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# ON THE INSOLUBILITY OF A CLASS OF DIOPHANTINE EQUATIONS AND THE NONTRIVIALITY OF THE CLASS NUMBERS OF RELATED REAL QUADRATIC FIELDS OF RICHAUD-DEGERT TYPE

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## Introduction

Many authors have studied the relationship between nontrivial class numbers $h(n)$ of real quadratic fields $\boldsymbol{Q}(\sqrt{n})$ and the lack of integer solutions for certain diophantine equations. Most such results have pertained to positive square-free integers of the form $n = \ell^2 + r$ with integer $\ell > 0$, integer $r$ dividing $4\ell$ and $-\ell < r \leq \ell$. For $n$ of this form, $\boldsymbol{Q}(\sqrt{n})$ is said to be of *Richaud-Degert* (R–D) type (see [3] and [8]; as well as [2], [6], [7], [12] and [13] for extensions and generalizations of R–D types.) For example, if $r = 1$ (respectively $r = 4$) Ankeny, Chowla and Hasse [1] (respectively S.D. Lang [5]) have shown that $x^2 - ny^2 = \pm m$ (respectively $x^2 - ny^2 = \pm 4m$) for a positive integer $m$ has integer solutions $(x, y)$ only when $m \geq 2l$ (respectively $m \geq l$). They used this fact to show that $h(p) > 1$ when $p = (2rq)^2 + 1$ is prime with prime $q$ and integer $r > 1$ (respectively $p = ((2r + 1)q)^2 + 4$ is prime with prime $q > 2$ and integer $r > 0$). For $r = -2$, H. Takeuchi [9] proved the insolubility in integers $(x, y)$ of $x^2 - py^2 = \pm 3$ in the special case where $12m + 7$ and $p = (3(8m + 5))^2 - 2$ are primes and $m \geq 0$ is an integer. More recently H. Yokoi [11] considered the case $|r| = 2$. He generalized Takeuchi's results by showing that for odd primes $p = ((2r + 1)q)^2 \pm 2$ (with $r \geq 0$) and $q > 2$ prime) $x^2 - py^2 = \pm q$ is solvable in integers $(x, y)$ if and only if $p = 7$ and $q = 3$. It is the purpose of this paper to generalize all of the above by considering arbitrary integers $r$ in the R–D types. The first section will deal with the insolubility in integers $(x, y)$ of $x^2 - ny^2 = \pm 4t$ where $t$ is a positive integer and $\boldsymbol{Q}(\sqrt{n})$ is of R–D type. We will use these results in the second section to establish $h(n) > 1$ for a wide class of R–D types including all those

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mentioned above from the literature. As an illustration we provide a table of values of $h(n) > 1$ for which our method applies.

## §1. Insolubility of $x^2 - ny^2 = \pm 4t$

Let $n$ be a square-free positive integer and let $t$ be any positive integer. Suppose that $(u, v)$ is a rational integral solution of $x^2 - ny^2 = \pm 4t$. We say that $(u, v)$ is a *trivial solution* when $t = m^2$ and $m$ divides both $u$ and $v$. Otherwise $(u, v)$ is called *nontrivial.*

The following result is attributable to Davenport, Ankeny and Hasse (D–A–H), (see [4]) in the case where $t$ is not a perfect square. We are indebted to Professor H. Yokoi for providing, in a recent letter to the author, a proof of the case where $t$ is a square. In granting the author permission to include the proof in this paper, Professor Yokoi has indicated that the proof was communicated to him by Humio Ichimura in relation to [12, Theorem 3, p. 147]. Note that this proof completes an omission in the proof of [12], (ibid.); and in fact the letter from Professor Yokoi to the author was a response to a query about that omission. We state the result in a fashion not usually found in the literature, but which will better serve our purposes later, and we have altered the aforementioned proof to suit the content of the paper.

