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# Colimits of algebras revisited

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It has been open for some time whether, given an algebraic theory (triple, monad) $\mathbb{T}$ in a cocomplete category $K$, also the category $K^{\mathbb{T}}$ of $\mathbb{T}$-algebras must be cocomplete. We solve this in the negative by exhibiting a free algebraic theory $\mathbb{T}$ in the category $\mathcal{Gra}$ of graphs such that $\mathcal{Gra}^{\mathbb{T}}$ is not cocomplete. Further, we improve somewhat the well-known colimit theorem of Barr and Linton by showing that the base category need not be complete.

## I. Categories of algebras ...

I.1. Is it true that an arbitrary theory of continuous (or ordered or compact, and so on) algebras allows the formation of sums? More generally: given an algebraic theory $\mathbb{T}$ in a "decent" cocomplete category $K$, is it true that the category $K^{\mathbb{T}}$ of $\mathbb{T}$-algebras is also cocomplete? While analogous questions about limits are elementary (the forgetful functor $K^{\mathbb{T}} \to K$ always creates limits), colimits present an interesting problem. Various sufficient conditions (which cover all of the important cases, in fact) have been found. For example, Linton proved in [7]:

THEOREM (Linton). *If* $K$ *has sums and* $K^{\mathbb{T}}$ *has coequalizers then* $K^{\mathbb{T}}$ *is cocomplete.*

I.2. Other conditions involve *factorization systems*. Let us recall (for example from [5] or [9]) that a factorization system $(E, M)$ in a category $K$ consists of classes $E$, $M$ of morphisms subject to the following conditions:

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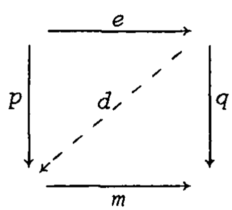
(i) all $M$-morphisms are monos, all $E$-morphisms are epis;

(ii) $M$ and $E$ are subcategories, that is, closed to composition, both containing all isomorphisms;

(iii) $K = M.E$, that is every morphism $f$ has a factorization $f = m.e$ with $e \in E$ and $m \in M$;

(iv) diagonal fill-in: for every commutative square




with $e \in E$, $m \in M$, there exists a (diagonal) morphism $d$, making both triangles commute.

Factorization systems have a lot of natural properties, easy to verify, such as the following:

(v) $E$ contains all coequalizers (this is an exercise in [5]);

(vi) opposite an $E$-morphism in a pushout there is an $E$-morphism (see [9]);

(vii) a multiple pushout of $E$-morphisms consists of $E$-morphisms (this is proved, more generally, in IV.1 below).

I.3. The following important theorem has been proved by Linton [7] and, in a different way, by Barr [4].

THEOREM (Barr and Linton). *Let* $K$ *be a category with a factorization system* $(E, M)$ *which is*

*(a) complete,*

*(b) cocomplete,*

*(c)* $E$*-cowell-powered.*

*Let* $\mathbb{T} = (T, \mu, \eta)$ *be an algebraic theory which preserves* $E$*; that is such that* $e \in E$ *implies* $Te \in E$. *Then the category* $K^{\mathbb{T}}$ *is cocomplete.*

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(Neither Barr nor Linton used the above definition of a factorization system; but we show in Section IV that their definitions are equivalent to ours. Linton supposed that $\mathbb{T}$ preserves also $M$ .)

I.4. Two of the assumptions in the above colimit theorem can be felt as not entirely natural: completeness (cannot we do without it in a colimit theorem?) and preservation of $E$ (is it necessary to assume things not only about $K$ but also about $\mathbb{T}$ ?). The aim of the present paper is to show that completeness is redundant (see Section II) while preservation of $E$ is not (see Section III).

Let us remark that Barr exhibits in [4] another colimit theorem: if $\mathbb{T}$ has rank then $K^{\mathbb{T}}$ is cocomplete. This covers all "natural" theories $\mathbb{T}$ . Thus, it is no surprise that the counterexample in Section III consists of an ugly algebraic theorem $\mathbb{T}$ (in a nice category $K$ , though).

## II. ... are often cocomplete ...

II.1. We shall consider not only $\mathbb{T}$-algebras of an algebraic theory but, more generally, $F$-algebras of an arbitrary endofunctor $F : K \to K$ . An *F-algebra* is a pair $(A, \alpha)$ , consisting of an object $A$ of $K$ and a morphism $\alpha : FA \to A$ (subject to no axioms). Given two $F$-algebras $(A, \alpha)$ and $(B, \beta)$ , by an *F-homomorphism* $f : (A, \alpha) \to (B, \beta)$ is meant a $K$-morphism $f : A \to B$ such that $f.\alpha = \beta.Ff$ . We denote by $K(F)$ the category of $F$-algebras and $F$-homomorphisms.

