbb{Z}[i]$. It then follows as in [VW] that $\mathrm{rank}(\mathrm{Ker}(\rho_*))$ is even. This establishes (1′). Compare with the alternate argument in Section 4.

For (2′) we proceed as above, noting that $\mathcal{L}$ is a unimodular form on $\bar{K}$, but define $\langle\ ,\ \rangle$ on $\bar{K}$ by

$$\langle x, y\rangle = \mathcal{L}(x, (s - s^{-1}) \cdot y).$$

We need only check that $s - s^{-1}$ or, equivalently, that $1 - s^2$ is an isomorphism on $\bar{K}$. If $x \in \bar{K}$ and $(1 - s^2) \cdot x = 0$, then since $(1 + s^2) \cdot x = 0$ we have $2 \cdot x = 0$. Since $\bar{K}$ has no 2-torsion, $x = 0$.

3. **Presentations for the Homology of Branched Covers.** In this Section we describe a procedure for calculating the first homology of a covering of a 3-manifold branched over a link. It comes from relating this homology to a certain relative homology group of the associated unbranched covering combined with a general method for relating relative first homology of covering pairs to presentations of the corresponding

fundamental group pairs of the base through a relative form of the free differential calculus.

Let G be a group and H a subgroup. A *presentation* for the group pair (G, H) (cf. [H_1]) is a system

$$P = \langle x_1, \ldots, x_{n+t}; x_{n+1}, \ldots, x_{n+t} : r_1, \ldots, r_{m+s}; r_{m+1}, \ldots, r_{m+s} \rangle$$

where

$$\langle x_1, \ldots, x_{n+t} : r_1, \ldots, r_{m+s} \rangle$$

presents G and

$$\langle x_{n+1}, \ldots, x_{n+t} : r_{m+1}, \ldots, r_{m+s} \rangle$$

presents H in a way that is consistent with the inclusion $H \longrightarrow G$.

The *relative Jacobian* of such a presentation is the matrix

$$J = \begin{bmatrix} \partial r_1/\partial x_1 & \cdots & \partial r_1/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial r_m/\partial x_1 & \cdots & \partial r_m/\partial x_n \end{bmatrix}$$

whose entries are the images in $\mathbb{Z}G$ of the free derivatives of the relators not involved with H with respect to the generators not involved with H.

Suppose (M, B) is a finite CW-pair with M connected. The associated *joined pair* (M^*, B^*) is obtained from (M, B) by fixing a component of B and joining the remaining components of B to a basepoint in the fixed component by arcs whose interiors are disjoint from M. This gives a well defined *joined fundamental group system* $\big(\pi_1(M^*), i_\sharp\pi_1(B^*)\big)$. Clearly $\pi_1(M^*) = \pi_1(M) * F$ where F is free of rank $\beta_0(B) - 1$ and $\pi_1(B^*)$ is the free product of the fundamental groups of the components of B.

Given a covering space $\rho: \tilde{M} \longrightarrow M$ with monodromy $\varphi: \pi_1(M) \longrightarrow S_d$ and a retraction $f: M^* \longrightarrow M$ the covering $\tilde{M}^* \longrightarrow M^*$ whose monodromy is $\varphi \circ f_\sharp$ is an extension of $\rho: \tilde{M} \longrightarrow M$. In the following it will not matter which retraction f is chosen, but in doing calculations a consistent choice must be made.

In this section and the next we always regard the symmetric group S_d on d symbols as being a subgroup of $GL(d, \mathbb{Z})$ by identifying a permutation with the linear transformation which so permutes the standard basis vectors:

$$\varphi \longmapsto (\delta_{i\varphi, j})$$

THEOREM 3.1. *Let*

$$\rho: \tilde{M} \longrightarrow M$$

be a d-sheeted covering space of a connected CW-complex M, B be a non-empty subcomplex of M, $\tilde{B} = \rho^{-1}(B)$, and

$$\varphi: \pi_1(M^*) \longrightarrow S_d \subset GL(d, \mathbb{Z})$$

be the monodromy of an extension of ρ to a cover of M^*. If J is the relative Jacobian matrix of a presentation of the joined fundamental group system $\big(\pi_1(M^*), i_\sharp\pi_1(B^*)\big)$, then the $m\cdot d\times n\cdot d$ matrix of integers $\varphi(J)$ presents $H_1(\tilde{M},\tilde{B})$ as an abelian group.