LEMMA 1.1. *Let $n$ be a square-free positive integer and $t$ any positive integer. Suppose that $(A + B\sqrt{n})/\sigma$ is the fundamental unit of $\boldsymbol{Q}(\sqrt{n})$ where $\sigma = 2$ if $n \equiv 1 \pmod 4$, $\sigma = 1$ otherwise and $A^2 - nB^2 = \sigma^2\delta$, (i.e., $\delta = \pm 1$). If there exists a nontrivial solution to the diophantine equation $x^2 - ny^2 = \pm \sigma^2 t$ then $t \geq ((2A/\sigma) - \delta - 1)/B^2$.*

*Proof.* Let $u^2 - nv^2 = \pm \sigma^2 t$ where $(u, v)$ is a nontrivial solution with $u \geq 0$ and the smallest possible $v > 0$. Thus:

$$\pm \sigma^2 t = (u^2 - nv^2)(A^2 - nB^2)/\sigma^2 = [(uA - nvB)/\sigma]^2 - n[(uB - vA)/\sigma]^2 .$$

It is easily seen that $(uA - nvB)/\sigma$ and $(uB - vA)/\sigma$ are rational integers. We claim that they provide a nontrivial solution. If not then $t = m^2$ and

(i) $(uA - nvB)/\sigma \equiv 0 \pmod m$

(ii) $(uB - vA)/\sigma \equiv 0 \pmod m$.

Multiplying (i) by $B/\sigma$, (ii) by $A/\sigma$, and subtracting we get:

$$0 \equiv v(A^2 - nB^2)/\sigma^2 \equiv v\delta \pmod m, \text{ implying } v \equiv 0 \pmod m .$$

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Similarly $u \equiv 0 \pmod m$ contradicting that $(u, v)$ is a nontrivial solution and securing the claim. Hence, by the minimality of $v$, $|(uB - vA)/\sigma| \geq v$. Therefore either:

(iii) $u \geq v(A + \sigma)/B$ or,

(iv) $u \leq v(A - \sigma)/B$.

If (iii) then:

$$\begin{aligned}\pm \sigma^2 t = u^2 - nv^2 &\geq (v^2(A + \sigma)^2/B^2) - nv^2 = v^2(A^2 + 2\sigma A + \sigma^2 - nB^2)/B^2 \\ &= v^2(\sigma^2\delta + 2\sigma A + \sigma^2)/B^2 \geq \sigma^2(\delta + 2A/\sigma + 1)/B^2 .\end{aligned}$$

Therefore $t \geq ((2A)/\sigma + \delta + 1)/B^2 \geq ((2A)/\sigma - \delta - 1)/B^2$. If (iv) then:

$$\begin{aligned}\pm \sigma^2 t = u^2 - nv^2 &\leq (v^2(A - \sigma)^2/B^2) - nv^2 = v^2(A^2 - 2\sigma A + \sigma^2 - nB^2)/B^2 \\ &\leq \sigma^2(1 + \delta - (2A)/\sigma)/B^2 .\end{aligned}$$

Therefore $t \geq ((2A/\sigma) - \delta - 1)/B^2$. Q.E.D.

The first theorem of this section encompasses and generalizes several results in the literature. Among these are: Ankeny, Chowla, Hasse [1, Lemma, p. 218], S.D. Lang [5, Lemma p. 70] and R. Mollin [7, Lemma 1.1, p. 8].