Thus, given an algebraic theory $\mathbb{T} = (T, \mu, \eta)$ in $K$ the category $K^{\mathbb{T}}$ of $\mathbb{T}$-algebras is a full subcategory of the category $K(T)$ of $T$-algebras.

II.2. Categories $K(F)$ were used by Barr [4] for the study of free algebraic theories - this study was then applied by Arbib and Manes [3] to automata in categories. The latter call $F$ an *input process* provided that the forgetful functor $K(F) \to K$ has a left adjoint, in other words, provided that each object $A$ in $K$ generates a *free F-algebra*. Explicitly, this free $F$-algebra consists of an $F$-algebra $(A^{\#}, \varphi^{A})$ and a morphism $s^{A} : A \to A^{\#}$ in $K$ which is universal in the following sense. Given an $F$-algebra $(B, \beta)$ , for every morphism $f : A \to B$ there is a unique $F$-homomorphism $f^{\#} : (A^{\#}, \varphi^{A}) \to (B, \beta)$ with $f = f^{\#}.s^{A}$ .

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For each input process $F$ there arises an algebraic theory $\mathbb{T}$ (freely generated by $F$ ) with

$TA = A^{\#}$ ;

$\mu^A : A^{\#\#} \to A^{\#}$ is the unique $F$-homomorphism

$$(A^{\#\#}, \varphi^{A^{\#}}) \to (A^{\#}, \varphi^A) \text{ with } \mu^A . s^A = 1_A ;$$

$\eta^A = s_A : A \to A^{\#}$ .

Barr [4] proves that, under additional assumptions on $K$ , these are the only free algebraic theories in $K$ .

PROPOSITION (Barr). *Let* $F$ *be an input process and let* $\mathbb{T}$ *be the corresponding free algebraic theory. Then the categories* $K(F)$ *and* $K^{\mathbb{T}}$ *are isomorphic.*

II.3. When aiming at a cocompleteness theorem for categories $K^{\mathbb{T}}$ , we can restrict our attention to coequalizers in $K^{\mathbb{T}}$ (I.1); it turns out that, sufficiently often, we can work with coequalizers in $K(T)$ :

LEMMA. *Let* $K$ *be a category with a factorization system* $(E, M)$ *, let* $\mathbb{T} = (T, \mu, \eta)$ *be an algebraic theory, preserving* $E$ *. Then for every coequalizer in* $K(T)$ *,*

$$(A, \alpha) \underset{g}{\overset{f}{\rightrightarrows}} (B, \beta) \xrightarrow{k} (C, \gamma)$$

*such that* $(B, \beta)$ *is a* $\mathbb{T}$*-algebra, also* $(C, \gamma)$ *is a* $\mathbb{T}$*-algebra.*

Proof. Let $E^T$ denote the class of all $T$-homomorphisms with underlying morphism in $E$ ; analogously $M^T$ . Then $(E^T, M^T)$ is a factorization system in $K(T)$ ; see [9], 3.4.17. Hence, by I.2 (v), $k \in E$ . By hypothesis, also $Tk \in E$ , $T^2k \in E$ , and so on.

To show that $(C, \gamma)$ is indeed a $\mathbb{T}$-algebra, consider the following diagrams, which clearly commute:

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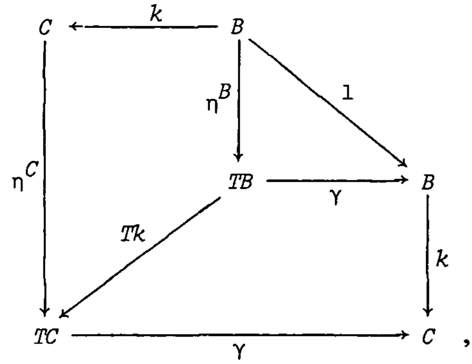


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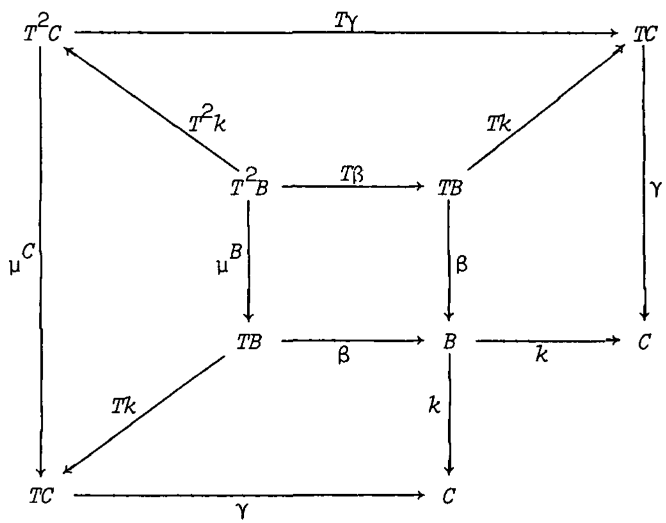



By the first one, $(\gamma.\eta^C).k = k$, hence $\gamma.\eta^C = 1$ ($k$ is epi). By the second one, $(\gamma.\mu^C).T^2k = (\gamma.T\gamma).T^2k$ ; hence $\gamma.\mu^C = \gamma\ T\gamma$ ($T^2k$ is epi).

