THE REGULAR MAPS ON A SURFACE OF GENUS THREE

F. A. SHERK

Introduction. A considerable volume of research on the theory of regular maps is now in existence. Systematic enumerations of regular maps on the surfaces of genus 1 and 2 were begun by Brahana **(1**; **2)** and completed by Coxeter **(6**; **7**, p. 141**)**. In addition Coxeter enumerated the regular maps on the simplest non-orientable surfaces **(7**, pp. 116, 139**)**, and constructed tables of some interesting families of regular maps **(3**; **7**, p. 140**)**.

Most of the regular maps on a surface of genus 3 have appeared in these papers, but no systematic enumeration of them seems to have been attempted. The ultimate goal of this paper is a complete list of these regular maps. However, the families of maps $\{j\cdot p, q\}$ and $\{j\cdot p, j\cdot q\}$ which are defined in § 4 and listed in Tables I and II are of considerable interest in themselves. Also of some importance is the complete list of regular maps of type $\{p, 3\}$ with six or fewer faces (§ 5 and Table III).

A method of deriving regular maps by identification of faces in a regular tessellation is introduced in § 2 and used in §§ 5 and 7. Although cumbersome in some cases, it is the only reliable tool which has yet been developed for completing a list of regular maps of genus $p > 1$ (Brahana's method **(2**, pp. 281–4**)** is dependent upon the completeness and accuracy of permutation group tables).

1. Elementary concepts and results. A *map* is a partitioning of an unbounded surface into N_2 simply-connected, non-overlapping regions called *faces* by means of N_1 lines called *edges*. The N_0 intersections of the edges are called *vertices*.

The Euler-Poincaré characteristic

1.1 $$\chi = N_0 - N_1 + N_2$$

has the same value for every map drawn on this surface. If the surface is orientable, then

1.2 $$\chi = 2 - 2\text{p},$$

where p is the genus of the surface.

Received July 25, 1958. The bulk of these results form a part of the author's Ph.D. thesis, which was written under the supervision of Professor H. S. M. Coxeter. The author wishes to express his sincere thanks to Professor Coxeter for his kind and generous help.

The research was carried on while the author held a research studentship from the National Research Council of Canada.

To every map there corresponds a *dual* map having N_0 faces, one surrounding each vertex or the original map, N_1 edges, one crossing each edge of the original map, and N_2 vertices, one contained in the interior of each face of the original map (**5**, p. 6).

With any map there is associated a group of transformations which leave the map invariant and preserve incidences, that is, a group of *automorphisms* (**7**, p. 100). An automorphism is determined by its effect on any one face. Suppose that the group contains, in particular, two automorphisms R and S, the first of which cyclically permutes the edges bounding a face F, while the other cyclically permutes the edges which meet at a vertex V of F. A map containing these two automorphisms is said to be *regular*.

It is immediately evident that if the face F is p-sided, and if q edges meet at the vertex V, then every face of the regular map is p-sided and exactly q edges meet at every vertex. Thus the regular map is composed of p-gons, q meeting at each vertex. Such a map is said to be a "map of type $\{p, q\}$," in analogy with Schläfli's notation for a regular polyhedron (**5**, p. 14). The dual map is of type $\{q, p\}$ and is, of course, also regular. It also follows from the definition of a regular map that the group of the map is transitive on its vertices, edges, and faces.

Suppose that we divide the surface of the regular map of type $\{p, q\}$ into pN_2 triangles by adding to the map the lines which join the vertices of each face to the corresponding vertex of the dual map (cf. Figure 1 for the case of a map of type $\{6, 3\}$). Thus each face of the map is made up of p triangles,

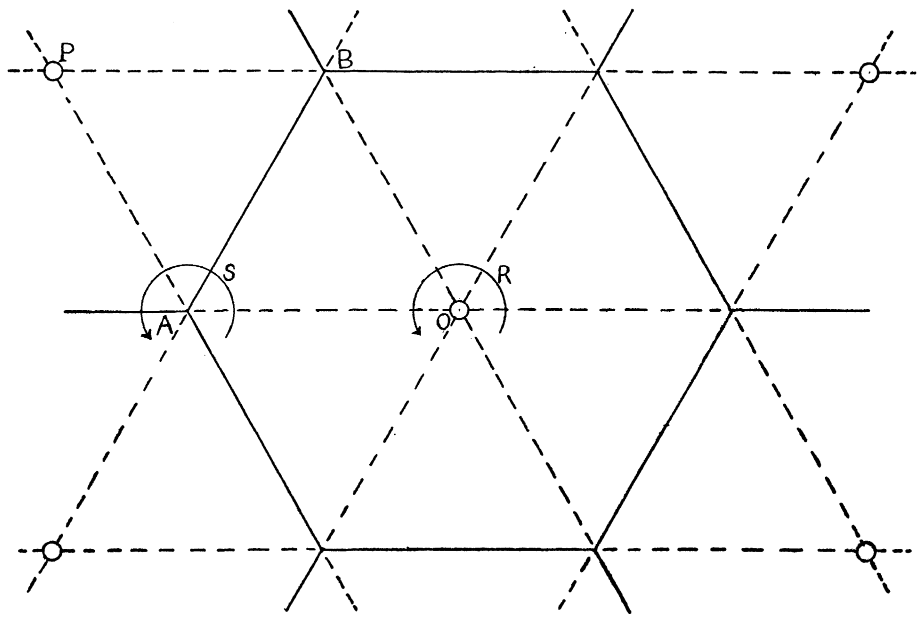

FIGURE 1

https://doi.org/10.4153/CJM-1959-046-9 Published online by Cambridge University Press

each edge borders on 2 triangles, and each vertex is surrounded by $2q$ triangles. It follows that

1.3 $$pN_2 = 2N_1 = qN_0.$$

Accordingly if the map has N_1 edges it has $2N_1/p$ faces and $2N_1/q$ vertices. Substituting in formula 1.1, we have for the surface of the regular map:

1.4 $$\chi = 2N_1\left(\frac{1}{p} + \frac{1}{q} - \frac{1}{2}\right).$$

We define the *group* of a regular map to be the group which is generated by the automorphisms R and S. Examining Figure 1, we note that the automorphism* RS interchanges the triangles OAB and PAB. Thus RS is of period 2. It is easy to see that this result is true for any regular map; the group of a regular map of type $\{p, q\}$ must satisfy the relations

1.5 $$R^p = S^q = (RS)^2 = E,$$

where E denotes the identity element. These relations are sufficient to define the group if the surface is simply-connected, but in any other case at least one extra relation is needed.

Looking again at Figure 1, we note that the edge AB is carried into itself by two automorphisms in the group, namely E and RS. When the surface on which the map lies is non-orientable, the group contains two other automorphisms which carry AB into itself. One of these will leave A and B invariant, interchanging O and P, while the other leaves O and P invariant and interchanges A and B. These automorphisms are called *reflections* since they operate in a manner analogous to the reflections of the Euclidean plane (**5**, p. 75). Since the group is transitive on the edges of the map it must be of order $4N_1$.

Any regular map whose automorphisms include reflections is said to be *reflexible* (**7**, p. 101). Certain non-reflexible regular maps do exist. Coxeter (**6**, p. 26; **7**, pp. 103, 107) exhibited the non-reflexible regular maps on a surface of genus 1 and stated that no others were known (**7**, p. 102). However, Frucht (**9**) discovered a non-reflexible regular map on a surface of genus 55 which is the embedding in that surface of a one-regular graph of degree three. Any non-reflexible regular map must lie on an orientable surface, since the group of a regular map on a non-orientable surface must contain reflections (**7**, p. 101).

If the map is on a non-orientable surface, or if it is non-reflexible, the group of the map is the complete group of automorphisms. Every map which is reflexible and lies on an orientable surface has a larger group of automorphisms which we shall call the *extended group* of the map (**4**, p. 125). The extended group includes reflections and is therefore of order $4N_1$. It contains "the group of the map" as a subgroup of index 2.

*By RS, the product of R and S, we mean the automorphism which is achieved by performing R first and then performing S.

The automorphisms that comprise the group of an orientable regular map are called *rotations*. By 1.3 the order of the group may be expressed in the forms pN_2 or qN_0 as well as in the form $2N_1$.

In virtue of relations 1.4 and 1.2, any regular map of type $\{p, q\}$ which has N_1 edges and is on an orientable surface is on a surface of genus

1.6 $$\mathrm{p} = 1 - N_1\left(\frac{1}{p} + \frac{1}{q} - \frac{1}{2}\right).$$

The expressions "regular map on a surface of genus p" will now be shortened to "regular map of genus p."

The regular maps of genus zero are simply the projections on concentric spheres of the 5 convex regular polyhedra, $\{3, 3\}$, $\{4, 3\}$, $\{3, 4\}$, $\{5, 3\}$, and $\{3, 5\}$. together with the "dihedral" maps $\{p, 2\}$ ($p \geqslant 1$) and their duals $\{2, p\}$. The groups of the regular maps of genus zero are the well-known polyhedral rotation groups (**11**, pp. 10–20; **5**, pp. 45–7), denoted by the symbols $[p, q]^+$ (**7**, p. 38), whose abstract definitions are given by 1.5 with appropriate values for p and q. Thus in the case of the regular maps of genus zero, the relations 1.5 are sufficient, as well as necessary, to define the group.

2. The regular tessellations. The above description of a regular map can be extended to include regular maps on an infinite surface. Thus, for example, we have in the Euclidean plane the regular maps $\{4, 4\}$, $\{6, 3\}$, and $\{3, 6\}$, more commonly called *regular tessellations* (**5**, pp. 58, 59). There are also regular tessellations in the hyperbolic plane (**7**, p. 53); they are of type $\{p, q\}$ for all p and q such that $(\mathrm{p} - 2)(q - 2) > 4$. The regular tessellations on the sphere are just the regular maps of genus zero. All regular tessellations are simply-connected maps.

As in the case of the regular maps on a sphere, the relations 1.5 are sufficient to define the group of a regular tessellation. It follows that the group of a regular map of type $\{p, q\}$ on an orientable surface is a factor group of the group of the regular tessellation $\{p, q\}$. It is also true that the plane of the tessellation is a universal covering surface for the surface in question (**7**, pp. 25, 26). These facts suggest a method of discovering regular maps. Beginning with a regular tessellation $\{p, q\}$ we add further relations to those of 1.5 by abstractly identifying certain faces of the tessellation (the exact procedure in this step will be outlined in the proof of Theorem 3). If the added relations do not effect the periods of R, S, and RS, and if they are sufficient to make the resulting group finite, let us say of order g, a regular map of type $\{p, q\}$ has been discovered. It has g/q vertices, $g/2$ edges, and g/p faces. It lies on a surface of genus

$$1 - g/2\left(\frac{1}{p} + \frac{1}{q} - \frac{1}{2}\right)$$

(cf. 1.6).

https://doi.org/10.4153/CJM-1959-046-9 Published online by Cambridge University Press

Furthermore, the above method will establish the existence or non-existence of all regular maps of type $\{p, q\}$ with given group order.