THEOREM 1.1. *Let $n = \ell^2 + r$ be a positive square-free integer with $r$ dividing $4\ell$ and $-\ell < r \leq \ell$; and let $t$ be any positive integer. If $x^2 - ny^2 = \pm \sigma^2 t$ has a nontrivial solution where $\sigma = 2$ if $n \equiv 1 \pmod 4$ and $\sigma = 1$ otherwise, then*:

( i ) *If $r = 1$ and $n \neq 5$ then $\sigma^2 t \geq 2\ell$.*

(ii) *If $r = -1$ then $t \geq 2(\ell - 1)$.*

(iii) *If $r = 4$ then $t \geq \ell$.*

(iv) *If $r = -4$ then $t \geq \ell - 2$.*

( v ) *If $|r| \neq 1$ or $4$ then $\sigma^2 t \geq |r|$.*

*Proof.* If $|r| = 1$ or 4 then the fundamental unit of $\boldsymbol{Q}(\sqrt{n})$ is $(l + \sqrt{n})/\beta$ where $\beta = 1$ if $|r| = 1$ and $n \neq 5$, whereas $\beta = 2$ if $|r| = 4$, (see [3] and [8]). If $r = 1$ and $n \equiv 1 \pmod 4$, $(n \neq 5)$, choose $A = 2l$ and $B = 2$, and otherwise choose $A = l$, $B = 1$. Then (i)–(iv) follow immediately from Lemma 1.1.

If $|r| \neq 1$ or 4 then the fundamental unit of $\boldsymbol{Q}(\sqrt{n})$ is $((2l^2 + r) + 2l\sqrt{n})/|r|$ by [3] and [8]. Now, choose $A = \sigma(2l^2 + r)/|r|$ and $B = 2l\sigma/|r|$. By Lemma 1.1: $t \geq ((2(2l^2 + r)/|r|) - 2)/(2l\sigma/|r|)^2$; i.e., $t \geq (2l^2|r| + r|r| - r^2)/2l^2\sigma^2$. If $r > 0$ then $\sigma^2 t \geq r$. If $r < 0$ then $\sigma^2 t \geq -(l^2 r + r^2)/l^2$. If $\sigma^2 t < -r - 1$ then $-r - 1 > -(l^2 r + r^2)/l^2$ which implies $l^2 < r^2$, a contradiction. Hence $\sigma^2 t \geq -r$. Q.E.D.

We are now in a position to prove the main result of this section.

THEOREM 1.2. *Let* $n = \ell^2 + r > 5$ *be a square-free integer such that the following conditions are satisfied*:

(1) $\ell = st$ *where* $s > 0$ *and* $t > 1$ *are integers with* $\text{g.c.d.}(t, r) = 1$.

(2) $r$ *divides* $4s$ *with* $-\ell < r \leq \ell$.

(3) *Either* $n \not\equiv 1 \pmod 4$, *or* $|r| = 1$ *or* 4.

(4) *If* $|r| = 4$ *then* $s > 1$.

(5) *If* $r = 1$ *and* $\ell$ *is even then* $s > 2$.

*Let* $\sigma = 2$ *if* $n \equiv 1 \pmod 4$ *and* $\sigma = 1$ *otherwise. The diophantine equation* $x^2 - ny^2 = \pm\sigma^2 t$ *has a nontrivial solution if and only if* $n = 7$ *and* $t = 3$; i.e., $x^2 - 7y^2 = -3$.

*Proof.* If $|r| = 4$ then by hypothesis (4) the result follows from Theorem 1.1 (iii)–(iv). If $r = 1$ then by hypothesis (5) the result follows from Theorem 1.1 (i). If $r = -1$ then the result is immediate from Theorem 1.1 (ii).

If $|r| \neq 1$ or 4 then by hypothesis (3) $n \not\equiv 1 \pmod 4$, so $\sigma = 1$. Let $u^2 - nv^2 = \pm t$ be a nontrivial solution with $u \geq 0$ and smallest possible $v > 0$. For the sake of convenience we let $w = \pm t$.