II.4. The following theorem is proved in [2] in a different manner, as a part of a more general study of colimits in $\mathrm{K}(F)$ . (An iterative colimit-construction is exhibited there, generalizing that used in universal algebra.) We present a straightforward proof. The help of Václav Koubek with this proof is gratefully acknowledged.

THEOREM. *Let* $\mathrm{K}$ *be a cocomplete category with a factorization system* $(\mathrm{E}, \mathrm{M})$ ; *let* $\mathrm{K}$ *be* E*-cowell-powered. Then for every functor* $F : \mathrm{K} \to \mathrm{K}$ *which preserves* $\mathrm{E}$ , *the category* $\mathrm{K}(F)$ *has coequalizers.*

Proof. Let $f, g : (A, \alpha) \to (B, \beta)$ be arbitrary $F$-homomorphisms.

Denote by $\Omega$ the class of all $E$-epis $t : B \to T$ in $K$ with the following property:

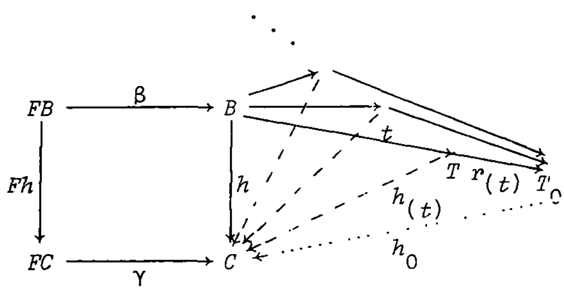
for every $F$-homomorphism $h : (B, \beta) \to (C, \gamma)$ with $h.f = h.g$ there exists $h_{(t)} : T \to C$ in $K$ such that $h = h_{(t)}.t$ .

Since $K$ is cocomplete and $E$-cowell-powered, the diagram $\Omega$ has a colimit (multiple pushout)

(1) $$r_0 = r_{(t)}.t : B \to T_0 \quad \left(r_{(t)} : T \to T_0 \text{ for each } t \in \Omega\right) .$$

Each $t \in \Omega$ is in $E$ , hence (by I.2 (vii)) each $r_{(t)}$ is in $E$ ; thus $t_0 \in E$ and $Ft_0 \in E$ .

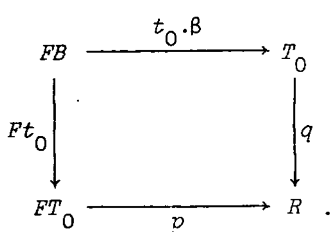
Fix a homomorphism $h : (B, \beta) \to (C, \gamma)$ with $h.f = h.g$ . Then we have a bound of $\Omega$ : $h_{(t)} : T \to C$ $(t \in \Omega)$ . Thus there exists




a unique $h_0 : T_0 \to C$ with

(2) $$h_0.r_{(t)} = h_{(t)} \quad (t \in \Omega) \quad \text{and} \quad h_0.t_0 = h .$$

Consider the pushout of $Ft_0$ and $t_0.\beta$ :




CLAIM. $q$ is an isomorphism. It suffices to show that $q.t_0 \in \Omega$ ; then by (1), $t_0 = r_{(q.t_0)}.q.t_0$ , which implies $1 = r_{(q.t_0)}.q$ since $t_0$

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is epi) and so $q$ is a split mono as well as an $E$-epi (opposite $Ft_0 \in E$ in a pushout, see I.2 (vi)) - thus, $q$ is an isomorphism. To show $q.t_0 \in \Omega$ we first remark that, since $q \in E$ and $t_0 \in E$ we have $q.t_0 \in E$. Secondly, consider any homomorphism $h : (B, \beta) \to (C, \gamma)$ with $h.f = h.g$ : we have $h_0.t_0 = h$ by (2) and $h.\beta = \gamma.Fh$, hence

$$h_0.(t_0.\beta) = \gamma.Fh = (\gamma.Fh_0).Ft_0 .$$

This implies that the pair $h_0$; $(\gamma.Fh_0)$ factorizes through $q$; $p$ above; that is, there is a unique morphism, denoted by $h_{(q.t_0)}$ from $R$ to $C$, with $h_0 = h_{(q.t_0)}.q$ and $\gamma.Fh_0 = h_{(q.t_0)}.p$. The first implies $h = h_{(q.t_0)}.(q.t_0)$, by (2). Thus, $q.t_0 \in \Omega$ and $q$ is an isomorphism.