PROOF. If J is the relative presentation for the pair $\big(\pi_1(M^*), i_\sharp\pi_1(B^*)\big)$ arising from a cell structure on (M^*, B^*), then as in [H$_2$; Theorem 3.1] $\varphi(J)$ is the matrix of

$$\partial\colon C_2(\tilde{M}^*,\tilde{B}^*)\longrightarrow C_1(\tilde{M}^*,\tilde{B}^*)$$

in terms of the natural $\mathbb{Z}$-bases arising from the lifted cell structure, and so presents

$$\operatorname{Coker}(\partial)=H_1(\tilde{M}^*,\tilde{B}^*)\cong H_1(\tilde{M},\tilde{B}).$$

By arguments of Fox [Fo] the relative Jacobians, J_1, J_2 of any two presentations are equivalent (see [H$_1$; Theorem 2.2 ff]) over $\mathbb{Z}\pi_1(M^*)$. This induces an equivalence between $\varphi(J_1)$ and $\varphi(J_2)$ and thus an isomorphism between the abelian groups which they present. ∎

Now consider an orientable 3-manifold M with a k-component link $L\subset \operatorname{Int}(M)$. Remove an open regular neighborhood of L from M to obtain a 3-manifold N. Let $\mu=\{\mu_1,\ldots,\mu_k\}$ be a system of meridians for L in ∂N. Let

$$\rho\colon\tilde{M}\longrightarrow M$$

be a degree d cover branched over L and with monodromy

$$\varphi\colon\pi_1(N)\longrightarrow S_d.$$

Let

$$\begin{aligned}c\big(\varphi(\mu)\big)&=\#\text{ of components of }(\rho^{-1}(\mu))\\&=\sum c\big(\varphi(\mu_i)\big)\end{aligned}$$

where $c\big(\varphi(\mu_i)\big)$ is the number of cycles of the permutation $\varphi(\mu_i)$. With this notation we have

THEOREM 3.2. *Let J be the relative Jacobian of any presentation of the joined fundamental group system $\big(\pi_1(N^*), i_\sharp\pi_1(\mu^*)\big)$. Then $\varphi(J)$ is a presentation matrix over $\mathbb{Z}$ of*

$$H_1(\tilde{M})\oplus\mathbb{Z}^{c(\varphi(\mu))-1}.$$

PROOF. Each meridian μ_i bounds a disk D_i in $M-\operatorname{Int}(N)$. Put $D=\bigcup D_i$. By 3.1 $\varphi(J)$ presents $H_1\big(\tilde{N},\rho^{-1}(\mu)\big)$ which by excision is isomorphic to

$$H_1\big(\tilde{N}\cup\rho^{-1}(D),\rho^{-1}(D)\big).$$

The exact sequence

$$0\longrightarrow H_1\big(\tilde{N}\cup\rho^{-1}(D)\big)\longrightarrow H_1\big(\tilde{N}\cup\rho^{-1}(D),\rho^{-1}(D)\big)\longrightarrow\tilde{H}_1\big(\rho^{-1}(D)\big)\longrightarrow 0$$

and the observation that $H_1(\tilde{M}) \cong H_1\big(\tilde{N} \cup \rho^{-1}(D)\big)$ completes the proof. ∎

For the remainder of this section we assume that M is compact and ∂M (possibly $\emptyset$) consists of tori. Then $\chi(N) = 0$ and $\partial N \neq 0$ so we have a presentation

$$\pi_1(N) = \langle x_1, \ldots, x_{n+1} : r_1, \ldots, r_n \rangle$$

with one more generator than relations. We may further suppose that for $1 \leq i \leq k$, x_i is represented by the meridian μ_i—properly joined to the base point.

Now in forming the joined fundamental group system $\big(\pi_1(N^*), i_\sharp \pi_1(\mu^*)\big)$ we add $k-1$ free generators, say $y_2, \ldots, y_k$ corresponding to the arcs joining $\mu_2, \ldots, \mu_k$ to a basepoint in μ_1.

