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EXISTENCE OF SOLUTIONS OF EXTREMAL PROBLEMS IN H^1

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An essentially bounded function on the unit circle gives a continuous linear functional on the Hardy space H^1. In this paper we study when there exists at least one function which attains its norm. We apply the results to an interpolation problem, Hankel operators and a characterization of exposed points of the closed unit ball of H^1.

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1. Introduction

Let H^p be the usual Hardy spaces on the unit circle T for $p \geqq 1$. If $\phi \in L^\infty$, we denote by T_ϕ the functional defined on H^1 by

$$T_\phi(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\phi(e^{i\theta})\, d\theta/2\pi.$$

Let S_ϕ be the set of functions in H^1 which satisfy $T_\phi(f) = \|T_\phi\|$ and $\|f\|_1 \leqq 1$. We define $\rho(\phi)$ to be the set of all complex numbers s for which $S_{\phi - s}$ is nonempty. If $\phi \in C$, then $T_{\phi - s}$ is weak-* continuous on H^1 for any $s \in \mathbb{C}$ and hence $S_{\phi - s}$ is nonempty, that is, $\rho(\phi) = \mathbb{C}$ where C denotes the space of continuous functions on the unit circle and $\mathbb{C}$ is the set of all complex numbers. S_ϕ can be empty for some $\phi \in L^\infty$ and hence $\rho(\phi) \neq \mathbb{C}$. Many mathematicians have studied the structure of S_ϕ when S_ϕ is nonempty (see [**1**], [**2**, Chapter 8], [**3**, Chapter IV], [**4**], [**9**] and [**10**]). Rogosinski and Shapiro, and Caughran gave the examples of ϕ with $0 \notin \rho(\phi)$ (see [**2**, Chapter 8]). However $\rho(\phi)$ has not been studied systematically. In this paper we describe $\rho(\phi)$ in general and apply our results to concrete ϕ.

In Section 2, we show that $\rho(\phi) = \mathbb{C}$ if $\|\phi + H^\infty\| \neq \|\phi + H^\infty + C\|$. In Section 3, we prove that $\rho(\phi) \supset \mathbb{C} \backslash E(\phi)$ where $E(\phi) = \{f(0) : \|\phi - f\|_\infty = \|\phi + H^\infty\|\}$. In Section 4, using a well known theorem of Adamyan, Arov and Krein (cf. [**3**, Chapter IV, Theorem 5. 3]) it is shown that $\rho(\phi) \subset \mathbb{C} \backslash E(\phi)^0$ if $\rho(\phi) \neq \mathbb{C}$. In Section 5, $E(\phi)$ is described, in fact, it is a closed disc. For special ϕ, an explicit description is given. In Sections 6 and 7 we consider $\rho(\phi)$ in case ϕ is a quotient of two inner functions. In Section 8 we give

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applications to a minimal interpolation problems, Hankel operators and a characterization of exposed points of ball (H^1), the closed unit ball of H^1.

2. $\rho(\phi)=\mathbb{C}$ if $\|\phi+H^\infty\|\neq\|\phi+H^\infty+C\|$

We denote the maximal ideal space of L^∞ by X and the Gelfand transform of the function ϕ in L^∞ by $\hat{\phi}$. Then L^∞ is isometrically isomorphic to the algebra $C(X)$ of all continuous functions on X, that is, $L^\infty\cong\hat{L}^\infty=C(X)$. Hence $(L^\infty)^*\cong M(X)$, where $M(X)$ is the set of all complex regular Borel measures on X. For each $\hat{\phi}\in C(X)$, if we assign the number $\int_{-\pi}^{\pi}\phi(e^{i\theta})\,d\theta/2\pi$ to it then there exists a probability measure m on X such that $\int_{-\pi}^{\pi}\phi\,d\theta/2\pi=\int_X\hat{\phi}\,dm$ for all ϕ. Let M^s be the set of all complex singular measures with respect to m, then $M(X)=L^1(m)\oplus M^s$. L^1 is canonically embedded into the bidual $(L^\infty)^*$ and $L^1\cong L^1(m)$. If we set

$$\mathscr{H}=(\hat{z}\hat{H}^\infty)^\perp\cap M(X)$$

$$=\{v\in M(X):\int_X\hat{f}\,dv=0\quad\text{for all}\quad f\in zH^\infty\},$$

then $\mathscr{H}\cong(zH^\infty)^\perp\cap(L^\infty)^*=(L^\infty/zH^\infty)^*=(H^1)^{**}$. By the F. and M. Riesz theorem for H^∞ (cf. [5, p. 186], $\mathscr{H}=\mathscr{H}\cap L^1(m)\oplus\mathscr{H}\cap M^s$. H^1 is canonically embedded into the bidual $(H^1)^{**}$ and $H^1\cong\mathscr{H}\cap L^1(m)$.