3. Some general lemmas. The following lemmas form the essential groundwork for all our results.

LEMMA 1. *For any map of type $\{p, q\}$ on a surface of Euler-Poincaré characteristic $\chi < 1$, $\min (p, q) \geqslant 3$.*

Proof. Consider a map of type $\{p, q\}$ which has k faces. It follows from 1.3 and 1.1 that

$$\chi = \frac{pk}{q} - \frac{pk}{2} + k.$$

Rearranging this equation, we obtain

$$k - \chi = pk\,(q - 2)/2q.$$

If $q \leqslant 2$, then $k - \chi \leqslant 0$ and $\chi \geqslant k \geqslant 1$. Thus if $\chi < 1$, $q \geqslant 3$. A similar argument holds for p when one considers the dual map, of type $\{q, p\}$.

LEMMA 2. *If two edges belonging to the same face of a regular map are identified, the map has only one face.*

Proof. We noted earlier that the group of a regular map is transitive on the edges of that map. Thus if two edges of a face are identified, then all the other edges of that face are also identified in pairs; the result is a one-faced map.

LEMMA 3. *If exactly two distinct faces come together at a vertex of a regular map of type $\{p, q\}$, the map is 2-faced, q is even, and the faces alternate around the vertex.*

Proof. If a face is contiguous to itself around a vertex, then by Lemma 2 the map is one-faced, contrary to our hypothesis. Thus q is even and the faces, α and β say, which surround a vertex alternate around that vertex (cf. Figure 2, where α and β surround the vertex V). Now consider any edge VV' (Figure 2). This edge borders on α and β, and hence α and β alternate around V' as well as around V. This happens at every vertex since the group of the map is transitive on its edges. Therefore α and β are the only faces.

LEMMA 4. *A one-faced map of type $\{p, q\}$ is regular if, and only if, one of the following two conditions is satisfied:*

(i) *$\frac{1}{2}p$ is an even integer and $q = p$;*
(ii) *$\frac{1}{2}p$ is an odd integer and $q = \frac{1}{2}p$.*

Proof. The single face of a one-faced map must have an even number of edges since these edges are identified in pairs to form the edges of the map. Thus $p = 2n$, where n is some integer, and the group of the one-faced regular map $\{2n, q\}$ is the cyclic group of order $2n$ generated by the rotation R of

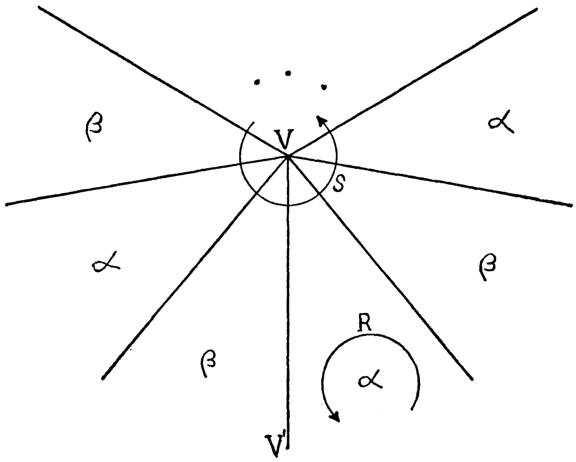

FIGURE 2

§ 1. Now the group of any regular map may be expressed in terms of the generators R and $T = RS$ instead of the R and S used earlier (**2**, p.269). The three relations of 1.5 are then equivalent to

3.1 $$R^p = T^2 = (RT)^q = E.$$

In the present case, T must be expressible in terms of R, and since $T^2 = E$, $T = R^n$. The existence of a regular map of type $\{2n, q\}$ depends upon the period of RT, which must be q. But $RT = R^{n+1}$. Hence if the map is regular

$$(RT)^q = R^{q(n+1)} = E = R^{2n},$$

and therefore $2n|q(n + 1)$. Now $(n, n + 1) = 1$, so that if n is even, $2n \mid q$, while if n is odd, $n \mid q$. Since the map has only n edges, $q \leqslant 2n$. Thus if n is even, $q = 2n$, while if n is odd,

$$(RT)^n = R^{(n+1)n} = (R^{2n})^{\frac{1}{2}(n+1)} = E,$$

and thus $q \mid n$. But $n \mid q$, therefore $q = n$. Conversely, any one-faced map of type $\{4\text{p}, 4\text{p}\}$ or $\{4\text{p} + 2, 2\text{p} + 1\}$ (p $= 0, 1, 2, \ldots$) is regular.

It is easily seen from 1.6 that the above two one-faced regular maps lie on a surface of genus p.

LEMMA 5. *If the rotation $R^h (1 < h < p$, where p is the period of $R)$ carries a vertex, edge, or face of a regular map into itself, while any rotation $R^i (0 < i < h)$ does not do so, then $p \equiv 0 \pmod h$.*

Proof. The integer p may be put into the form

$$p = mh + n$$

where m and n are integers and $0 \leqslant n < h$. Since both R^h and R^p $(= E)$ carry the vertex, edge, or face into itself, so also must R^n. Therefore $n = 0$.

https://doi.org/10.4153/CJM-1959-046-9 Published online by Cambridge University Press

LEMMA 6. *The abstract definition**

3.2 $R^{jp} = S^q = (RS)^2 = E, \quad R^p \rightleftarrows S \quad (0 \leqslant (p-2)(q-2) < 4)$

is significant only if $j \mid Q$, *where* $Q = 4q/[4 - (p-2)(q-2)]$, *and then defines a group of order* jpQ.

Proof. If we exclude the first relation in 3.2 we have

3.3 $$S^q = (RS)^2 = E, \quad R^p \rightleftarrows S.$$

If $SR = T$, this becomes

$$T^2 = S^q = E, \qquad (TS)^p = (ST)^p.$$

These relations define the group $\langle\langle 2, q \mid p\rangle\rangle$ of order PQ^2, which was introduced by Coxeter and Moser (**7**, p. 79). The period of R $(= TS^{-1})$ is pQ (**7**, p. 71) and therefore the abstract definition 3.2 is significant only if this period is a multiple of jp. If we add to 3.3 the relation $R^{jp} = E$, where $j \mid Q$, the only effect is to change the period of R to jp; the periods of S and RS will remain unchanged. Now it is easily shown that the number of cosets of $\{R\}$ in 3.2 remains the same, no matter what the particular choice of j is. When $j = 1$, the group is $[p, q]^+$, the group of the regular map $\{p, q\}$ and $\{R\}$ has Q cosets (**7**, p. 38). Thus the group defined by 3.2 has order jpQ.

To the group defined by 3.2 or

$$(TS)^p = (ST)^p = Z, \qquad T^2 = S^q = Z^j = E,$$

we assign the symbol $\langle\langle 2, q \mid p; j\rangle\rangle$. In particular, $\langle\langle 2, q \mid p; 1\rangle\rangle = [p, q]^+$.

4. Two new families of regular maps. Coxeter and Moser (**7**, § 8.8) introduced the regular map $\{p + p, q\}$ $(0 \leqslant (p-2)(q-2) < 4)$ and its dual $\{q, p + p\}$, whose group has the abstract definition.

$$R^{2p} = T^2 = (RT)^q = (R^pT)^2 = E,$$

or, in terms of R and $S = RT$,

$$R^{2p} = S^q = (RS)^2 = E, \qquad R^p \rightleftarrows S.$$

We generalize this notion by considering the regular map of type $\{jp, q\}$ $(0 \leqslant (p-2)(q-2) < 4)$ and its dual, of type $\{q, jp\}$, whose group G has the following property: the centre of G is a cyclic group, generated by R^p, where R has its usual meaning as a generator of the group. Such a group G will satisfy the following four relations:

4.1 $$R^{jp} = S^q = (RS)^2 = E, \qquad R^p \rightleftarrows S.$$

By Lemma 6, these relations are significant only if $j \mid Q$, where $Q = 4q/[4 - (p-2)(q-2)$, and then they define the group $\langle\langle 2, q \mid p; j\rangle\rangle$ of order jpQ.

*The notation $A \rightleftarrows B$ means that A and B commute.

https://doi.org/10.4153/CJM-1959-046-9 Published online by Cambridge University Press

Now the central quotient group of G is the group $[p, q]^+$ of order pQ. Thus G is of order jpQ, which is precisely the order of the group $\langle\langle 2, q \mid p; j\rangle\rangle$. Therefore the relations 4.1 define G.

To the above map of type $\{jp, q\}$ we assign the symbol $\{j\cdot p, q\}$, and denote its dual by $\{q, j\cdot p\}$. The map occurs for all integer values $p \geqslant 2$, $q \geqslant 2$ and $j > 0$ satisfying the two conditions $0 \leqslant (p - 2)(q - 2) < 4$ and $j \mid Q$. Its group is $\langle\langle 2, q \mid p; j\rangle\rangle$.

In particular, the maps $\{2\cdot p, q\}$ and $\{q, 2\cdot p\}$ are the $\{p + p, q\}$ and $\{q, p + p\}$ respectively of Coxeter and Moser **(7**, § 8.8**)** who pointed out that these maps may be drawn on a two-sheeted Riemann surface of the proper genus in a remarkably symmetrical manner. This construction is capable of generalization to the case of the regular maps $\{j.p, q\}$ and $\{q, j.p\}$ ($j \neq 2$).

In the proof of Lemma 6 it was shown that when $j = Q$, the groups 3.2 are the groups $\langle\langle 2, q \mid p\rangle\rangle$ of order pQ^2. They were shown by Coxeter and Moser **(7**, pp. 79–80**)** to be the groups of the regular complex polygons $2\{2p\}q$, discovered by Shephard **(12**, p. 92**)**. When these complex polygons are compared with the corresponding regular maps $\{pQ\cdot q, p\}$, it can be shown by the proper interpretation of the group generators in each case that the vertices and edges of the polygon form the same graph as the vertices and edges of the map. Thus the map may be regarded as a real representation of the complex polygon.

Generalizing in another direction, we consider the regular map of type $\{jp, jq\}$ $(0 \leqslant (p - 2)(q - 2) < 4)$ and its dual, of type $\{jq, jp\}$, whose group G' has the following properties: the centre of G' is a cyclic group generated by R^p; and $R^p = S^q$, where R and S have their usual meanings as generators of G'. Such a group will have among its defining relations the following:

4.2 $$R^p = S^q = Z, \qquad (RS)^2 = Z^j = E.$$

These relations are significant only if

$$j \,\Big|\, \frac{p + q}{q} Q,$$

where $Q = 4q/[4 - (p - 2)(q - 2)]$, and then they define the group $\langle p, q \mid 2; j\rangle$ of order jpQ **(7**, pp. 71–3**)**. This is a factor group of Miller's group $\langle p, q \mid 2\rangle$ which is defined by the relations

$$R^p = S^q, \qquad (RS)^2 = E.$$

Now the centre $\{R^p\}$ of G' is of order j, and the central quotient group of G' is $[p, q]^+$, of order pQ. Thus G' is of order jpQ, which is precisely the order of the group $\langle p, q \mid 2; j\rangle$. Therefore G' is defined by 4.2.