Let us show that the $F$-homomorphism $t_0 : (B, \beta) \to \left(T_0, q^{-1}.p\right)$ is a coequalizer of $f$ and $g$ in $K(F)$.

Firstly, $t_0.f = t_0.g$ : indeed, consider the coequalizer $c$ of $f$ and $g$ in $K$; $c \in E$ by I.2 (v), and clearly $c \in \Omega$. Hence $t_0 = r_{(c)}.c$, which proves $t_0.f = t_0.g$.

Secondly, for every homomorphism $h : (B, \beta) \to (C, \gamma)$ with $h.f = h.g$ we have $h_0 : T_0 \to C$ with $h = h_0.t_0$, by (2). This $h_0$ is unique, because $t_0$ is epi. To conclude the proof we only have to show that $h_0$ is a homomorphism; that is, that $h_0.(q^{-1}.p) = \gamma.Fh_0$. We use (2) and the fact that $Ft_0 \in E$ is epi, and that $p.Ft_0 = q.t_0.\beta$ (see the pushout above):

$$\begin{aligned}\left[h_0.(q^{-1}.p)\right].Ft_0 &= h_0.(q^{-1}.q).t_0.\beta \\ &= h.\beta \\ &= \gamma.Fh \\ &= [\gamma.Fh_0].Ft_0 .\end{aligned}$$

II.5. **COROLLARY.** *Let* $K$ *and* $(E, M)$ *be as in* II.4. *Then for every*

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*algebraic theory* $\Pi$ *which preserves* $E$ *, the category* $K^{\Pi}$ *is cocomplete.*

Proof. By II.4, the category $K(T)$ has coequalizers, hence (by II.3) so does $K^{\Pi}$. By I.1 this implies the cocompleteness.

COROLLARY. *Let* $K$ *be a cowell-powered, cocomplete category. Then for every algebraic theory* $\Pi$ *preserving epis, also* $K^{\Pi}$ *is cocomplete.*

Proof. It is proved in [5] (the dual to 34.1) that $K$ has a factorization system $(E, M)$ with $E$ equal to all epis, $M$ equal to all extremal monos.

II.6. The latter corollary is proved in [1] in the same way as in the present paper. The first corollary was first formulated by Reiterman. See [6], where a completely different method is used (related to that used in [2] to prove Theorem II.4 above).

## III. ... but not always!

III.1. We denote by $\mathcal{G}ra$ the category of graphs and compatible mappings. A graph is a pair $A = \langle A, K \rangle$ consisting of a set $A$ and a subset $K$ of $A \times A$. A compatible mapping $f : \langle A, K \rangle \to \langle B, L \rangle$ is a mapping $f : A \to B$ for which $(x, y) \in K$ implies $\big(f(x), f(y)\big) \in L$.

$\mathcal{G}ra$ is a complete and cocomplete concrete category, with underlying functor $\mathcal{G}ra \to \mathcal{S}et$ creating all limits and colimits; it is also a well-powered and cowell-powered category and is, in one word, decent.

III.2. We shall define an input process $F$ in $\mathcal{G}ra$ such that the category $\mathcal{G}ra(F)$ of $F$-algebras is not cocomplete. Before doing this, we shall make a simple observation about $P$-algebras, where $P : \mathcal{S}et \to \mathcal{S}et$ is the power-set functor (sending a set $X$ to the power set $PX = 2^X$ and a mapping $f : X \to Y$ to the mapping

$$Pf : A \mapsto \{f(a);\ a \in A\}\ ).$$

We recall that an object $O$ of a category is *weak initial* if for every other object $X$ there exists at least one morphism from $O$ to $X$.

LEMMA. *The category* $\mathcal{S}et(P)$ *of* $P$*-algebras has no weak initial object.*

Proof. It is easy to see that $\mathcal{S}et(P)$ is a complete category (with

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limits created by the forgetful functor $\mathcal{S}et(P) \to \mathcal{S}et$ ). Thus the existence of a weak initial object would imply the existence of an initial object; see [8].

Now let $(A, \alpha)$ be an initial $P$-algebra. Barr proves in [4] that $\alpha$ is then an isomorphism. But there exists no isomorphism from a power set $PA$ to $A$, of course; a contradiction.