Observe that $w + rv^2 = (u - \ell v)(u + \ell v)$ and let $a = |u - \ell v| > 0$ and $b = (u + \ell v) < 0$. Set $\alpha = 1$ if $w > -rv^2$ and $\alpha = -1$ otherwise. Thus: $(a-1)(b+\alpha) = ab + a\alpha - b - \alpha \geq 0$; i.e., $ab - \alpha \geq a\alpha$. Also $|w| = ab - \alpha rv^2$. Hence: $0 \leq |w|(s-1) = \ell - |w| = ((b - \alpha a)/2v) - ab + \alpha rv^2 = (b - \alpha a - 2vab + 2\alpha rv^3)/2v \leq (ab - \alpha - 2vab + 2\alpha rv^3)/2v = -(\alpha(1 - 2rv^3) + ab(2v-1))/2v$ which is less than zero if $r\alpha < 0$. Thus we assume henceforth that $r\alpha > 0$. Since we will need it later in the proof we label the following which follows from the above inequality:

$$(*) \qquad ab \leq -\alpha(1 - 2rv^3)/(2v - 1).$$

Now, let $A = (2\ell^2 + r)/|r|$ and $B = 2\ell/|r|$. As in the proof of Lemma 1.1 we obtain that $(uA - nvB,\ uB - vA)$ is a nontrivial solution. Hence by the minimality of $v$, $|uB - vA| \geq v$. It follows that either:

(a) $2\ell(u - \ell v) \geq v(r + |r|)$ or

(b) $2\ell(u - \ell v) \leq v(r - |r|)$.

*Case* I. $w < -rv^2$; (i.e., $\alpha = -1$). Hence $r < 0$, since $r\alpha > 0$.

(i) If (*a*) then $u \geq \ell v$. Thus:

$w = u^2 - (\ell^2 + r)v^2 \geq \ell^2 v^2 - (\ell^2 + r)v^2 = -rv^2$ a contradiction.

(ii) If (b) then $\ell u \leq (\ell^2 + r)v$. Thus:

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$\ell^2 w = \ell^2 u^2 - \ell^2(\ell^2 + r)v^2 \leq (\ell^2 + r)^2 v^2 - \ell^2(\ell^2 + r)v^2 = r(\ell^2 + r)v^2$, so:

$$(**) \qquad \ell^2 w \leq r(\ell^2 + r)v^2$$

If $w > 0$ then by (**) $0 < \ell^2 w \leq r(\ell^2 + r)v^2 < 0$, a contradiction. We assume for the remainder of case I that $w < 0$.

Suppose $v \geq -w$ then by (**): $\ell^2 v \geq -\ell^2 w \geq -r(\ell^2 + r)v^2$. Therefore: $r^2 v^2 \geq -\ell^2(rv + 1)v \geq -w^2(rv + 1)v$. Hence $w^2 \leq -r^2v^2/(rv + 1)v < 1 - r$, contradicting Theorem 1.1 (v). (Observe that $v > 1$ is forced by our supposition, and the fact that $r \neq -1$).

Now assume $v(-r - 1) \geq -w > v$ then by (**): $\ell^2 v(-r - 1) \geq -\ell^2 w \geq -r(\ell^2 + r)v^2$. Hence: $r^2v^2 \geq -r\ell^2 v^2 + \ell^2 v(r + 1) = \ell^2(v(r + 1) - rv^2) \geq w^2(v(r + 1) - v^2 r)$. Thus: $w^2 \leq r^2v^2/(v(r + 1) - v^2 r) = r^2 v/(r + 1 - vr)$. By Theorem 1.1 (v) the fact that $v(-r - 1) > -w$ forces $v > 1$. Hence: $w^2 \leq r^2 v/(r + 1 - vr) < r^2$. This, however, contradicts Theorem 1.1 (v).

Assume for the remainder of case I that $-w > v(-r - 1)$. Recall that from (*) we have: $ab \leq (1 - 2rv^3)/(2v - 1)$. Hence:

$$\begin{aligned} -w = ab + rv^2 &\leq ((1 - 2rv^3)/(2v - 1)) + rv^2 = (1 - rv^2)/(2v - 1) \\ &= v(-r - 1) + (v^2(r + 2) - v(r + 1) + 1)/(2v - 1). \end{aligned}$$

Let $c = v^2(r + 2) - v(r + 1) + 1$. If $r < -4$ then $c < 1$ if $v > 1$. If $r = -3$ then $c = -v^2 + 2v + 1 = -(v - 1)^2 + 2 < 1$ if $v > 2$. If $r = -2$ then $c/(2v - 1) = (v + 1)/(2v - 1) < 1$ if $v > 2$. We have shown that if either $r < -4$ and $v > 1$, or $-2 \geq r \geq -3$ and $v > 2$ then $v(-r - 1) < -w < v(-r - 1) + 1$, a contradiction.