The y_i can be chosen so that $i_\sharp \pi_1(\mu^*)$ is freely generated by

$$x_1, y_2x_2y_2^{-1}, \ldots, y_kx_ky_k^{-1}$$

To present the pair we need named free generators for $i_\sharp \pi_1(\mu^*)$. Call these $z_1, \ldots, z_k$. Then we add the relations:

$$x_1z_1^{-1} = 1, y_2x_2y_2^{-1}z_2^{-1} = 1, \ldots, y_kx_ky_k^{-1}z_k^{-1} = 1$$

We may assume that $\pi_1(\mu^*) \longrightarrow \pi_1(M^*)$ is monic; otherwise some μ_i bounds a disk in N and the relation $z_i = 1$ would be added in both the presentation of $\pi_1(N^*)$ and the presentation of $i_\sharp\big(\pi_1(\mu^*)\big)$ and would not be involved in the relative Jacobian. Hence this relative Jacobian comes from taking free derivatives of the relations (old and new) with respect to the x_i's and y_j's. It is:

$$\begin{bmatrix}
\frac{\partial r_1}{\partial x_1} & \frac{\partial r_1}{\partial x_2} & \cdots & \frac{\partial r_1}{\partial x_k} & \cdots & \frac{\partial r_1}{\partial x_{n+1}} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial r_n}{\partial x_1} & \frac{\partial r_n}{\partial x_2} & \cdots & \frac{\partial r_n}{\partial x_k} & \cdots & \frac{\partial r_n}{\partial x_{n+1}} & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & y_2 & & 0 & & 0 & 1 - y_2x_2y_2^{-1} & & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & y_k & \cdots & 0 & 0 & \cdots & 1 - y_kx_ky_k^{-1}
\end{bmatrix}$$

We may project to $\pi_1(N)$ by $y_i \longmapsto 1$ and then eliminate the last k rows and first k columns by row and column operations to get an equivalence over $\mathbb{Z}\pi_1(N)$ between the above matrix and the $n \times n$ matrix

$$(3.3) \qquad J = \begin{bmatrix}
\frac{\partial r_1}{\partial x_{k+1}} & \cdots & \frac{\partial r_1}{\partial x_{n+1}} & \frac{\partial r_1}{\partial x_2}(x_2 - 1) & \cdots & \frac{\partial r_1}{\partial x_k}(x_k - 1) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial r_n}{\partial x_{k+1}} & \cdots & \frac{\partial r_n}{\partial x_{n+1}} & \frac{\partial r_n}{\partial x_2}(x_2 - 1) & \cdots & \frac{\partial r_n}{\partial x_k}(x_k - 1)
\end{bmatrix}$$

This has the form

$$J = [J_0, J_1]$$

where J_0 is an $n \times (n+1-k)$ matrix.

If we apply 3.2 to the augmentation map $\epsilon: \pi_1(N) \longrightarrow \mathbb{Z}$ which maps each group element to 1 (and is the monodromy of the trivial cover), we see that $\epsilon(J)$ presents $H_1(M) \oplus \mathbb{Z}^{k-1}$.

Since $\epsilon(J_1) = 0$ we have

LEMMA 3.4. *$\epsilon(J_0)$ presents $H_1(M)$.*

4. **Theorem A Revisited.** In this section we continue with the notation of the previous section with the following additions.

$$\rho: \tilde{M} \longrightarrow M$$

will be a cyclic cover of prime degree d branched over a k component link $L \subset M$ corresponding to an epimorphism

$$\theta: \pi_1(N) \longrightarrow C = \langle t : t^d = 1 \rangle.$$

We can identify the monodromy φ of this cover with the composition of θ with the action of C on itself by right multiplication.

We assume that branching occurs over each component of L and that $L \neq \emptyset$. The case of a true cover ($L = \emptyset$) is slightly different—see 4.5 below. Thus φ maps each meridian of L to a d-cycle and so

$$c\big(\varphi(\mu)\big) = k.$$

Let

$$J = [J_0, J_1]$$

be the $n \times n$ matrix over $\mathbb{Z}\pi_1(N)$ given in 3.3, and let

$$J(t) = [J_0(t), J_1(t)] \overset{\text{def}}{=} [\theta(J_0), \theta(J_1)]$$

be its image over $\mathbb{Z}C$.