III.3. We start by defining a functor $F : \mathcal{G}ra \to \mathcal{G}ra$. First, for every graph $\mathsf{A} = \langle A, K\rangle$, define a set

$$\mathsf{A}^{(3)} = \{(x, y, z) \in A \times A \times A;\ (x, y) \in K \text{ and } (y, z) \in K\} .$$

Given a compatible mapping $f : \mathsf{A} \to \mathsf{B}$, define a mapping $f^{(3)} : \mathsf{A}^{(3)} \to \mathsf{B}^{(3)}$ by

$$f^{(3)} : (x, y, z) \mapsto (f(x), f(y), f(z)) .$$

Now define $F$ as follows: for each graph $\mathsf{A}$ put

$$F\mathsf{A} = \left\langle P\mathsf{A}^{(3)}, M_{\mathsf{A}}\right\rangle \text{ where } (X, Y) \in M \text{ iff } X = \emptyset \text{ and } Y \neq \emptyset \quad \left(X, Y \subset \mathsf{A}^{(3)}\right);$$

for each compatible map $f : \mathsf{A} \to \mathsf{B}$ put

$$Ff = Pf^{(3)} .$$

Clearly, $Pf^{(3)} : F\mathsf{A} \to F\mathsf{B}$ is compatible and $F$ is a correctly defined functor.

III.4. LEMMA. *$F$ is an input process.*

Proof. For every graph $\mathsf{A}$ define a new graph

$$\mathsf{A}^{\#} = \mathsf{A} \vee F\mathsf{A}$$

and notice that $(F\mathsf{A})^{(3)} = \emptyset$; hence $F\mathsf{A}^{\#} = F\mathsf{A}$. Denote by $\delta^{\mathsf{A}} : \mathsf{A} \to \mathsf{A}^{\#}$, $\varphi^{\mathsf{A}} : F\mathsf{A}^{\#} = F\mathsf{A} \to \mathsf{A}^{\#}$ the canonical injections. Then $(\mathsf{A}^{\#}, \varphi^{\mathsf{A}})$ is a free $F$-algebra generated by $\mathsf{A}$ with universal morphism $\delta^{\mathsf{A}}$.

Indeed, let $(\mathsf{B}, \beta)$ be an $F$-algebra and let $f : \mathsf{A} \to \mathsf{B}$ be a morphism (that is a compatible mapping). Define $f^{\#} : \mathsf{A}^{\#} \to \mathsf{B}$ by

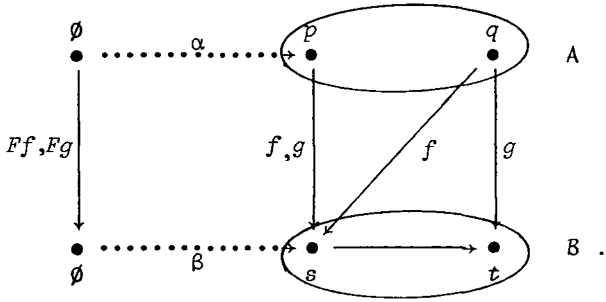
$$f^{\#}.\varphi^{\mathsf{A}} = \beta.Ff , \tag{3}$$

(4) $$f^{\#}.s^{A} = f\ .$$

Since $(FA)^{(3)} = \emptyset$, clearly $Ff^{\#} = Ff$, and so (3) means that $f^{\#} : (A^{\#}, \varphi^{A}) \to (B, \beta)$ is an $F$-homomorphism; by (4), $f^{\#}$ extends $f$. The uniqueness of $f^{\#}$ follows from the fact that (3) and (4) are actually necessary.

III.5. We define a pair $f, g : (A, \alpha) \to (B, \beta)$ of $F$-homomorphisms of which we shall prove that they do not have a coequalizer in $\mathcal{G}ra(F)$.

Let $A = \langle \{p, q\}, \emptyset \rangle$ and $B = \langle \{s, t\}, \{(s, t)\} \rangle$.




Clearly $A^{(3)} = B^{(3)} = \emptyset$, hence $FA = FB = \langle \{\emptyset\}, \emptyset \rangle$. Define $\alpha : FA \to A$ by $\alpha(\emptyset) = p$; $\beta : FB \to B$ by $\beta(\emptyset) = s$.

Finally, define $f, g : \{p, q\} \to \{s, t\}$ by

$$f(p) = g(p) = f(q) = s \quad \text{and} \quad g(q) = t\ .$$

Clearly, $f$ is a homomorphism with $f.\alpha = \beta.Ff : \emptyset \mapsto s$, analogously $g$.

III.6. Assuming that $f, g$ have a coequalizer $c : (B, \beta) \to (C, \gamma)$ in $\mathcal{G}ra(F)$, we shall find a weak initial object in $\mathcal{S}et(P)$ – a contradiction.

We have $C = \langle C, K \rangle$. Put $\bar{s} = c(s)$ $(= c(t)$, because $c.f = c.g)$. Since $c : B \to C$ is compatible, clearly $(\bar{s}, \bar{s}) \in K$.