Now, if $r < -4$ and $v = 1$ then: $3 < v(-r - 1) < -w \leq (1 - rv^2)/(2v - 1) = 1 - r$. Therefore $-w = 1 - r$; i.e., $t = 1 - r$, whence $u^2 - \ell^2 = 2r - 1$. Recall that $a = \ell v - u$ and $b = \ell v + u$, whence $u = (b - a)/2$ and $\ell = (b + a)/2$. Hence $s = (b + a)/(ab + 1)$ and $r = (1 - ab)/2$, whence $4s/(-r) = 8(a + b)/((ab)^2 - 1)$ which must be an integer by hypothesis (2). In particular, $8(a + b) \geq (ab)^2 - 1$. Since $b > a$ then $16b > (ab)^2 - 1$, whence $1 > b(a^2 b - 16)$, i.e., $a^2 b \leq 16$. Since $a$ and $b$ are both odd then $a = 1 < b$ is forced and $b \in \{3, 5, 7, 9, 11, 13, 15\} = S$. Of the values in $S$ only $b = 3, 5$ or $9$ yields that $4s/(-r)$ is an integer. If $b = 9$ then $r = -4$; if $b = 5$ then $r = -2$, and if $b = 3$ then $r = -1$; all of which are contradictions.

If $r = -3$ and $v = 2$ then $v(-r - 1) = 4 < -w \leq (1 - rv^2)/(2v - 1) = 13/3 < 5$, a contradiction. If $r = -3$ and $v = 1$ then $v(-r - 1) = 2 <$

$-w \leq (1 - rv^2)/(2v - 1) = 4$. If $-w = 3$ then we have a contradiction to hypothesis (1). It $-w = 4$ then $-4 = u^2 - n = u^2 - \ell^2 + 3$. Thus $-7 = u^2 - \ell^2 = (u - \ell)(u + \ell)$ which forces $u = 3$ and $\ell = 4$. This contradicts hypothesis (2). If $r = -2$ and $v = 2$ then: $2 = v(-r - 1) = v < -w \leq (1 - rv^2)/(2v - 1) = 3$. Therefore $-w = 3$. By (**) we have: $3\ell^2 \geq 8(\ell^2 - 2)$ which implies $16 \geq 5\ell^2$ forcing $\ell = 1$, a contradiction. If $r = -2$ and $v = 1$, then $1 = v(-r - 1) < -w \leq (1 - rv^2)/(2v - 1) = 3$ whence: $-3 = u^2 - n = u^2 - \ell^2 + 2$, i.e., $u^2 - \ell^2 = -5$ which forces $\ell = 3$, $u = 2$, and $n = 7$. This completes case I.

*Case* II. $w > -rv^2$, i.e. $\alpha = 1$. Therefore $r > 0$. If (b) holds then $u \leq \ell v$. Thus: $w = u^2 - (\ell^2 + r)v^2 \leq \ell^2v^2 - \ell^2v^2 - rv^2 = -rv^2$ contradicting our assumption. We assume for the remainder of the proof that (a) holds: i.e., $\ell u \geq v(\ell^2 + r)$. If $w < 0$ then $0 > \ell^2 w = \ell^2 u^2 - \ell^2(\ell^2 + r)v^2 \geq (\ell^2 + r)^2 v^2 - \ell^2(\ell^2 + r)v^2 = r(\ell^2 + r)v^2 > 0$, a contradiction.