To the above type of map whose group is $\langle p, q \mid 2; j\rangle$ we assign the symbol $\{j\cdot p, j\cdot q\}$, and denote its dual by $\{j\cdot q, j\cdot p\}$. The map occurs for all integer values $p \leqslant 2$, $q \geqslant 2$ and $j > 0$ satisfying the two conditions $0 \leqslant (p-2)(q-2) < 4$ and

$$j \,\Big|\, \frac{p+q}{q} Q.$$

The members $\{(r+1)\cdot(r-1),\ (r+1)\cdot 2\}$ of this family were noted by Coxeter and Moser (**7**, p. 114).

The regular maps $\{p, q\}$ of genus zero are members of the family $\{j\cdot p, q\}$ as well as of the family $\{j\cdot p, j\cdot q\}$. With these exceptions, all the regular maps $\{j\cdot p, q\}$ and $\{j\cdot p, j\cdot q\}$ $(p \geqslant q)$ are listed in Tables I and II respectively. The sixth column in both tables exhibits some interesting isomorphisms between the group of the map and certain well-known groups. The information for the sixth column of Table I was kindly supplied by W. O. J. Moser; in the case of Table II the source is 7, § 6.6.

TABLE I

THE REGULAR MAPS $\{j\cdot p, q\}$ $(j \geqslant 2)$

Map	N_0	N_1	N_2	Genus	Group	Order
$\{j\cdot 2, q\}$ $(j\vert q)$	$2j$	jq	q	$\frac{1}{2}(j-1)(q-2)$	$\langle\langle 2, q \mid 2; j\rangle\rangle$	$2jq$
$\{q\cdot 2, q\}$	$2q$	q^2	q	$\frac{1}{2}(q-1)(q-2)$	$\langle\langle 2, q \mid 2\rangle\rangle$	$2q^2$
$\{2\cdot p, 2\}$	$2p$	$2p$	2	0	$\langle\langle 2, 2 \mid p\rangle\rangle \cong \mathfrak{D}_{2p}$	$4p$
$\{2\cdot 3, 3\}$	8	12	4	1	$\langle\langle 2, 3 \mid 3; 2\rangle\rangle \cong \mathfrak{A}_4\times\mathfrak{C}_2$	24
$\{4\cdot 3, 3\}$	16	24	4	3	$\langle\langle 2, 3 \mid 3\rangle\rangle$	48
$\{2\cdot 4, 3\}$	16	24	6	2	$\langle\langle 2, 3 \mid 4; 2\rangle\rangle$	48
$\{3\cdot 4, 3\}$	24	36	6	4	$\langle\langle 2, 3 \mid 4; 3\rangle\rangle$	72
$\{6\cdot 4, 3\}$	48	72	6	10	$\langle\langle 2, 3 \mid 4\rangle\rangle$	144
$\{2\cdot 5, 3\}$	40	60	12	5	$\langle\langle 2, 3 \mid 5; 2\rangle\rangle \cong \mathfrak{A}_5\times\mathfrak{C}_2$	120
$\{3\cdot 5, 3\}$	60	90	12	10	$\langle\langle 2, 3 \mid 5; 3\rangle\rangle \cong \mathfrak{A}_5\times\mathfrak{C}_3$	180
$\{4\cdot 5, 3\}$	80	120	12	15	$\langle\langle 2, 3 \mid 5; 4\rangle\rangle$	240
$\{6\cdot 5, 3\}$	120	180	12	25	$\langle\langle 2, 3 \mid 5; 6\rangle\rangle \cong \mathfrak{A}_5\times\mathfrak{C}_6$	360
$\{12\cdot 5, 3\}$	240	360	12	55	$\langle\langle 2, 3 \mid 5\rangle\rangle$	720
$\{2\cdot 3, 4\}$	12	24	8	3	$\langle\langle 2, 4 \mid 3; 2\rangle\rangle \cong \mathfrak{S}_4\times\mathfrak{C}_2$	48
$\{4\cdot 3, 4\}$	24	48	8	9	$\langle\langle 2, 4 \mid 3; 4\rangle\rangle \cong \mathfrak{S}_4\times\mathfrak{C}_4$	96
$\{8\cdot 3, 4\}$	48	96	8	21	$\langle\langle 2, 4 \mid 3\rangle\rangle$	192
$\{2\cdot 3, 5\}$	24	60	20	9	$\langle\langle 2, 5 \mid 3; 2\rangle\rangle \cong \mathfrak{A}_5\times\mathfrak{C}_2$	120
$\{4\cdot 3, 5\}$	48	120	20	27	$\langle\langle 2, 5 \mid 3; 4\rangle\rangle$	240
$\{5\cdot 3, 5\}$	60	150	20	36	$\langle\langle 2, 5 \mid 3; 5\rangle\rangle \cong \mathfrak{A}_5\times\mathfrak{C}_5$	300
$\{10\cdot 3, 5\}$	120	300	20	81	$\langle\langle 2, 5 \mid 3; 10\rangle\rangle \cong \mathfrak{A}_5\times\mathfrak{C}_{10}$	600
$\{20\cdot 3, 5\}$	240	600	20	171	$\langle\langle 2, 5 \mid 3\rangle\rangle$	1200

5. Regular maps of type $\{p, 3\}$. In some respects the most interesting regular maps are those which have 3 faces at a vertex. We shall now proceed to enumerate these maps when the number of faces is small.

To facilitate reference to it, a map of type $\{p, q\}$ having k faces will be denoted by the symbol ${}^k\{p, q\}$. In particular we shall now study the regular maps ${}^k\{p, 3\}$ for small values of k.

TABLE II

THE REGULAR MAPS $\{j\cdot p, j\cdot q\}$ $(p \geqslant q; j \geqslant 2)$

Map	N_0	N_1	N_2	Genus	Group	Order
$\{j\cdot p, j\cdot 2\}$ $(j \mid p+2)$	p	jp	2	$\frac{1}{2}(j-1)p$	$\langle p, 2 \mid 2; j\rangle$	$2jp$
$\{(p+2)\cdot p, (p+2)\cdot 2\}$	p	$p(p+2)$	2	$\frac{1}{2}p(p+1)$	$\langle p, 2 \mid 2\rangle$	$2p(p+2)$
$\{2\cdot 3, 2\cdot 3\}$	4	12	4	3	$\langle 3, 3 \mid 2; 2\rangle \cong \mathfrak{A}_4\times\mathfrak{C}_2$	24
$\{4\cdot 3, 4\cdot 3\}$	4	24	4	9	$\langle 3, 3 \mid 2; 4\rangle \cong \mathfrak{A}_4\times\mathfrak{C}_4$	48
$\{8\cdot 3, 8\cdot 3\}$	4	48	4	21	$\langle 3, 3 \mid 2\rangle$	96
$\{2\cdot 4, 2\cdot 3\}$	8	24	6	6	$\langle 4, 3 \mid 2; 2\rangle$	48
$\{7\cdot 4, 7\cdot 3\}$	8	84	6	36	$\langle 4, 3 \mid 2; 7\rangle$	168
$\{14\cdot 4, 14\cdot 3\}$	8	168	6	78	$\langle 4, 3 \mid 2\rangle$	336
$\{2\cdot 5, 2\cdot 3\}$	20	60	12	15	$\langle 5, 3 \mid 2; 2\rangle \cong \mathfrak{A}_5\times\mathfrak{C}_2$	120
$\{4\cdot 5, 4\cdot 3\}$	20	120	12	45	$\langle 5, 3 \mid 2; 4\rangle \cong \mathfrak{A}_5\times\mathfrak{C}_4$	240
$\{8\cdot 5, 8\cdot 3\}$	20	240	12	105	$\langle 5, 3 \mid 2; 8\rangle \cong \mathfrak{A}_5\times\mathfrak{C}_8$	480
$\{16\cdot 5, 16\cdot 3\}$	20	480	12	225	$\langle 5, 3 \mid 2; 16\rangle \cong \mathfrak{A}_5\times\mathfrak{C}_{16}$	960
$\{32\cdot 5, 32\cdot 3\}$	20	960	12	465	$\langle 5, 3 \mid 2\rangle$	1920

From Lemma 4 we deduce

THEOREM 1. *The only regular map* $^1\{p, 3\}$ *is the map* $\{6, 3\}_{1,0}$ *of genus* 1 (**6**, p. 25).

Lemmas 2 and 3 imply

THEOREM 2. *There is no regular map* $^2\{p, 3\}$.

Turning now to the case $k = 3$, we exhibit in Figure 3 a part of the regular tessellation $\{p, 3\}$. In virtue of Lemmas 2 and 3 the three faces of a regular

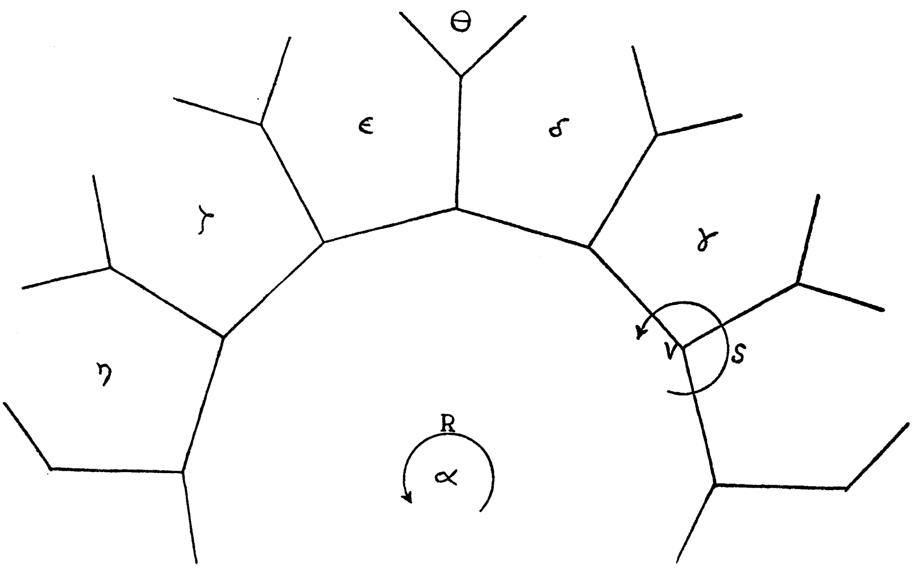

FIGURE 3

https://doi.org/10.4153/CJM-1959-046-9 Published online by Cambridge University Press

map ${}^{3}\{p, 3\}$ are situated in the manner of the faces α, β, and γ. Since the map is 3-faced, δ must be identified with β. Thus, representing faces by right cosets* of $\{R\}$, where the automorphisms R and S act in the indicated manner, we have

$$\{R\}SR^2 = \{R\}S.$$

In particular, there is an integer l such that

$$SR^2 = R^l S.$$

Thus the generators of the group of ${}^{3}\{p, 3\}$ must satisfy this relation as well as

$$R^p = S^3 = (RS)^2 = E.$$

It follows that

$$\begin{aligned} R^l &= SR^2S^{-1} = S^{-2}R^2S^2 = S^{-1}RSR^2S^{-1}R^{-1}S \\ &= S^{-1}RR^lR^{-1}S = S^{-1}R^lS = R^2, \end{aligned}$$

and the extra relation reduces to

$$R^2 \rightleftarrows S.$$

Moreover, Lemma 5 shows that p is even. Thus the abstract definition

5.1 $$R^p = S^3 = (RS)^2 = E, \qquad R^2 \rightleftarrows S$$

is a special case of 3.2, and Lemma 6 shows that $p = 2$ or 6. Thus we have

THEOREM 3. *There are exactly two regular maps* ${}^{3}\{p, 3\}$, *namely* $\{2, 3\}$ *of genus zero and* $\{3\cdot 2, 3\}$ *of genus* 1.