Put

$$C_0 = \{x \in C;\ (\bar{s}, x) \in K\}\ .$$

For every subset $X \subset C_0$ put

$$\hat{X} = \{(\bar{s}, \bar{s}, x);\ x \in X\} \in PC^{(3)}\ .$$

We have $\gamma(\hat{X}) \in C$ – let us show that, in fact, $\gamma(\hat{X}) \in C_0$ . If $X = \emptyset$ , then $\hat{X} = \emptyset$ and $\gamma(\emptyset) = \bar{s} \in C_0$, because $\gamma . Fc = c . \beta$ and $Fc(\emptyset) = \emptyset$ ; thus

$$\gamma(\emptyset) = c(\beta(\emptyset)) = c(s) = \bar{s} .$$

If $X \neq \emptyset$, then $\hat{X} \neq \emptyset$ and so $(\emptyset, \hat{X}) \in M_C$ (see III.3). Since $\gamma : FC \to C$ is compatible, this yields $(\gamma(\emptyset), \gamma(\hat{X})) \in K$ ; that is,

$$(\bar{s}, \gamma(\hat{X})) \in K \text{ ; thus } \gamma(\hat{X}) \in C_0 .$$

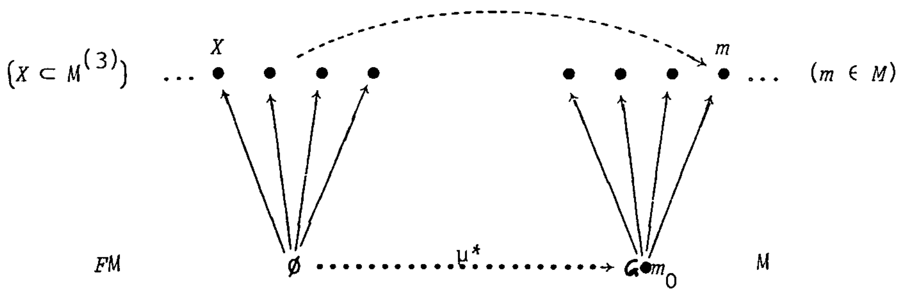
Now we define a $P$-algebra $(C_0, \hat{\gamma})$ by

$$\hat{\gamma}(X) = \gamma(\hat{X}) \quad (X \subset C_0) .$$

This $P$-algebra is weak initial.

Proof. Let $(M, \mu)$ be another $P$-algebra, that is, a set $M$ and a mapping $\mu : PM \to M$ ; put $m_0 = \mu(\emptyset)$ . Define an $F$-algebra $(M, \mu^*) : M = \langle M, \{(m_0, m) ; m \in M\} \rangle$ where $\mu^* : FM \to M$ is defined by

$$\mu^*(X) = \mu(\{m \in M; (m_0, m_0, m) \in X\}) .$$




Particularly, $\mu^*(\emptyset) = m_0$ . Thus $h : (B, \beta) \to (M, \mu^*)$ , defined by $h(s) = h(t) = m_0$ , is an $F$-homomorphism. Since $h.f = h.g$ , there exists an $F$-homomorphism $k : (C, \gamma) \to (M, \mu^*)$ such that $k.c = h$ – particularly, $k(\bar{s}) = m_0$ .

The proof will be concluded when we show that the restriction $k_0 : C_0 \to M$ of $k$ is a $P$-homomorphism; that is, that $k_0 . \hat{\gamma} = \mu . Pk_0$ . Given $X \subset C_0$ we have

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$$k_0.\hat{\gamma}(X) = k.\gamma(\hat{X}) = \mu^*.Fk(\hat{X}) .$$

Furthermore, $\hat{X} = \{(\bar{s}, \bar{s}, x);\ x \in X\}$ implies

$$Fk(\hat{X}) = \{(m_0, m_0, k(x));\ x \in X\}$$

and $k(x) = k_0(x)$ for $x \in X$ (since $X \subset C_0$); thus

$$\mu^*.Fk(\hat{X}) = \mu(\{k_0(x);\ x \in X\}) = \mu.Pk_0(X) .$$

Thus $(C_0, \hat{\gamma})$ is a weak initial $P$-algebra, in contradiction to Lemma III.2.

CONCLUSION. The free algebraic theory $\mathbb{T}$ generated by the above input process $F$ in $\mathcal{G}ra$ is such that $\mathcal{G}ra^{\mathbb{T}}$ is not complete. Explicitly, $\mathbb{T} = (T, \mu, \eta)$ with

$$TA = A \vee FA \quad (\eta^A : A \to TA \text{ and } \varphi^A : FA \to TA \text{ canonical})$$

and $\mu^A : T^2A = (A \vee FA) \vee FA \to TA$ is defined on $A$ as $\eta^A$ and on both copies of $FA$ as $\varphi^A$.

## IV. Appendix on factorizations

IV.1. **Barr's right factorization systems.** For the colimit theorem of I.3, Barr [4] uses a right factorization system, which is a pair $(E, M)$ as in I.2, except that $E$-morphisms need not be epis. More precisely, a right factorization system consists of a class $E$ of morphisms and a class $M$ of monos such that conditions (ii)-(iv) of I.2 are fulfilled.