We assume for the remainder of the proof that $w > 0$. Now assume $w > (r - 1)v$. Recall that from (*): $ab \leq (2rv^3 - 1)/(2v - 1)$. Therefore: $w = ab - rv^2 \leq ((2rv^3 - 1)/(2v - 1)) - rv^2 = (rv^2 - 1)/(2v - 1) = (r - 1)v + (v(r - 1) - v^2(r - 2) - 1)/(2v - 1) < (r - 1)v + 1$, a contradiction to our assumption. Hence $0 < w \leq (r - 1)v$. Recall that from (a) we have: $\ell^2 w \geq r(\ell^2 + r)v^2$. Hence: $l^2 w \geq r(\ell^2 + r)w^2/(r - 1)^2 > \ell^2 w^2/r$, which implies $r > w$ contradicting Theorem 1.1(v). This completes case II, and secures the result. Q.E.D.

The following result is immediate from Theorem 2.1.

COROLLARY 1.1 (H. Yokoi [11, Theorem 2, p. 153]). *Let $p$ and $q$ be odd primes satisfying $p = ((2n + 1)q)^2 \pm 2$ with $n \geq 0$. Then the diophantine equation $x^2 - py^2 = \pm q$ has a solution $(x, y)$ in integers if and only if $p = 7$ and $q = 3$, $(n = 0)$.*

Yokoi's result was in turn a generalization of H. Takeuchi [9, Lemma, p. 55].

We note that Theorem 1.1 fails for $n \equiv 1 \pmod 4$ in general. For example it fails when $n \pm 4t$ are perfect squares. An instance of this is $n = 33$, with $\ell = 6$, $r = -3$ and $t = 2$ for which we have $u^2 - nv^2 = 5^2 - 33 = -8 = -\sigma^2 t$.

## § 2. Class numbers of real quadratic fields

We now employ the machinery established in Section 1 to study a certain class of real quadratic fields and determine that they have non-trivial class numbers.

THEOREM 2.1. *Let* $n = \ell^2 + r > 7$ *be a square-free integer with* $r|4\ell$, $-\ell < r \leq \ell$, *and either* $n \not\equiv 1 \pmod 4$ *or* $|r| = 1$ *or* 4. *Suppose there exists a prime* $q$ *dividing* $\ell$ *such that* $q < \ell$ *if* $|r| = 4$. *Then* $h(n) > 1$ *whenever any of the following conditions are satisfied*:

(i) g.c.d. $(q, r) = 1, q > 2$ *and* $(r/q) = 1$ *where* $(\ /\ )$ *denotes the Legendre symbol. Moreover if* $r = 1$ *and* $\ell$ *is even then* $\ell > 2q$.

(ii) $q = 2$ *and* $r \neq 1$ *is odd.*

(iii) $q = 2$, $r = 1$, $\ell \equiv 0 \pmod 4$, *and* $\ell > 4$.

(iv) $q|r$, $|r| > q$, *and* $|r| \neq 4$.

(v) $|r| = q > 2$.

*Proof.* Assume $h(n) = 1$. Therefore there exist integers $(x, y)$ such that:

(a) In case (i) $x^2 - ny^2 = \pm 4q$ if $n \equiv 1 \pmod 4$ and $x^2 - ny^2 = \pm q$ if $n \not\equiv 1 \pmod 4$, since $q$ splits in $\boldsymbol{Q}(\sqrt{n})$.

(b) In case (ii) $x^2 - ny^2 = \pm 2$, since 2 ramifies in $\boldsymbol{Q}(\sqrt{n})$.

(c) In case (iii), $x^2 - ny^2 = \pm 8$, since 2 splits in $\boldsymbol{Q}(\sqrt{n})$.

(d) In case (iv) $x^2 - ny^2 = \pm q$, since $q$ ramifies in $\boldsymbol{Q}(\sqrt{n})$.