In the notation of Coxeter (**6**, p. 25), $\{3\cdot 2, 3\}$ is the map $\{6, 3\}_{1,1}$.

Lemma 6 also shows that the identification of faces carried out in the above case can yield only a 3-faced regular map (of type $\{p, 3\}$). Thus the four faces of any map ${}^{4}\{p, 3\}$ are situated in the manner of α, β, γ, and δ in Figure 3, and ϵ must be identical with β. By similar reasoning to that used in proving Theorem 3, we now have

THEOREM 4. *There are exactly three regular maps* ${}^{4}\{p, 3\}$, *namely* $\{3, 3\}$ *of genus zero,* $\{2\cdot 3, 3\}$ *of genus* 1, *and* $\{4\cdot 3, 3\}$ *of genus* 3.

In the notation of Coxeter (**6**, p. 25; cf. **7**, p. 116), $\{2\cdot 3, 3\}$ is the map $\{6, 3\}_{2,0}$.

Turning now to the case ${}^{5}\{p, 3\}$, we prove

THEOREM 5. *There is no regular map* ${}^{5}\{p, 3\}$.

*Since the rotation R carries α into itself, α may be represented in the group by the subgroup $\{R\}$ while the other faces are represented by right cosets of $\{R\}$ (**2**, p. 270). Thus there is a (1, 1) correspondence between the faces of a regular map and the right cosets of $\{R\}$ in its group.

The reader is requested to insert the letter β in the face to the right of α (Figure 3).

Proof. Suppose that a regular map ${}^5\{p, 3\}$ exists. Then in view of the results of the two previous theorems, its faces must be situated in the manner of α, β, γ, δ, and ϵ in Figure 3, and ζ must be identical with β. The group of the map must therefore satisfy the relations

5.2 $$R^p = S^3 = (RS)^2 = E, \qquad R^4 \rightleftarrows S,$$

where $p \equiv 0 \pmod 4$. But by Lemma 6, 5.2 defines a group of order $6p$, while a regular map ${}^5\{p, 3\}$ must have a group of order $5p$. The group defined by 5.2 cannot have a factor group of order $5p$; hence there is no regular map ${}^5\{p, 3\}$.

It was noted in the above proof that 5.2 defines a group of order $6p$. Hence the identification of ζ with β in Figure 3 yields regular maps ${}^6\{p, 3\}$. We ask if any other identification of faces in the tessellation $\{p, 3\}$ will yield 6-faced regular maps. The only other possible arrangement is to let α, β, γ, δ, ϵ, and ζ be the 6 faces and identify η with β. This gives rise to a group satisfying the relations

5.3 $$R^p = S^3 = (RS)^2 = E, \qquad R^5 \rightleftarrows S$$

where $p \equiv 0 \pmod 5$. By Lemma 6 this defines a group of order $12p$, while a regular map ${}^6\{p, 3\}$ must have a group of order $6p$. Thus relations 5.3 are insufficient to define the group which we seek, and we must add a further relation. Since the regular map we seek is 6-faced, the face θ of Figure 3 must be identified with α, β, γ, δ, ϵ, or ζ. Each face is surrounded by 5 different faces, and therefore θ can only be identified with β; in symbols

$$\{R\}SR^{-1}SR^2 = \{R\}S.$$

In particular, there is an integer l such that $SR^{-1}SR^2 = R^lS$. However, if we add this relation to those of 5.3 and enumerate right cosets of $\{R\}$ by the Todd-Coxeter method (**7**, p. 12), a collapse occurs in the tables which reduces the number of cosets of $\{R\}$ to one. Thus we eliminate the possibility of a group of order $6p$.

Since the Todd-Coxeter method will be employed many times in similar situations, it is perhaps advisable to exhibit the tables in this case. They are

$R\,R\,R\,R\,R \ldots R$	$S\,S\,S$	$R\,S\,R\,S$	$R^5\,S$	$S\,R^5$
1 1 1 1 1 1 1 1	1 2 3 1	1 1 2 3 1	1 1 2	1 2 2
2 3 4 5 6 2 6 2	4 6 5 4	3 4 6 2 3	2 2 3	2 3 3
		4 5 4 5 4	3 3 1	3 1 1
		5 6 5 6 5		

$S\,R^{-1}\,S\,R^2$	$R^l\,S$
1 2 6 5 2	1 1 2
2 3 2 3 5	2 6 5
3 1 1 2 4	3 5 4

The table for the fifth relation indicates that R^l carries coset 2 into coset 6

https://doi.org/10.4153/CJM-1959-046-9 Published online by Cambridge University Press

and at the same time carries coset 3 into coset 5. Transferring this information to the table for the first relation, we see that cosets 2, 3, 4, 5, and 6 are identical. But the table for the second relation then indicates that coset 2 = coset 1 and the collapse is complete. It is important to notice that the enumeration of cosets is carried out without knowing the specific values of p and l. This time-saving fact should be kept in mind and applied to any particular case when an enumeration of cosets is desired in the following pages.

Collecting the above results, and taking Lemma 6 into account, we have

THEOREM 6. *The only regular maps* ${}^{6}\{p, 3\}$ *are* $\{4, 3\}$ *of genus zero,* $\{2\cdot4, 3\}$ *of genus* 2, $\{3\cdot4, 3\}$ *of genus* 4, *and* $\{6\cdot4, 3\}$ *of genus* 10.

It is not difficult to classify completely the regular maps ${}^{k}\{p, 3\}$ for other small values of k by using the above methods. For example, it can be shown quite easily that there is only one regular map ${}^{7}\{p, 3\}$, namely (**6**, p. 25) the map $\{6, 3\}_{2,1}$ of genus 1. For the present, however, we shall confine ourselves to the following result, obtained by an examination of the proofs of Theorems 3–6.

THEOREM 7. *The faces of any regular map,* ${}^{k}\{p, 3\}$ $(k > 6)$ *are surrounded by at least five other distinct faces.*

The regular maps ${}^{k}\{p, 3\}$ $(k \leqslant 6)$ are listed in Table III.

TABLE III

THE REGULAR MAPS OF TYPE $\{p, 3\}$ WITH SIX OR FEWER FACES

Symbol	N_0	N_1	N_2	Genus	Group
$\{6, 3\}_{1,0}$	2	3	1	1	$\mathfrak{C}_6$
$\{2, 3\}$	2	3	3	0	$[2, 3]^+ \cong \mathfrak{D}_3$
$\{3\cdot2, 3\}$	6	9	3	1	$\langle\langle 2, 3 \mid 2\rangle\rangle$
$\{3, 3\}$	4	6	4	0	$[3, 3]^+ \cong \mathfrak{A}_4$
$\{2\cdot3, 3\}$	8	12	4	1	$\langle\langle 2, 3 \mid 3; 2\rangle\rangle \cong \mathfrak{A}_4 \times \mathfrak{C}_2$
$\{4\cdot3, 3\}$	16	24	4	3	$\langle\langle 2, 3 \mid 3\rangle\rangle$
$\{4, 3\}$	8	12	6	0	$[4, 3]^+ \cong \mathfrak{S}_4$
$\{2\cdot4, 3\}$	16	24	6	2	$\langle\langle 2, 3 \mid 4; 2\rangle\rangle$
$\{3\cdot4, 3\}$	24	36	6	4	$\langle\langle 2, 3 \mid 4; 3\rangle\rangle$
$\{6\cdot4, 3\}$	48	72	6	10	$\langle\langle 2, 3 \mid 4\rangle\rangle$

6. The arithmetically possible maps of genus 3. The first step in determining the regular maps of genus 3 is to list all the maps of type $\{p, q\}$ whose vertices, edges, and faces satisfy 1.1 with $\chi = -4$. We call them the *arithmetically possible* maps of genus 3.

To facilitate the enumeration of these maps we prove the following theorem, due to Coxeter:

THEOREM 8. *For any map of type* $\{p, q\}$ *on a surface of characteristic* $\chi \leqslant 0$, *if* $p \geqslant q$, *then* $q \leqslant 2(2-\chi)$.

Proof. In terms of p, q, and k (the number of faces of $\{p, q\}$), formula 1.1 is

$$\frac{kp}{q} - \frac{kp}{2} + k = \chi,$$

that is,

6.1 $$2kp - kpq + 2kq - 2\chi q = 0.$$

Now $p \geqslant q$, $k \geqslant 1$, and $\chi \leqslant 0$; hence

$$\begin{aligned}(1-\chi)p - q &\geqslant \chi q,\\ k[(1-\chi)p - q] &\geqslant -\chi q,\\ (1-\chi)kp &\geqslant kq - \chi q,\\ 2(1-\chi)kp &\geqslant 2kq - 2\chi q.\end{aligned}$$

But by 6.1, $2kq - 2\chi q = kpq - 2kp$. Therefore

$$\begin{aligned}2(1-\chi)kp &\geqslant kpq - 2kp,\\ [2(2-\chi) - q]kp &\geqslant 0,\\ q &\leqslant 2(2-\chi).\end{aligned}$$

In particular, when the map of type $\{p, q\}$ lies on a surface of genus 3, $\chi = -4$, and $q \leqslant 12$. Now 6.1 with $\chi = -4$ may be written in the form

$$kp(q-2) = 8q + 2kq.$$

6.2 $$p = (8q/k + 2q)/(q-2).$$

We tabulate the solutions of 6.2 for specified values of q. Since $3 \leqslant q \leqslant 12$ (cf. Lemma 1 of § 3) when $p \geqslant q$, we have only 10 diophantine equations to consider in order to list all the arithmetically possible maps of type $\{p, q\}$ ($p \geqslant q$) and genus 3. The maps omitted (those for which $p < q$) are simply the duals of maps already listed.

From 1.3 we see that pk must be even; hence any solution of 6.2 for which pk is odd does not yield an arithmetically possible map. With this in mind, a complete list of the arithmetically possible maps of type $\{p, q\}$ ($p \geqslant q$) on a surface of genus 3 is given by Table IV. The final column of the table indicates the order which the group of the map must have if it happens to be regular. The rows are numbered for easier reference.