Let $\xi = e^{2\pi i/d}$ and consider the rings

$$\mathbb{Z} \subset \Lambda \overset{\text{def}}{=} \mathbb{Z}[\xi] \subset \Lambda_1 \overset{\text{def}}{=} \mathbb{Z}[1/d, \xi].$$

The following is well known (see [H_2; 3.4]) and doesn't require that d be prime.

THEOREM 4.1. *There is a matrix $W \in GL(n\cdot d, \Lambda_1)$ such that $W\varphi(J)W^{-1}$ is the block diagonal*

$$D = \begin{bmatrix} J(1) & 0 & \cdots & 0 \\ 0 & J(\xi) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & J(\xi^{d-1}) \end{bmatrix}$$

Now by 3.2 the abelian group presented by $\varphi(J)$ is $H_1(\tilde{M}) \oplus \mathbb{Z}^{k-1}$. By 4.1 the Λ_1-modules presented by $\varphi(J)$ and D are isomorphic. We want to compare these modules as abelian groups. First we note

LEMMA 4.2. *Let H be an abelian group presented by a matrix K of integers. Then the Λ_1-module presented by K is $\mathbb{Z}$-isomorphic to* $\{d\text{-red}(H)\}^{d-1}$.

PROOF. This follows from the observation that as an abelian group Λ_1 is free on $1/d, \xi/d, \ldots, \xi^{d-2}/d$. ∎

Now consider the blocks $J(\xi), \ldots, J(\xi^{d-1})$ of D. *A fortiori* they present Λ-modules.

LEMMA 4.3. *The matrices $J(\xi), \ldots, J(\xi^{d-1})$ present mutually isomorphic Λ-modules.*

PROOF. Since d is prime $\xi \mapsto \xi^q$, $2 \le q \le d-1$ induces an isomorphism of Λ which in turn induces the desired isomorphism. ∎

Let K be the Λ-module presented by $J(\xi)$. Using 3.4, 4.1, 4.2 applied to each of the matrices of integers $\varphi(J)$ and $J(1)$, and 4.3 we get a $\mathbb{Z}$-isomorphism

$$d\text{-red}[H_1(\tilde{M}) \oplus \mathbb{Z}^{k-1}]^{d-1} \cong d\text{-red}[H_1(M) \oplus \mathbb{Z}^{k-1} \oplus K]^{d-1}.$$

This simplifies to

LEMMA 4.4. *As abelian groups we have an isomorphism*

$$d\text{-red}\big(H_1(\tilde{M})\big) \cong d\text{-red}(H_1(M) \oplus K)$$

Recall that the above statement depends on the assumption that branching actually occurs. In the case of a true cover we modify the argument as follows. Take any 1-component link $L \subset M$ and regard the cover as being branched over L with trivial branching. So the monodromy is trivial on a meridian of L and so $c\big(\varphi(\mu)\big) = d$. One then continues the argument as above to obtain

LEMMA 4.5. *If $\tilde{M} \longrightarrow M$ is a true covering (cyclic of prime degree d) then we have an isomorphism of abelian groups*

$$d\text{-red}(H_1(\tilde{M})) \oplus \mathbb{Z}^{d-1} \cong d\text{-red}(H_1(M) \oplus K)$$

We now review the conclusions of Theorem A in the context of these calculations.

A1. This follows directly from 4.1 and 4.3 by comparing the nullities of the matrices $\varphi(J)$ and D; although the case of a true cover must also use 4.5. For a branched cover 4.4 shows that

$$\beta_1(\tilde{M}) - \beta_1(M) = \text{nullity}\big(J(\xi)\big)$$

In particular there is a generalized *Alexander polynomial*

$$\Delta = \det\big(\eta(J)\big) \in \mathbb{Z}H_1(N), \quad \eta : \pi_1(N) \longrightarrow H_1(N) \text{ abelianization}$$

whose vanishing under the specialization

$$\mathbb{Z}H_1(N) \longrightarrow \mathbb{Z}C \longrightarrow \mathbb{Z}[\xi]$$

is necessary in order to have $\beta_1(\tilde{M}) > \beta_1(M)$. By contrast, 4.5 shows that in the case of a true cover $J(\xi)$ is always singular.