There always exists a simple right factorization system: $E$ equals all morphisms, $M$ equals all isomorphisms. In this case, $K$ is seldom $E$-cowell-powered (as required in the colimit theorem I.3). We shall show that this is no coincidence.

LEMMA. *For each right factorization system* $(E, M)$ *and each multiple pushout*

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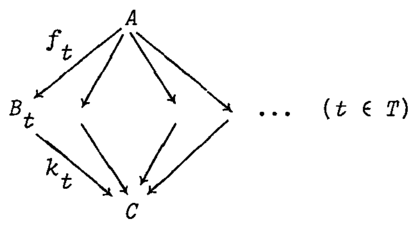
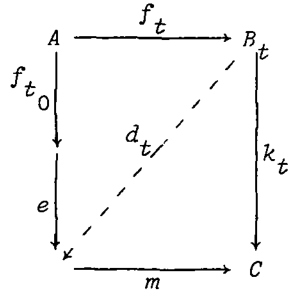



*with* $f_t \in E$ *for* $t \in T$ *, also* $k_t \in E$ *for* $t \in T$ *.*

Proof. Choose $t_0 \in T$ and let $k_{t_0} = m.e$ be an $E$-$M$-factorization. For every $t \in T$ use the diagonal fill-in:




to obtain $d_t$ with $d_t.f_t = e.f_{t_0}$ (hence $d_t$ is a bound of the pushout) and $m.d_t = k_t$. There exists a unique $d$ with $d_t = d.k_t$ $(t \in T)$. Then $(m.d).k_t = k_t$ $(t \in T)$; hence $m.d = 1$. Since $m \in M$, $m$ is a mono as well as a split epi - thus $m$ is an isomorphism. This shows that $k_{t_0} = m.e$ is in $E$.

PROPOSITION. *Every right factorization system* $(E, M)$ *in a cocomplete,* $E$*-cowell-powered category is a factorization system (that is, all* $E$*-morphisms are epis).*

Proof. Assume that $K$ is a cocomplete category with a right factorization system $(E, M)$. Given $f : A \to B$ in $E$ which is not epi, we shall show that $K$ is not $E$-cowell-powered. Indeed, if $K$ is $E$-cowell-powered, there exist $q_i : B \to Q_i$ $(i \in I$, $I$ a set) in $E$ such that each $E$-morphism with domain $B$ is isomorphic to some $q_i$. Choose a cardinal $\lambda$ such that $\operatorname{card} \hom(B, Q_i) < \lambda$ for each $i \in I$.

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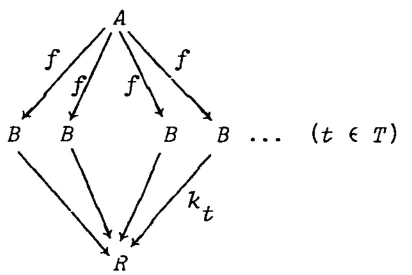
Let $\{k_t\}_{t \in T}$ be the multiple




pushout of a $T$-indexed family of copies of $f$, where $T$ is a set of power $\lambda$. By the above lemma, $k_t \in E$ for each $t \in T$. To conclude the proof it suffices to show that the $k_t$'s are pairwise distinct: then card hom$(B, R) \geq \lambda$; hence $R$ is not isomorphic to any of $Q_i$.

Since $f$ is not epi, there exist distinct morphisms $g_1, g_2 : B \to C$ with $g_1 \cdot f = g_2 \cdot f$. Consider the following bound $g_{n_t} : B \to C$ of the above pushout: for a given $t_0 \in T$, $n_{t_0} = 1$; else $n_t = 2$. There exists a unique $h : R \to C$ with $g_{n_t} = h \cdot k_t$ $(t \in T)$. Since $g_{n_{t_0}} \neq g_{n_t}$, we have $k_{t_0} \neq k_t$ for each $t_0 \neq t$. This shows that the $k_t$'s are pairwise distinct.

**IV.2. Linton's factorization functors.** Denote by $K^2$ the morphism category of $K$ and by $K^3$ the triangle category of $K$ (objects are triples $(f; g, h)$ of $K$-morphisms with $f = g \cdot h$; morphisms are triples $(p, r, q) : (f; g, h) \to (f'; g', h')$ of $K$-morphisms with $r \cdot h = h' \cdot p$ and $q \cdot g = g' \cdot q$). There is a natural forgetful functor $\gamma : K^3 \to K^2$ (sending $(f; g, h)$ to $f$).