(e) In case (v) $x^2 - ny^2 = \pm 2$, since 2 ramifies in $\boldsymbol{Q}(\sqrt{n})$.

Now, (a)–(c) contradict Theorem 1.2, and (d)–(e) contradict Theorem 1.1 (v). Q.E.D.

*Remark.* Known results contained in Theorem 2.1 are [7, Theorem 1.1, p. 8] as well as the following two results.

COROLLARY 2.1 (H. Yokoi [11, Theorem 3, p. 157]).

(a) *If* $n = (st)^2 - 2$ *is an odd prime where* $t$ *is an odd prime*, $t \equiv 1$ *or* 3 (mod 8) *and* $s \geq 1$ *is an odd integer then* $h(n) > 1$ *except when* $n = 7$.

(b) *If* $n = (st)^2 + 2$ *is an odd prime where* $t$ *is an odd prime satisfying* $t \equiv 1$ *or* 7 (mod 8) *and* $s \geq 1$ *is an odd integer then* $h(n) > 1$.

COROLLARY 2.2 (Hasse [4]; c.r. Ankeny-Chowla-Hasse [4], S.D. Lang [5] and Mollin [6]–[7]).

*If* $n$ *satisfies the hypotheses of Theorem* 2.1 *and is either a prime congruent to* 1 (mod 4) *or of the form* $qq'$ *for primes* $q \equiv q' \equiv 3 \pmod 4$ *then* $h(n) > 1$.

We now illustrate the above in the form of a table. The class numbers

Table 2.1

| $l$ | $r$ | $n$ | $h(n)$ |
|---|---|---|---|
| 3 | 1 | 10 | 2 |
| 4 | −1 | 15 | 2 |
| 5 | 1 | 26 | 2 |
| 5 | 5 | 30 | 2 |
| 6 | −2 | 34 | 2 |
| 6 | −1 | 35 | 2 |
| 6 | 3 | 39 | 2 |
| 6 | 6 | 42 | 2 |
| 7 | 2 | 51 | 2 |
| 8 | 1 | 65 | 2 |
| 9 | −3 | 78 | 2 |
| 9 | −2 | 79 | 3 |
| 9 | 1 | 82 | 4 |
| 9 | 4 | 85 | 2 |
| 9 | 6 | 87 | 2 |
| 10 | −5 | 95 | 2 |
| 15 | −2 | 223 | 3 |
| 15 | 4 | 229 | 3 |
| 16 | 1 | 257 | 3 |
| 19 | −2 | 359 | 3 |
| 20 | 1 | 401 | 5 |
| 21 | −2 | 439 | 5 |
| 21 | 2 | 443 | 3 |
| 268 | −2 | 71822 | 19 |
| 270 | 2 | 72902 | 15 |
| 271 | 1 | 73442 | 24 |
| 274 | −2 | 75074 | 20 |
| 275 | 1 | 75626 | 34 |
| 279 | 1 | 77842 | 52 |
| 280 | −2 | 78398 | 13 |
| 282 | 2 | 79522 | 48 |
| 286 | −2 | 81794 | 18 |
| 289 | 1 | 83522 | 32 |
| 299 | 1 | 89402 | 38 |
| 306 | −2 | 93634 | 56 |
| 306 | 2 | 93638 | 15 |
| 308 | 2 | 94862 | 15 |
| 309 | 1 | 95482 | 70 |
| 310 | −2 | 96098 | 14 |
| 312 | 2 | 97346 | 28 |
| 314 | −2 | 98594 | 20 |
| 315 | 1 | 99226 | 94 |
| 316 | −2 | 99854 | 21 |

in the following are taken from [10]. The first 16 entries are all those integers less than 100 which are available by this method. The next 10 entries are the remaining *primes* less than 500 which are available. Note that we miss only one of them namely 499. The remaining entries are the available integers of the form $2q$ where $q$ is an odd prime and 70,000 $< 2q <$ 100,000.

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