A1$'$. Let J_1 be the relative Jacobian for a presentation of the pair $\big(\pi_1(N_1), i_\sharp\pi_1(\mu_1)\big)$ associated with the manifold $M_1 = \tilde{M}/C_1$. Note that $\rho_1^{-1}(\mu_1)$ and $(\rho_1 \circ \rho)^{-1}(\mu_1)$ have the same number, say c, of components. Consider the block matrix

$$D = \begin{bmatrix} J(1) & 0 & 0 & 0 \\ 0 & J(-1) & 0 & 0 \\ 0 & 0 & J(i) & 0 \\ 0 & 0 & 0 & J(-i) \end{bmatrix}$$

Then nullity$(D) = \beta_1(\tilde{M}) + c - 1$ and for the submatrix D_1 consisting of the first two blocks nullity$(D_1) = \beta_1(M) + c - 1$. Then $(1')$ follows from the observation that $J(i)$ and $J(-i)$ have the same nullity.

A2. There are two aspects to this statement. The part about the direct double comes from symmetry imposed by that of the linking form. The other comes from the fact that $\mathrm{Ker}(\rho_*)$ is a Λ-module. In the present context we have that K is a Λ-module whose d-reduction, by 4.4 or 4.5, is $\mathbb{Z}$-isomorphic to the d-reduction of $\mathrm{Ker}(\rho_*)$. The advantage here is that we obtain a specific presentation for K. It is natural to ask whether K and $\mathrm{Ker}(\rho_*)$ are isomorphic as Λ-modules. The answer is no, they are not even isomorphic as abelian groups, as example 5.4 shows. There is, however, some relation between the d-torsion of these modules as given in Lemma 4.6 below.

A3. This is best proved as in the beginning of Section 2. Lemma 4.6 gives some additional information about the d-torsion subgroup $H_1(\tilde{M})$.

A4. If ρ is a true cover, then $\pi_1(M)$ maps onto $\mathbb{Z}_d$ and the result is trivial. Otherwise we have $k = 1$ and so $J = J_0$. Since $\beta_1(\tilde{M}) > 0$, we must have $\det\big(J(\xi)\big) = 0$ and thus $\Phi(t)$ divides $\det\big(J_0(t)\big)$. Then $d = \Phi(1)$ divides $\det\big(J_0(1)\big) = o\big(H_1(M)\big)$.

This is false for links with more than one component. See Example 5.5.

LEMMA 4.6. *In the case ρ has nontrivial branching there is a map*

$$e\colon \mathrm{Ker}(\rho_*) \longrightarrow K = \mathbb{Z}[\xi]\text{-}module\ presented\ by\ J(\xi)$$

such that $d \cdot \mathrm{Ker}(e) = d \cdot \mathrm{Coker}(e) = 0$.

If $H_1(M) = 0$ *then* $\mathrm{Ker}(e) = 0$; *if in addition L is connected then e is an isomorphism.*

PROOF. Let $\epsilon\colon \mathbb{Z}C \longrightarrow \mathbb{Z}$ be the augmentation map and let $\mathcal{A} = \mathrm{Ker}(\epsilon)$ be the augmentation ideal of $\mathbb{Z}C$. Recall that in the case of a prime degree cyclic cover with actual branching $c\big(\varphi(\mu)\big) = k =$ the number of components of L.

From the arguments of Section 3 we get a commutative diagram with exact rows and columns

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{A}^n \arrow[r] & {[\mathbb{Z}C]^n} \arrow[r, "\epsilon^n"] \arrow[d, "J(t)"'] & \mathbb{Z}^n \arrow[r] \arrow[d, "J(1)"] & 0 \\
0 \arrow[r] & \mathcal{A}^n \arrow[r] \arrow[d] & {[\mathbb{Z}C]^n} \arrow[r, "\epsilon^n"] \arrow[d, "\eta"'] & \mathbb{Z}^n \arrow[r] \arrow[d] & 0 \\
 & X \arrow[r, "h"] \arrow[d] & H_1(\tilde{M}) \oplus \mathbb{Z}^{k-1} \arrow[r, "{\rho_*,1}"] \arrow[d] & H_1(M) \oplus \mathbb{Z}^{k-1} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The two vertical sequences on the right come from 3.1 and 3.2, together with the identification of the groups

$$H_1(\tilde{M}, \tilde{\mu}) = H_1(\tilde{M}) \oplus \mathbb{Z}^{k-1} \quad H_1(M, \mu) = H_1(M) \oplus \mathbb{Z}^{k-1}.$$