Linton [7] uses a factorization functor, that is a functor

$$\Delta : K^2 \to K^3$$

such that

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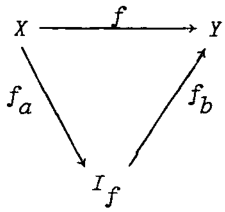
(1) $\gamma \circ \Delta = 1$ ; for $f : X \to Y$ in $K^2$ , $\Delta(f)$ is denoted by




;

(2) $f_a$ is epi, $f_b$ is mono, for each $f \in K^2$ ;

(3) $(f_a)_b$ and $(f_b)_a$ are isomorphisms, for each $f \in K^2$ .

PROPOSITION. *(A) Given a factorization functor* $\Delta$ *, the pair* $(E_\Delta, M_\Delta)$ *with*

$$E_\Delta = \{f \in K^2;\ f_b \text{ is an isomorphism}\} ,$$

$$M_\Delta = \{f \in K^2;\ f_a \text{ is an isomorphism}\} ,$$

*is a factorization system in* $K$ .

*(B) If* $\Delta, \Delta'$ *are naturally equivalent factorization functors then* $E_\Delta = E_{\Delta'}$ *, and* $M_\Delta = M_{\Delta'}$ .

*(C) For every factorization system* $(E, M)$ *there exists a factorization functor* $\Delta$ *, unique up to natural equivalence, with* $(E, M) = (E_\Delta, M_\Delta)$ .

Proof. *(A)*. Conditions (i), (iii) in I.2 are evident. Condition (iv) follows from the fact that $(p, q) : e \to m$ is a morphism in $K^2$ ; hence we have $\Delta(p, q) : \Delta(e) \to \Delta(m)$ in $K^3$ .

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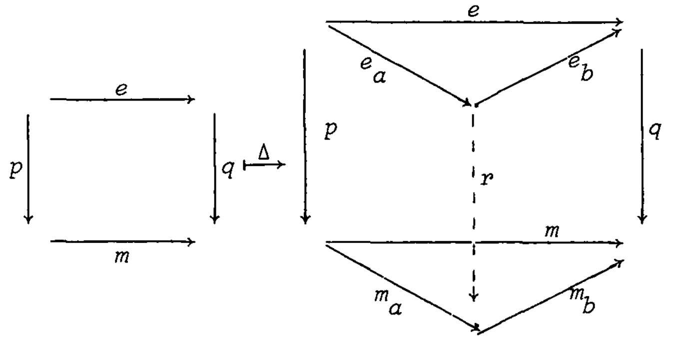
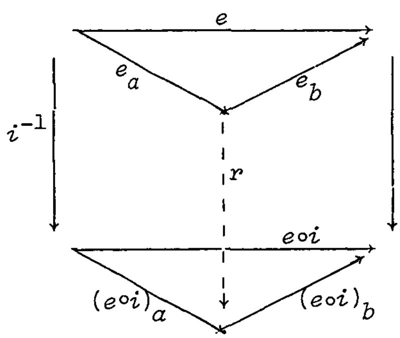



Since $\gamma \circ \Delta = 1$ , clearly $\Delta(p, q) = (p, r, q)$ for some morphism $r : I_e \to I_m$ , making the above diagram commute. The diagonal morphism is then

$$d = m_a^{-1}.r.e_b^{-1} .$$

Finally, to verify condition (ii) it suffices to show that $E$ and $M$ are closed to composition with isomorphisms; see [5], 33.3. Let us verify, for example, that $e \in E$ implies $e \circ i \in E$ whenever $i$ is an isomorphism; the rest is analogous. Consider $(i^{-1}, 1) : e \to e.i$ in $K^2$ ; we have $\Delta(i^{-1}, 1)$ of the form $(i^{-1}, r, 1) : \Delta(e) \to \Delta(e \circ i)$ in $K^3$ :




Since $(e \circ i)_b \circ r \circ e_b^{-1} = 1$ , we see that $(e \circ i)_b$ is a split epi as well as a mono by (2) above . Hence $(e \circ i)_b$ is an isomorphism; thus $e \circ i \in E$ .

*(B)* follows easily from the fact that both $E$ and $M$ are closed to

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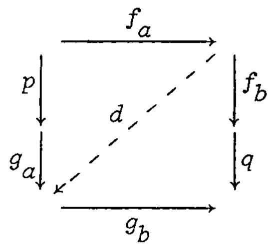
composition with isomorphisms.

*(C)*. For every morphism $f$ choose a fixed factorization $f = f_b \cdot f_a$ with $f_a \in E$, $f_b \in M$. Put $\Delta(f) = (f;\ f_b,\ f_a)$. Given a morphism $(p,\ q) : f \to g$ in $K^2$,




,

use the diagonal fill-in on the square above to find $d$ and define

$$\Delta(p,\ q) = (p,\ d,\ q) : \Delta(f) \to \Delta(g) .$$

This gives rise to a factorization functor $\Delta : K^2 \to K^3$ with $E = E_\Delta$ and $M = M_\Delta$. Uniqueness follows from *(B)*.

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