7. The regular maps of genus 3. The problem now is to isolate the regular maps which lie among the arithmetically possible maps in Table IV. We determine first the regular maps $^1\{p, q\}$. Then, using the method of Brahana (**2**, p. 280) we determine the regular maps $^2\{p, q\}$. We then note the regular maps of genus 3 which occur in the tables of Coxeter mentioned previously. Finally, the remaining possibilities in Table IV will be tested by recourse to the results of §§ 3, 4, and 5, and by methods not unlike those used there.

https://doi.org/10.4153/CJM-1959-046-9 Published online by Cambridge University Press

TABLE IV

THE ARITHMETICALLY POSSIBLE MAPS OF TYPE $\{p, q\}$ $(p \geqslant q)$ AND GENUS 3

	Type	N_0	N_1	N_2	g
1.	{12, 12}	1	6	1	12
2.	{14, 7}	2	7	1	14
3.	{20, 4}	5	10	1	20
4.	{30, 3}	10	15	1	30
5.	{8, 8}	2	8	2	16
6.	{9, 6}	3	9	2	18
7.	{10, 5}	4	10	2	20
8.	{12, 4}	6	12	2	24
9.	{18, 3}	12	18	2	36
10.	{14, 3}	14	14	3	42
11.	{6, 6}	4	12	4	24
12.	{8, 4}	8	16	4	32
13.	{12, 3}	16	24	4	48
14.	{6, 5}	6	15	5	30
15.	{10, 3}	20	30	6	60
16.	{5, 5}	8	20	8	40
17.	{6, 4}	12	24	8	48
18.	{9, 3}	24	36	8	72
19.	{8, 3}	32	48	12	96
20.	{5, 4}	20	40	16	80
21.	{7, 3}	56	84	24	168

Applying Theorem 1, we discover two 1-faced regular maps of genus 3, namely $^1\{12, 12\}$ and $^1\{14, 7\}$, and exclude possibilities 3 and 4 in Table IV. The regular maps $^1\{12, 12\}$ and $^1\{14, 7\}$ may be denoted by the symbols $\{12, 12\}_{1,0}$ and $\{14, 7\}_2$, in analogy with the corresponding cases of regular maps of genus 2 (**7**, p. 141). The group of $\{12, 12\}_{1,0}$ is the cyclic group of order 12, while the group of $\{14, 7\}_2$ is the cyclic group of order 14.

In virtue of Lemmas 2 and 3 we may immediately rule out numbers 7 and 9 in Table IV as possibilities for regular maps. To determine whether the remaining 2-faced maps are regular or not, we use the method initiated by Brahana (**2**, p. 280), that is, given the 2-faced map of type $\{p, q\}$, we look for a group generated by R and $T = RS$ (cf. 3.1), with the defining relations

$$R^p = T^2 = E, \qquad TRT = R^n$$

where $n^2 \equiv 1 \pmod{p}$, and implying that RT is of the desired period, namely q.

In case no. 5 we have $p = 8$, and hence

7.1 $$n^2 \equiv 1 \pmod{8}.$$

Solutions are $n \equiv 1, 3, 5,$ and 7. If $n = 1$, then $RT = TR$ and RT is of period 8. Thus there exists a regular map of type $\{8, 8\}$ and genus 3 whose group is defined by the relations

$$R^8 = T^2 = E, \qquad R \rightleftarrows T.$$

The map is analogous to the regular map $\{6, 6\}_2$ of genus 2 **(7**, p. 141**)**; accordingly we denote it by the symbol $\{8, 8\}_2$. The solution $n = 3$ of 7.1 gives no further regular map of type $\{8, 8\}$, nor does the solution $n = 7$. But when $n = 5$, RT is again of period 8 and hence there exists another regular map of type $\{8, 8\}$ and genus, 3, whose group has the abstract definition

$$R^8 = T^2 = E, \qquad TRT = R^5.$$

It was shown by Coxeter and Moser **(7**, p. 114**)** that this abstract definition may be put in the form

7.2 $$T^2 = E, \qquad TST = S^{-3}$$

and that the above relations define Miller's group $\langle 2, 2 \mid 2\rangle$. Accordingly, the map is denoted by the symbol $\{4\cdot 2, 4\cdot 2\}$ (cf. Table II). This is the "map of type $\{8, 8\}$" mentioned by Coxeter and Moser **(7**, p. 114**)**, a member of the sub-family of regular maps $\{(r + 1)\cdot(r - 1), (r + 1)\cdot 2\}$ on a surface of genus $\frac{1}{2}r(r - 1)$.

Proceeding in the manner outlined above, we eliminate case 6 in Table IV, but discover corresponding to case 8 a regular map of type $\{12, 4\}$. Its group has the abstract definition

$$R^{12} = T^2 = E, \qquad TRT = R^5.$$

This is the group $\langle 6, 2 \mid 2; 2\rangle$ **(7**, p. 114**)**, and therefore the map is denoted by the symbol $\{2\cdot 6, 2\cdot 2\}$. It is a member of the sub-family of maps $\{2\cdot 2\text{p}, 2\cdot 2\}$, to which Coxeter and Moser give the symbol $\{4\text{p}, 4\}_{1,1}$ **(7**, p. 115**)**. Another symbol for the group $\langle 6, 2 \mid 2; 2\rangle$ is $\mathfrak{C}_4 \times \mathfrak{D}_3$ **(**cf. **7**, p. 10, (1.861) when $r = 5$, $m = 3$, $n = 2$**)**.

An important family of regular maps is the family whose members are characterized by specified Petrie polygons. A *Petrie polygon* of a map is a "zig-zag" along its edges such that every two but no three successive edges of the polygon are edges of a single face. For example the path $ABCDEF\ldots$ of Figure 4 is a Petrie polygon. A regular map of type $\{p, q\}$ characterized by its r-gonal Petrie polygons is denoted by the symbol $\{p, q\}_r$. If the map is on an orientable surface, then r is even **(7**, p. 111**)** and the group of the map has the abstract definition

7.3 $$R^p = S^q = (RS)^2 = (R^2S^2)^n = E$$

where $n = \frac{1}{2}r$ **(4**, p. 126**)**. The dual of $\{p, q\}_r$ also has r-gonal Petrie polygons and is denoted by the symbol $\{q, p\}_r$.

The previous use of the symbols $\{14, 7\}_2$ and $\{8, 8\}_2$ is easily shown to be justified. In addition to these two cases of regular maps $\{p, q\}_{2n}$ of genus 3, the tables of Coxeter and Moser **(7**, p. 140**)** contain $\{8, 3\}_6$ and $\{7, 3\}_8$, corresponding to entries 19 and 21 in Table IV. We thus establish the existence of two more regular maps of genus 3. The map $\{7, 3\}_8$ was discussed in 1879 by Klein **(10)**; Dyck **(8)** examined $\{8, 3\}_6$ in 1880.

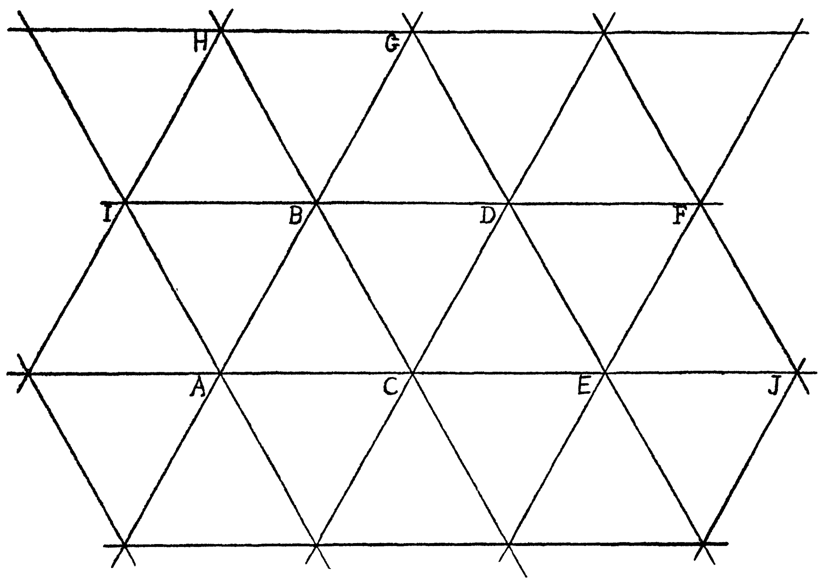

FIGURE 4

The relations 7.3 form an abstract definition for the group $(2, q, p; n)$ (**4**, p. 86). Thus in particular $(2, 3, 8; 3)$ is the group of $\{8, 3\}_6$ and $(2, 3, 7; 4)$ is the group of $\{7, 3\}_8$. The latter is the simple group $LF(2, 7)$ (**7**, p. 96).

We have not yet shown that $\{8, 3\}_6$ and $\{7, 3\}_8$ are the only regular maps of types $\{8, 3\}$ and $\{7, 3\}$ on this surface. We shall postpone the proof until we are ready to make a systematic study of all the regular maps of type $\{p, 3\}$ and genus 3.

We now seek regular maps of genus 3 among the regular maps having specified holes. A *hole* is a path along the edges of a map such that at each vertex visited we leave two faces on (say) the left (**3**, p. 38). Thus, for example, the path *ACDGHIA* in Figure 4 is a hexagonal hole. A regular map of type $\{p, q\}$ characterized by its n-gonal holes is denoted by the symbol $\{p, q \mid n\}$ and its group, denoted by $(p, q \mid 2, n)$, has the abstract definition

7.4 $$R^p = S^q = (RS)^2 = (R^{-1}S)^n = E$$

(**4**, p. 74). If $n = 2$, then p and q are even (**7**, p. 109). Suppose that $n = 2$, $p = 4$, and q is any even number. Then the final relation in 7.4 is

$$(R^{-1}S)^2 = E.$$

Since $R^4 = (RS)^2 = E$, this relation implies

$$R^2SR^2 = RS^{-1}R = S,$$

whence

$$R^2 \rightleftarrows S.$$

https://doi.org/10.4153/CJM-1959-046-9 Published online by Cambridge University Press

Thus

$$(4, q \mid 2, 2) \cong \langle\langle 2, q \mid 2; 2\rangle\rangle$$

and

$$\{4, q \mid 2\} = \{2\cdot 2, q\}$$

(cf. Table I). Dually

$$\{q, 4 \mid 2\} = \{q, 2\cdot 2\}.$$

Consulting Coxeter and Moser (**7**, p. 109), we discover the regular map $\{8, 4 \mid 2\} = \{8, 2\cdot 2\}$, of genus 3, which occurs in Table IV as case 12.