The identification of the bottom right map with $(\rho_*, 1)$ comes from the fact that ρ maps distinct components of $\tilde{\mu}$ to distinct components of μ. Thus

$$\mathrm{Image}(h) = \mathrm{Ker}(\rho_*)$$

Now consider the diagram

$$\begin{tikzcd}
\mathcal{A}^n \arrow[r] & {[\mathbb{Z}C]^n} \arrow[r, "\zeta^n"] \arrow[d, "J(t)"] & {\mathbb{Z}[\xi]^n} \arrow[d, "J(\xi)"] \\
\mathcal{A}^n \arrow[r] \arrow[d] & {[\mathbb{Z}C]^n} \arrow[r, "\zeta^n"] \arrow[d, "\eta"] & {\mathbb{Z}[\xi]^n} \arrow[d, "\bar{\eta}"] \\
X \arrow[r] \arrow[d] & H_1(\tilde{M}) \oplus \mathbb{Z}^{k-1} \arrow[r, "E"] \arrow[d] & K \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

Note that

(1) For $z \in \mathbb{Z}C$, $z \cdot \Phi(t) = \epsilon(z) \cdot \Phi(t)$;

(2) $\zeta \,|\, \mathcal{A} : \mathcal{A} \longrightarrow \mathbb{Z}[\xi]$ is monic, since non-trivial elements of $\mathrm{Ker}(\zeta)$ map to non-zero multiples of d under ϵ.

(3) $\mathrm{Image}(\zeta) = d \cdot \mathbb{Z}[\xi]$, since given $f(\xi) \in \mathbb{Z}[\xi]$, $d \cdot f(\xi) = \zeta\big(d \cdot f(t) - \epsilon\big(f(t)\big) \cdot \Phi(t)\big)$.

The desired map is

$$e = E \big| h(X) \colon \mathrm{Ker}(\rho_*) \longrightarrow K.$$

Since $\zeta^n(\mathcal{A}^n) = d \cdot \mathbb{Z}[\xi]^n$, $\mathrm{Image}(e) \supset d \cdot K$; so $d \cdot \mathrm{Coker}(e) = 0$.

If $x \in \mathcal{A}^n$ and $\eta(x) \in \mathrm{Ker}(e)$, then $\zeta^n(x) = J(\xi) \cdot y$ for some $y \in \mathbb{Z}[\xi]^n$. Choose $z \in \mathcal{A}^n$ such that $\zeta^n(z) = d \cdot y$. Then $J(t) \cdot z$ and $d \cdot x$ both map to the same element of $\mathbb{Z}[\xi]^n$ and both are in $\mathcal{A}^n$; so they are equal. Thus $d \cdot \eta(x) = 0$. This shows that $d \cdot \mathrm{Ker}(e) = 0$.

Now suppose that $H_1(M) = 0$. For x and y as above, there is some $w \in \mathbb{Z}C^n$ with $\zeta^n(w) = y$. Now $J(t) \cdot w - x \in \mathrm{Ker}(\zeta^n)$ and so has the form $\big(r_1\Phi(t), \ldots, r_n\Phi(t)\big)$ for some $r_1, \ldots, r_n \in \mathbb{Z}$. By assumption there is some $(s_1, \ldots, s_n) \in \mathbb{Z}^n$ such that $J(1) \cdot (s_1, \ldots, s_n) = (r_1, \ldots, r_n)$.

Then for $w_1 = w - \big(s_1\Phi(t), \ldots, s_n\Phi(t)\big)$, we have $J(t) \cdot w_1 = x$. So $\eta(x) = 0$. Finally if L is connected, $E = e$ is an epimorphism of $\mathrm{Ker}(\rho_*) = H_1(\tilde{M})$ to K ■

We conclude this section with yet one more interpretation of the connection with the module K and $H_1(\tilde{M})$. Let

$$\Xi = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -1 & -1 & -1 & \cdots & -1 \end{bmatrix}$$

be the companion matrix of the polynomial $1 + x + x^2 + \cdots + x^{d-1}$ which we may also think of the matrix of the action, by multiplication, of ξ on $\mathbb{Z}[\xi]$ in terms of the $\mathbb{Z}$-basis $1, \xi, \ldots, \xi^{d-1}$. If we replace each occurrence of ξ in $J(\xi)$ by Ξ, we get an $n \cdot (d-1) \times n \cdot (d-1)$ matrix of integers which we denote by $\psi(J)$. It is straightforward to show

LEMMA 4.7. *The abelian group presented by $\psi(J)$ is $\mathbb{Z}$-isomorphic to K.*

Now the matrix $\varphi(J)$ is obtained from $J(t)$ by replacing each occurrence of t by the matrix of the action of t on $\mathbb{Z}C$ in the $\mathbb{Z}$-basis $1, t, \ldots, t^{d-1}$ which t cyclically permutes. In terms of the equivalent $\mathbb{Z}$-basis

$$1, t, \ldots, t^{d-2}, 1 + t + \cdots + t^{d-1}$$

the matrix of t is

$$\Upsilon = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ -1 & -1 & -1 & \cdots & -1 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} = \begin{bmatrix} & & & 0 \\ & \psi(J) & & \vdots \\ & & & 1 \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

If we replace each occurrence of t in $J(t)$ by this matrix we get something which, after permuting rows and columns we can identify by

LEMMA 4.8. *There is a matrix $V \in SL(n \cdot d, \mathbb{Z})$ such that*

$$V\varphi(J)V^{-1} = \begin{bmatrix} \psi(J) & * \\ 0 & J(1) \end{bmatrix}.$$

This lemma could be used to give an alternate proof of the last part of Lemma 4.6; since an equivalence between $J(1)$ and I will induce an equivalence between $V\varphi(J)v^{-1}$ and $\psi(J)$. In general it seems to be weaker than 4.6; except that in certain calculations specific information about the nature of the upper righthand block marked $*$ may be used to draw additional conclusions.

5. **Examples.**

5.1. For the q-fold cyclic cover M_q of S^3 branched over the trefoil knot we have

$$H_1(M_2) = \mathbb{Z}_3, \quad H_1(M_3) = \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad H_1(M_6) = \mathbb{Z} \oplus \mathbb{Z}.$$

Since M_6 is a 3-fold cyclic cover of M_2 and a 2-fold cyclic cover of M_3, this shows the necessity of the condition $\beta_1 = 0$ in conclusion A3. This also illustrates conclusion A4.

5.2. Suppose M is a q-fold irregular dihedral cover of S^3 branched over a knot L. Then a 2-fold cyclic branched cover $\tilde{M}$ of M is a $2q$-fold regular cover of S^3 branched over L which is solvable—it can be factored

$$\tilde{M} \longrightarrow M_1 \longrightarrow S^3$$

where M_1 is the 2-fold cyclic cover of S^3 branched over L and $\tilde{M}$ is a q-fold cyclic true cover of M_1. It is well known that $H_1(M_1)$ always has (finite) odd order.

For $q = 3$ any closed, orientable 3-manifold can be represented as M as above. This can be used to illustrate the differences between even and odd degree cyclic covers—in particular that the existence of the square root is necessary for conclusions A1′ and A2′.

5.3. Take the two component link $L \subset S^3$ formed by linking two unknotted circles by r full twists as shown in Figure 1.

The d-sheeted cyclic covers $\tilde{M}_d$ of S^3 branched over L corresponding to $x \longmapsto t, \quad y \longmapsto t^{-1}$ has (see [VW; Prop. 3.1])

$$H_1(\tilde{M}_d) = \mathbb{Z}_r{}^{d-2} \oplus \mathbb{Z}_{rd}$$

for r prime to d this is

$$H_1(\tilde{M}_d) = \mathbb{Z}_r{}^{d-1} \oplus \mathbb{Z}_d.$$

This nicely illustrates conclusion 2 of Theorem A; since for d prime $g_d(r)$ always divides $d - 1$, and so for appropriate r we can exhibit all possibilities.