Having found one regular map of type $\{8, 4\}$ and genus 3, we ask if there are any others. To answer this question we exhibit in Figure 5 a diagram of the arrangement of the four faces α, β, γ, and δ of any regular map ${}^4\{8, 4\}$,

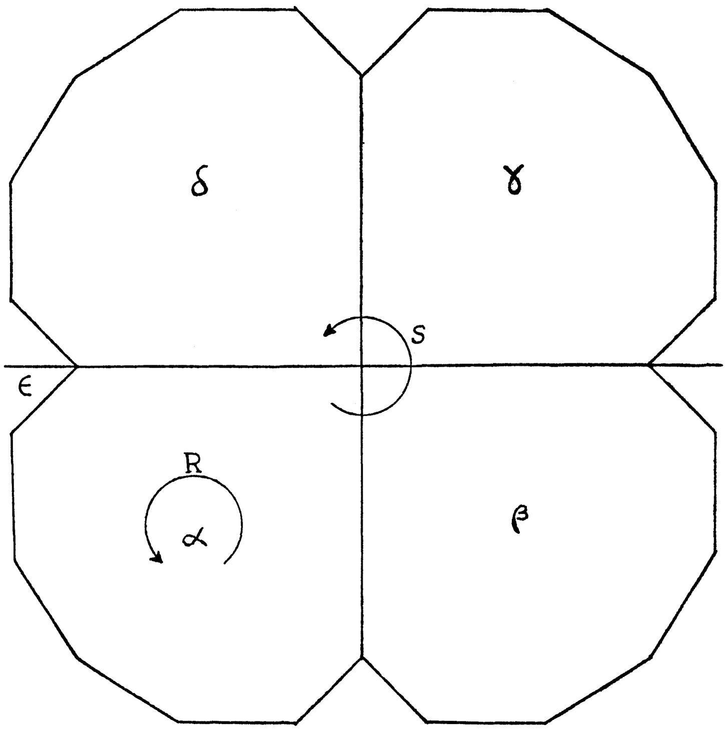

FIGURE 5

this arrangement being the only one possible because of Lemmas 2 and 3 of § 3. Again by Lemma 2 and 3 the face ϵ of the regular tessellation $\{8, 4\}$, which must now be identified with one of the former 4 faces, cannot be identified with α or δ. Applying Lemma 5 to the face α, we see that ϵ cannot be identified with γ. Thus ϵ must be identified with β; in symbols

$$\{R\}SR^2 = \{R\}S.$$

https://doi.org/10.4153/CJM-1959-046-9 Published online by Cambridge University Press

In particular for some integer l,

$$SR^2 = R^lS.$$

Since R^l is of the same period as R^2, $l = \pm 2$. If $l = -2$. the group of the map must satisfy the following relations:

7.5 $$R^8 = S^4 = (RS)^2 = E, \qquad SR^2 = R^{-2}S.$$

The final relation rewritten is

$$S^{-1}R(RSR)R = E,$$
$$S^{-1}RS^{-1}R = E \qquad \text{(since } (RS)^2 = E).$$

Thus 7.5 is identical with 7.4 when $p = 8$, $q = 4$, and $n = 2$. We have, therefore, no new regular map for the case $l = -2$. When $l = 2$, the group of the map must satisfy the relations

$$R^8 = S^4 = (RS)^2 = E, \qquad R^2 \rightleftarrows S,$$

which define the group $\langle\langle 2, 4 \mid 2; 4\rangle\rangle \cong \langle\langle 2, 4 \mid 2\rangle\rangle$. Thus the above choice of l yields a second regular map of type $\{8, 4\}$ and genus 3, which is denoted by the symbol $\{4 \cdot 2, 4\}$ (cf. Table I).

In an extension of his concept of a hole, Coxeter (**3**, p. 59) introduced the notion of a *second hole*. This is a path along the edges of a map such that at each vertex visited we leave three faces on (say) the left. Thus, for example, the path $ACEJ\ldots$ of Figure 4 is a second hole. A regular map of type $\{p, q\}$ characterized by its n-gonal second holes is denoted by the symbol $\{p, q \mid, n\}$, and its group has the abstract definition

7.6 $$R^p = S^q = (RS)^2 = (RS^{-2})^n = E.$$

Coxeter (**3**, p. 61) compiled a list of regular maps $\{p, q \mid, n\}$. There are three unfortunate omissions in this table, which were later corrected by Coxeter. They are

$\{4, 6 \mid, 2\}$	12	24	8	3	$\mathfrak{S}_4 \times \mathfrak{C}_2$	48
$\{5, 6 \mid, 2\}$	24	60	20	9	$\mathfrak{A}_5 \times \mathfrak{C}_2$	120
$\{3, 11 \mid, 4\}$	2024	3036	552	231	$LF(2, 23)$	6072

In the complete table there are three regular maps of genus 3, namely $\{3, 8 \mid, 3\}$, $\{3, 7 \mid, 4\}$, and $\{4, 6 \mid, 2\}$, corresponding to entries 19, 21, and 17 in Table IV. The group of $\{3, 8 \mid, 3\}$ has the abstract definition 7.6 with $p = 3$, $q = 8$, and $n = 3$ while the group of $\{3, 7 \mid, 4\}$ is 7.6 with $p = 3$, $q = 7$, and $n = 4$. It is easily seen by comparing 7.6 with 7.3 that any map $\{3, q \mid, n\}$ is identical with the map $\{3, q\}_{2n}$. Thus $\{3, 8 \mid, 3\}$ is $\{3, 8\}_6$ and $\{3, 7 \mid, 4\}$ is $\{3, 7\}_8$. Because of the ease with which it may be dualized, the latter symbol in each case is used exclusively to denote the map.

The group of the regular map $\{4, 6 \mid, 2\}$ has the abstract definition

$$R^6 = S^4 = (RS)^2 = (R^{-2}S)^2 = E.$$

It is easily shown that the above relations are equivalent to

7.7 $$R^6 = S^4 = (RS)^2 = E, \qquad R^3 \rightleftarrows S.$$

But these relations define the group $\langle\langle 2, 4 \mid 3; 2\rangle\rangle$ and therefore $\{4, 6 \mid, 2\}$ will be denoted by the symbol $\{4, 2\cdot 3\}$, which dualizes more easily.

We wish to determine whether or not there are any other regular maps of type $\{6, 4\}$ and genus 3. To this end we exhibit in Figure 6 a part of the

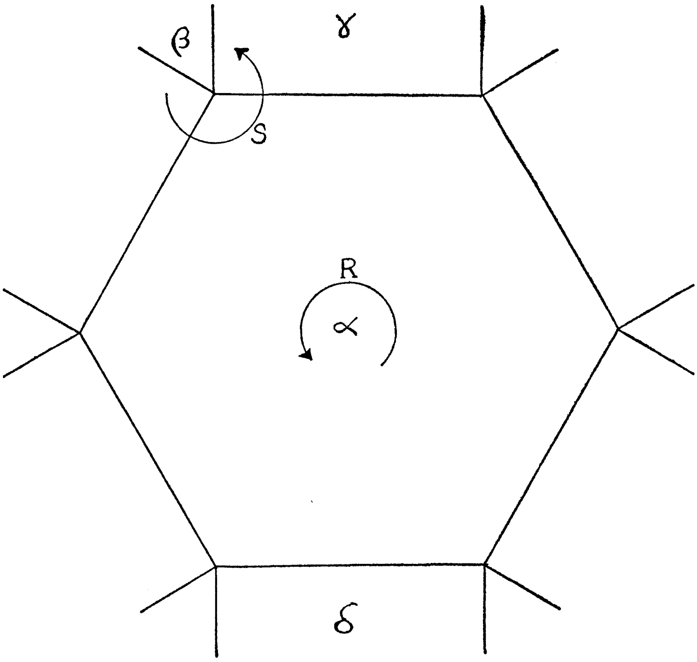

FIGURE 6

regular tessellation $\{6, 4\}$ in the hyperbolic plane. There are three possible arrangements of the faces of a regular map around the face α, namely

(i) only two distinct faces are contiguous with α,
(ii) only three distinct faces are contiguous with α,
(iii) six distinct faces are contiguous with α.

Case (i) is easily dispensed with, for if there were such a regular map, we would have the relation

$$\{R\}S = \{R\}SR^2,$$

which implies, for some integer l, the relation

$$R^l S = SR^2.$$

When this is added to

7.8 $$R^6 = S^4 = (RS)^2 = E$$

it is not hard to show that there are only 4 right cosets of the subgroup $\{R\}$. This eliminates the possibility of an 8-faced map. Case (iii) is likewise easily dispensed with, for it is readily seen that the face β (Figure 6) must be one of the six faces which surround α. Taking into account the symmetry of the map, this implies

$$\{R\}S^2 = \{R\}SR^{-j} \qquad (j = 1 \text{ or } 2),$$

which in turn implies either

$$R^l S^2 = SR^{-1}$$

or

$$R^m S^2 = SR^{-2}$$

for some integers l and m. But if we add either one of these relations to 7.8 and enumerate cosets of $\{R\}$, we obtain a collapse which implies a breakdown of the proposed structure of the map. Thus case (ii) is the only possible arrangement of faces surrounding α that can yield a regular map. In this case γ and δ must be identical; in symbols

$$\{R\}S = \{R\}SR^3.$$

This implies in particular that

$$R^l S = SR^3$$

for some integer l. If the period of R is to remain at 6, then $l = 3$ and the only regular map that this case yields is the map whose group has the abstract definition 7.7, that is, the group $\langle\langle 2, 4 \mid 3; 2\rangle\rangle$. Therefore $\{2\cdot 3, 4\}$ is the only regular map of type $\{6, 4\}$ and genus 3.

We shall now consider the maps ${}^k\{p, 3\}$ with $k \geqslant 3$ in the order in which they appear in Table IV, applying the results of §§ 3 and 5.

By Theorem 3, case no. 10 is eliminated; there is no regular map ${}^3\{14, 3\}$. There is, however, by the result of Theorem 4, exactly one regular map of type $\{12, 3\}$ and genus 3. It is the regular map $\{4\cdot 3, 3\}$, and corresponds to case 13 in Table IV.

Theorem 6 eliminates case 15 as a possibility for a regular map.

Applying Theorem 7 and Lemma 5 to case 18, we see that if the number of faces of a regular map of type $\{9, 3\}$ is > 6 it must be $\geqslant 10$. But the arithmetically possible map of type $\{9, 3\}$ and genus 3 has only 8 faces and therefore cannot be regular.

Having already found one regular map of type $\{8, 3\}$ and genus 3, namely $\{8, 3\}_6$, we now check the possibility of others. In view of Theorem 7 and Lemma 5 we know that a face α of a regular map ${}^{12}\{8, 3\}$ must be surrounded by 8 other distinct faces. Now either the face β of the tessellation $\{8, 3\}$

https://doi.org/10.4153/CJM-1959-046-9 Published online by Cambridge University Press

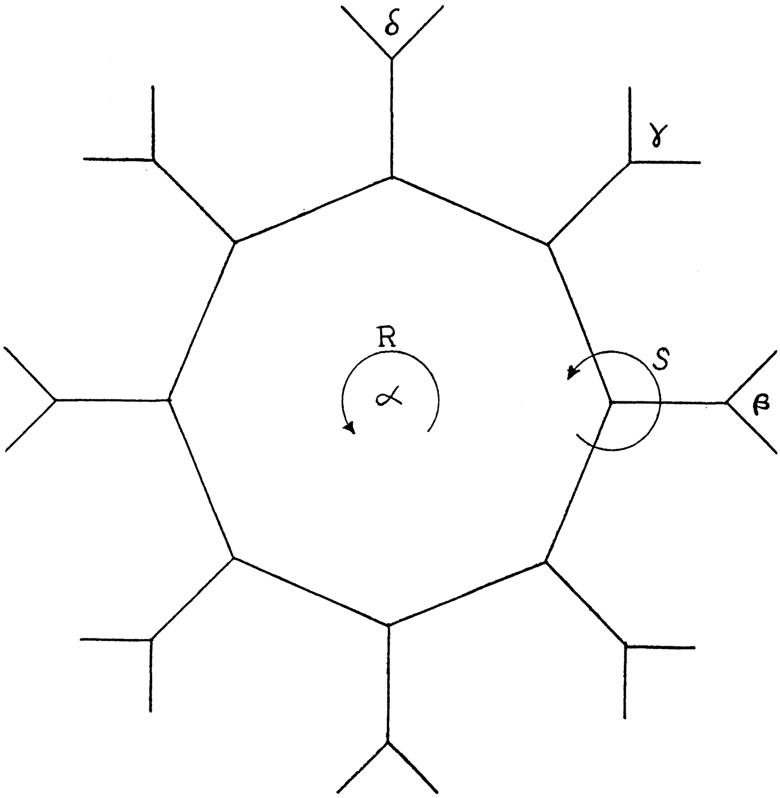

FIGURE 7

(cf. Figure 7) is one of these 8 faces, or else it is one of the 3 remaining faces. If the former alternative is true we have, taking into account the symmetry of the map,

$$\{R\}SR^{-1}S = \{R\}SR^{-j} \qquad (j = 2 \text{ or } 3).$$

In particular, we have for some integers l and m,

$$R^l SR^{-1}S = SR^{-2}$$

or else

$$R^m SR^{-1}S = SR^{-3}.$$

Adding each of these in turn to the relations

7.9 $$R^8 = S^3 = (RS)^2 = E,$$

and enumerating cosets of $\{R\}$, we prove in both cases that the structure of the faces surrounding α is broken down by the proposed identification of β. Hence β must be a tenth face of the map. If β is not one of the faces surrounding α then neither is γ. Moreover γ is not identical to β since both