5.4. Take the link from 5.3 with $r = 3$ then

$$H_1(\tilde{M}_3) = \mathbb{Z}_3 \oplus \mathbb{Z}_9, \quad H_1(\tilde{M}_9) = \mathbb{Z}_3{}^7 \oplus \mathbb{Z}_{27}$$

For the branched cover $\rho\colon \tilde{M}_3 \longrightarrow S^3$ one calculates (say using 4.7) that

$$K = \mathbb{Z}_9 \oplus \mathbb{Z}_9$$

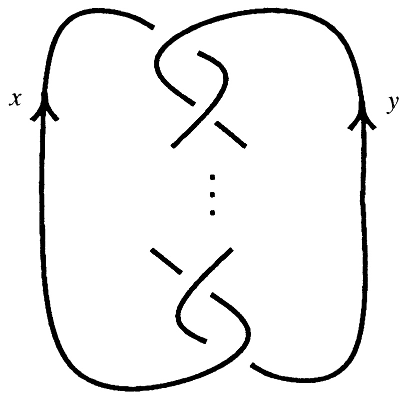

Figure 1.

Thus the map e: Ker$(\rho_*) \longrightarrow K$ need not be epic.

Now consider that we also have a branched covering $\rho: \tilde{M}_9 \longrightarrow \tilde{M}_3$. By Theorem A $\rho_*: H_1(\tilde{M}_9) \longrightarrow H_1(\tilde{M}_3)$ must be epic. This forces

$$\mathrm{Ker}(\rho_*) = \mathbb{Z}_3{}^7.$$

Again we can calculate K for this cover by applying 4.7 to a presentation of $\pi_1(\tilde{M}_3 -$ branch set) and get

$$K = \mathbb{Z}_3{}^4 \oplus \mathbb{Z}_9{}^2.$$

Thus we see that e: Ker$(\rho_*) \longrightarrow K$ need not be monic.

5.5. Let L be a split link of (say) two components in S^3. Then any branched cover $\tilde{M}$ of S^3 branched over L can be decomposed

$$\tilde{M} = \tilde{M}_1 \# \tilde{M}_2 \# S^2 \times S^1 \# \cdots \# S^2 \times S^1.$$

In particular $\beta_1(\tilde{M}) > 0$.

Similar statements can be made in more general situations—e.g. $\beta_1(\tilde{M}) > 0$ for every cyclic cover of S^3 branched over a link which is split by a surface S such that $H_1(S) \longrightarrow H_1(S^3 - L)$ is trivial.

References

[CM] Valerio Chumillas and José Montesinos, *The homology of cyclic and irregular dihedral coverings branched over homology spheres*, Math. Ann. **280**(1988), 483–500.

[F] E. E. Floyd, *Orbit spaces of finite transformation groups I*, Duke Math. J. **20**(1953), 563–567.

[Fo] R. H. Fox, *Free differential calculus V, The Alexander matrices re-examined*, Ann. of Math. **71**(1960), 408–422.

[H_1] John Hempel, *Intersection Calculus on Surfaces with Applications to 3-manifolds*, Memoirs of the Amer. Math Soc. No. 282 **43**(1983), 1–48.

[H2]———, *Homology of coverings*, Pacific J. Math. **112**(1984), 83–113.

[H3]———, *Coverings of Dehn fillings of surface bundles*, Topology and its applications **24**(1986), 157–170.

[P] A. Plans, *Aportacion al astudio de los grupos de homologia de los recubrimientos circlicos ramificados correspondientes a un nudo*, Rev. Real Acad. Cienc. Exact. Fis. Natur. Madrid **47**(1953), 161–193 .

[dR] G. de Rham, *Sur l'analysis situs des variétés à n dimensions*, J. Math. Pures Appl. **10**(1931), 115–200.

[VW] Pablo del Val and Claude Weber, *Plans' theorem for links*, Topology and its Applications **34**(1990), 247–255.

Rice University
Houston, Texas
USA 77251
email: hempel@rice.edu