β and γ border on one and the same face. Thus γ is an eleventh face. However, Lemma 5 implies that δ must be identical with β; hence

$$\{R\}SR^{-1}S = \{R\}SR^{-1}SR^2.$$

Thus, for some integer l,

$$R^lSR^{-1}S = SR^{-1}SR^2.$$

If we add this relation to 7.9 and enumerate cosets of $\{R\}$ in the group thus defined, we obtain 12 cosets and the correct group structure if, and only if, $l = 2$. The relation $R^2SR^{-1}S = SR^{-1}SR^2$ rewritten is

$$R^2SR^{-1}S^{-1} = SR^{-1}SRRS.$$

Since $(RS)^2 = E$, this relation becomes

$$R^2SSR = SR^{-2}S^{-1}S^{-1}R^{-1}$$
$$(R^2S^{-1})^3 = E \qquad \text{(since } S^3 = E).$$

This is the fourth defining relation of the group of the regular map $\{8, 3\}_6$ (cf. 7.3 when $p = 8$, $q = 3$, $n = 3$). Hence $\{8, 3\}_6$ is the only regular map of type $\{8, 3\}$ and genus 3.

In a similar fashion we may prove that $\{7, 3\}_8$ is the only regular map of type $\{7, 3\}$ and genus 3. The method is now apparent; we build up the structure of the map step by step, using the theory of §§ 3 and 5, and testing each step by examining its effect on the group of the regular tessellation $\{7, 3\}$.

The only entries in Table IV which remain to be considered are 11, 14, 16, and 20. In case 11 we seek a 4-faced regular map of type $\{6, 6\}$. In virtue of Lemmas 2, 3, and 5 of § 3, and the fact that the map has only 4 faces, it follows that three distinct faces surround a vertex, and their arrangement must be like that of α, β, and γ in Figure 8. Thus we have the relation

$$\{R\}S^3 = \{R\}.$$

In particular there is an integer l such that

$$S^3 = R^l.$$

Since S and R are both of period 6, $l = 3$. Adding the relation $S^3 = R^3$ to

$$R^6 = S^6 = (RS)^2 = E$$

we note that the relations may be put into the form

$$R^3 = S^3 = Z, \qquad (RS)^2 = Z^2 = E,$$

which defines the group $\langle 3, 3,| 2; 2\rangle$. This is the group of the regular map $\{2\cdot3, 2\cdot3\}$, isomorphic to the group $\mathfrak{A}_4 \times \mathfrak{C}_2$ (**7**, p. 73). The map $\{2\cdot3, 2\cdot3\}$ is the only regular map of type $\{6, 6\}$ and genus 3.

The remaining three cases, 14, 16, and 20 in Table IV, yield no regular maps. As in previous cases this fact may be verified by assuming in each

https://doi.org/10.4153/CJM-1959-046-9 Published online by Cambridge University Press

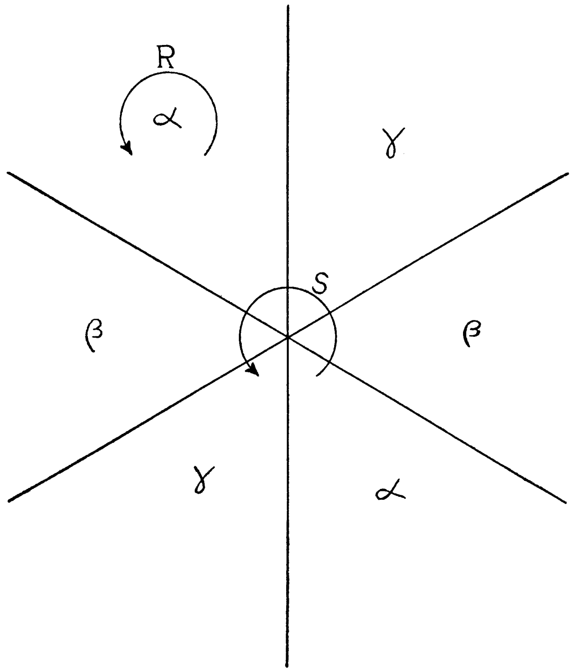

FIGURE 8

case that such a map exists, and then attempting to find its group by identifying faces of the regular tessellation from which the map would arise. Every identification that is possible proves to be unfruitful. With the help of the lemmas of § 3, this procedure is neither difficult nor unduly long.

TABLE V

THE REGULAR MAPS OF GENUS 3

Map	N_0	N_1	N_2	Dual	Group	Order	Reference
$\{12, 12\}_{1,0}$	1	6	1	Self-dual	$\mathfrak{C}_{12}$	12	(7, p. 61)
$\{14, 7\}_2$	1	7	2	$\{7, 14\}_2$	$\mathfrak{C}_{14}$	14	(7, p. 140)
$\{8, 8\}_2$	2	8	2	Self-dual	$\mathfrak{C}_8\times\mathfrak{C}_2$	16	(7, p. 140)
$\{4\cdot 2, 4\cdot 2\}$	2	8	2	Self-dual	$\langle 2, 2 \mid 2\rangle$	16	(7, p. 114)
$\{2\cdot 6, 2\cdot 2\}$	2	12	6	$\{2\cdot 2, 2\cdot 6\}$	$\langle 6, 2 \mid 2; 2\rangle \cong \mathfrak{C}_4\times\mathfrak{D}_3$	24	(7, p. 115)
$\{2\cdot 3, 2\cdot 3\}$	4	12	4	Self-dual	$\langle 3, 3 \mid 2; 2\rangle \cong \mathfrak{A}_4\times\mathfrak{C}_2$	24	
$\{8, 2.2\}$	4	16	8	$\{2\cdot 2, 8\}$	$\langle\langle 2, 8 \mid 2; 2\rangle\rangle \cong (8, 4 \mid 2, 2)$	32	(7, p. 109)
$\{4\cdot 2, 4\}$	4	16	8	$\{4, 4\cdot 2\}$	$\langle\langle 2, 4 \mid 2\rangle\rangle$	32	
$\{4\cdot 3, 3\}$	4	24	16	$\{3, 4\cdot 3\}$	$\langle\langle 2, 3 \mid 3\rangle\rangle$	48	
$\{2\cdot 3, 4\}$	8	24	12	$\{4, 2\cdot 3\}$	$\langle\langle 2, 4 \mid 3; 2\rangle\rangle \cong \mathfrak{S}_4\times\mathfrak{C}_2$	48	(7, p. 115)
$\{8, 3\}_6$	12	48	32	$\{3, 8\}_6$	$(2, 3, 8; 3)$	96	(7, p. 140)
$\{7, 3\}_8$	24	84	56	$\{3, 7\}_8$	$(2, 3, 7; 4) \cong \mathrm{LF}(2, 7)$	168	(2, p. 278) (7, p. 140)

The 12 regular maps of genus 3 (a map and its dual counted as one) are listed in Table V. Figures 9–20 are drawings of the maps in which the edges are numbered. Those bordering edges which are numbered alike are to be identified.

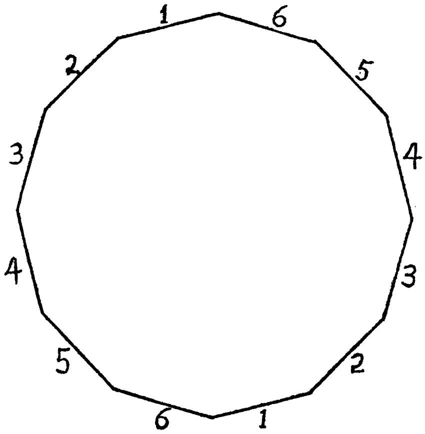

FIGURE 9: $\{12, 12\}_{1,0}$

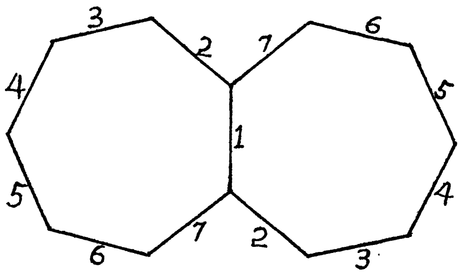

FIGURE 10: $\{14, 7\}_2$

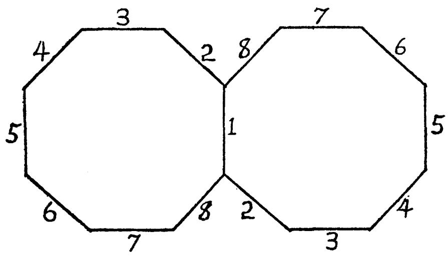

FIGURE 11: $\{8, 8\}_2$

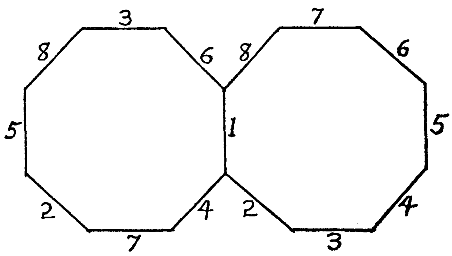

FIGURE 12: $\{4\cdot 2, 4\cdot 2\}$

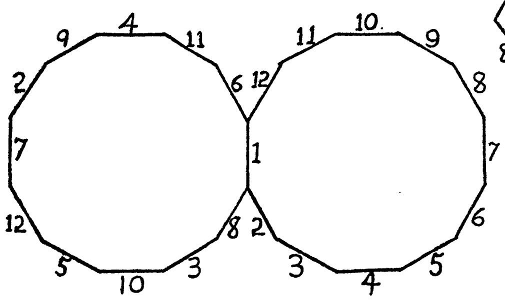

FIGURE 13: $\{2\cdot 6, 2\cdot 2\}$

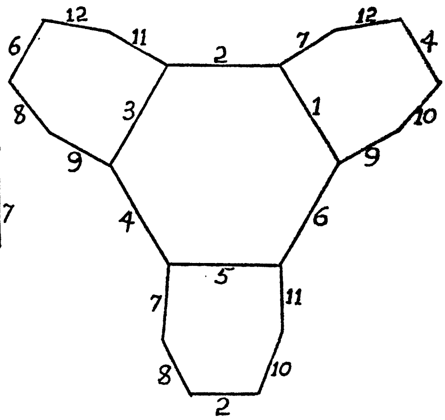

FIGURE 14: $\{2\cdot 3, 2\cdot 3\}$

https://doi.org/10.4153/CJM-1959-046-9 Published online by Cambridge University Press

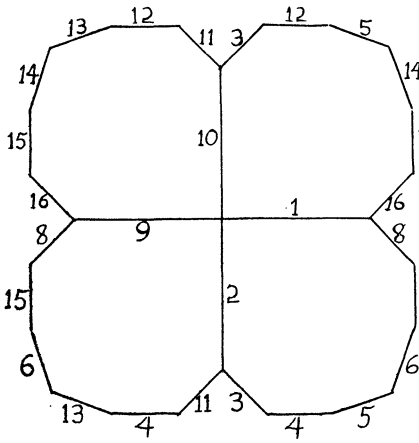

FIGURE 15: {8·2, 2}

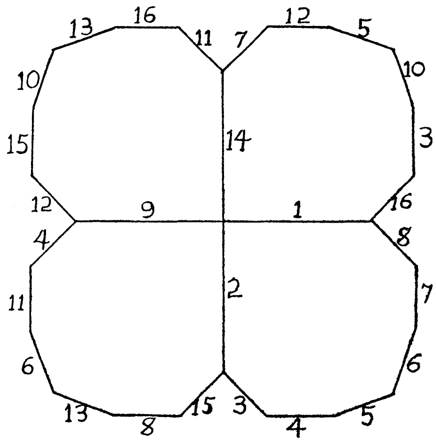

FIGURE 16: {4·2, 4}

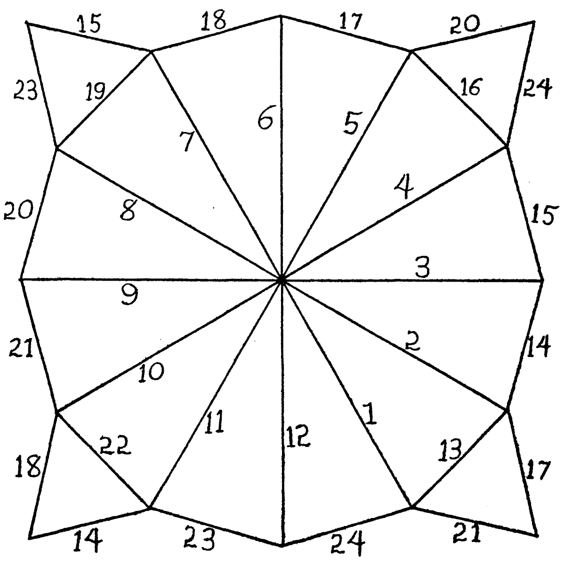

FIGURE 17: {3, 4·3}

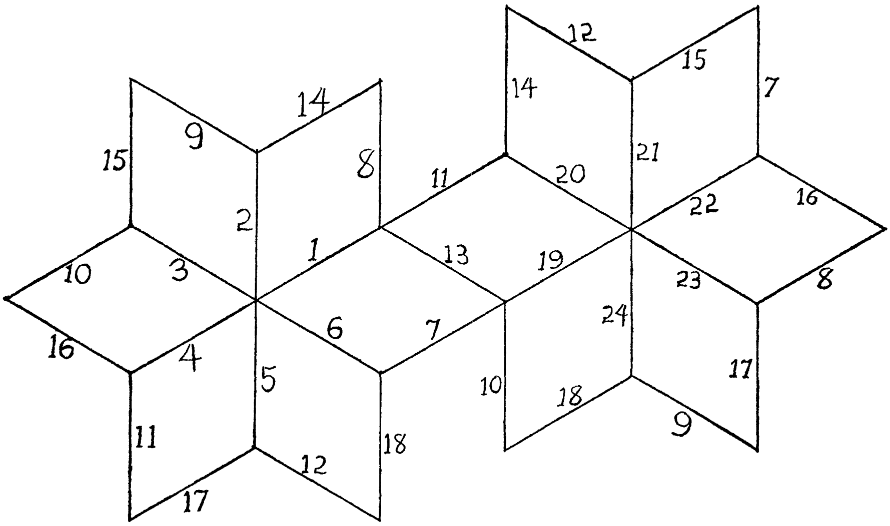

FIGURE 18: {4, 2·3}

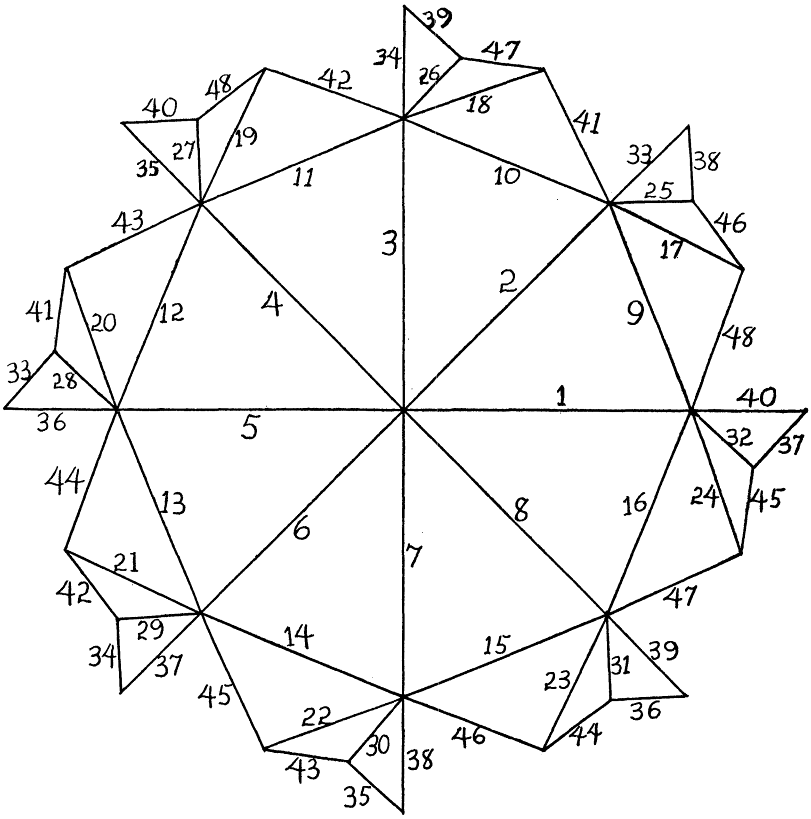

FIGURE 19: {3, 8}₆

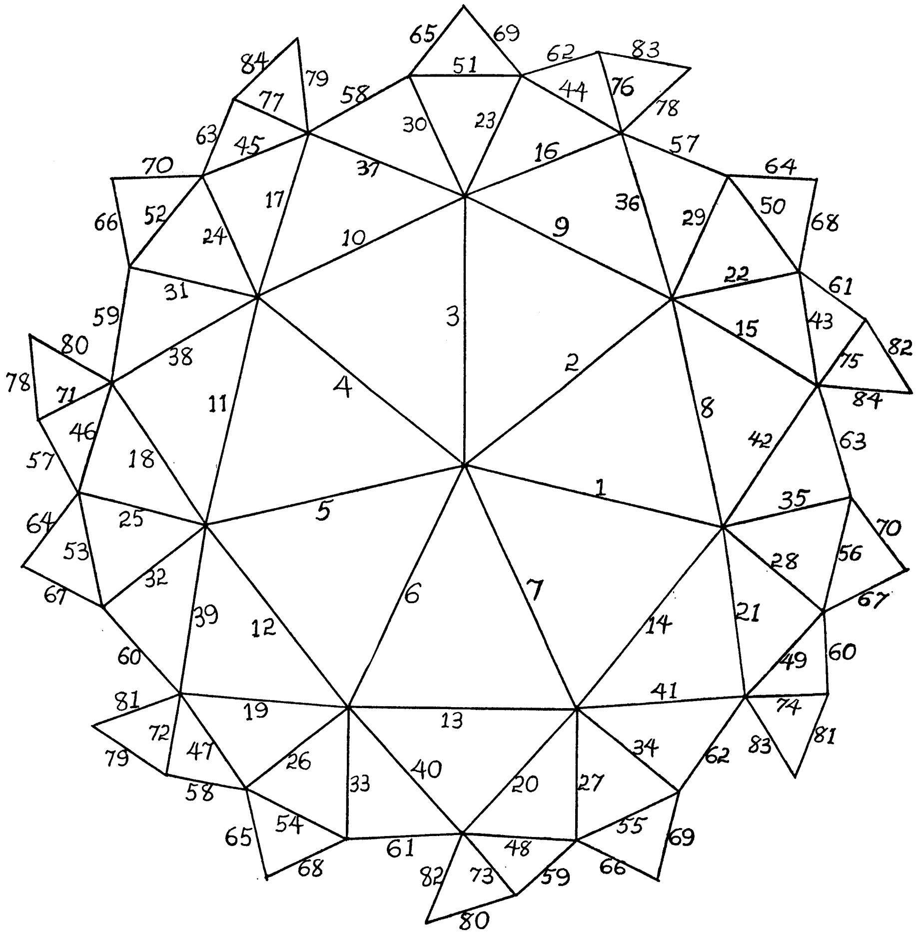

FIGURE 20: $\{3, 7\}_8$

https://doi.org/10.4153/CJM-1959-046-9 Published online by Cambridge University Press

References

1. H. R. Brahana, *Regular maps on an anchor ring*, Amer. J. Math., *48* (1926), 225–40.
2. ———, *Regular maps and their groups*, Amer. J. Math., *49* (1927), 268–84.
3. H. S. M. Coxeter, *Regular skew polyhedra in three and four dimensions and their topological analogues*, Proc. London Math. Soc. (2), *43* (1937), 33–62.
4. ———, *The abstract groups* $G^{m,n,p}$, Trans. Amer. Math. Soc., *45* (1939), 73–150.
5. ———, *Regular polytopes* (London, 1948).
6. ———, *Configurations and maps*, Reports of a Math. Colloq. (2), *8* (1948), 18–38.
7. H. S. M. Coxeter and W. O. J. Moser, *Generators and relations for discrete groups*, Ergebn. Math., *14* (1957).
8. W. Dyck, *Notiz ueber eine reguläre Riemann'sche Fläche vom geschlechte drei und die zugehörige "normalcurve" vierter ordnung*, Mat. Ann., *17* (1880), 510–16.
9. R. Frucht, *A one-regular graph of degree three*, Can. J. Math., *4* (1952), 240–7.
10. F. Klein, *Ueber die transformationen siebenter ordnung der elliptischen functionen*, Mat. Ann., *14* (1879), 428–71.
11. ———, *Lectures on the icosahedron and the solution of equations of the fifth deegree* trans. G. C. Morrice (London, 1913).
12. G. C. Shephard, *Regular complex polytopes*, Proc. London Math. Soc. (3), *2* (1952), 82–97.

University of Toronto

https://doi.org/10.4153/CJM-1959-046-9 Published online by Cambridge University Press