If $\phi\in L^\infty$, we denote by $\mathscr{T}_\phi$ the functional defined on $\mathscr{H}$ by

$$\mathscr{T}_\phi(v)=\int_X\hat{\phi}\,dv.$$

The norm of $\mathscr{T}_\phi$ is $\|\mathscr{T}_\phi\|=\sup\{|\mathscr{T}_\phi(v)|:v\in\mathscr{S}\}$ and let $\mathscr{S}_\phi$ denote the set of all $v\in\mathscr{S}$ for which $\mathscr{T}_\phi(v)=\|\mathscr{T}_\phi\|$, where $\mathscr{S}$ is a unit ball of $\mathscr{H}$. Set $\mathscr{S}_\phi^a=\mathscr{S}_\phi\cap L^1(m)$ and $\mathscr{S}_\phi^s=\mathscr{S}_\phi\cap M^s$, then $\mathscr{S}_\phi^a\cong S_\phi$. Since $\mathscr{H}\cong(L^\infty/zH^\infty)^*$, $\mathscr{S}_\phi$ is not empty and $\|T_\phi\|=\|\phi+zH^\infty\|=\|\mathscr{T}_\phi\|$.

Lemma 1. *If $\phi\in L^\infty$, then*

$$\max\{|\mathscr{T}_\phi(v)|:v\in\mathscr{S}\cap M^s\}=\|\phi+H^\infty+C\|$$

Proof. If $v\in\mathscr{S}\cap M^s$ then the v annihilates $\hat{C}$ by the F. and M. Riesz theorem for H^∞ (cf. [5, p. 186]) and so $\sup|\mathscr{T}_\phi(v)|=\|\phi+H^\infty+C\|$. If $v_n\in S\cap M^2$ and $|\mathscr{T}_\phi(v_n)|\to\sup|\mathscr{T}_\phi(v)|$ as $n\to\infty$, there exists $v_\infty\in\mathscr{S}$ such that $|\mathscr{T}_\phi(v_\infty)|=\sup|\mathscr{T}_\phi(v)|$ and $v_{nj}\to v_\infty$ in the weak-* topology of $\mathscr{H}$, where $\{v_{nj}\}$ is a subsequence of $\{v_n\}$. Since v_{nj} annihilates $\hat{C}$, v_∞ annihilates $\hat{C}$, too, and so $v_\infty\in\mathscr{S}\cap M^s$.

Proposition 1. *Let $\phi\in L^\infty$. Then*

(1) *$\mathscr{S}_\phi$ is nonempty;*

(2) $\mathscr{S}_\phi = \{\gamma\mathscr{S}^a_\phi + (1-\gamma)\mathscr{S}^s_\phi : 0 \leqq \gamma \leqq 1\}$;

(3) $\mathscr{S}_\phi = \mathscr{S}^a_\phi$ *if and only if* $\|\phi + zH^\infty\| \gneqq \|\phi + H^\infty + C\|$.

Proof. (1) was proved already. (2) It is clear that $\mathscr{S}_\phi \supseteq \{\gamma\mathscr{S}^a_\phi + (1-\gamma)\mathscr{S}^s_\phi\}$. If $v \in \mathscr{S}_\phi$, we can write $v = k\,dm + v^s$ for $k \in \mathscr{H} \cap L^1(m)$ and $v^s \in \mathscr{H} \cap M^s$ and then $\log|k| \in L^1(m)$. For $\mathscr{H} = \mathscr{H} \cap L^1(m) \oplus \mathscr{H} \cap M^s$ and $\mathscr{H} \cap L^1(m) \cong H^1$. We can show that $\hat{\psi}v = |v|$ a.e. $|v|$ for the extremal kernel ψ of ϕ. Hence $\hat{\psi}k = |k|$ a.e. m and $\hat{\psi}v^s = |v^s|$ a.e. $|v^s|$. Since $k \in \mathscr{H} \cap L^1(m)$ and $v^s \in \mathscr{H} \cap M^s$, $k/\|k\|_1$ belongs to $\mathscr{S}^a_\phi$ and $v^s/\|v^s\|$ belongs to $\mathscr{S}^s_\phi$ and $\|k\|_1 + \|v^s\| = \|v\| = 1$. Thus $v \in \{\gamma\mathscr{S}^a_\phi + (1-\gamma)\mathscr{S}^s_\phi : 0 \leqq \gamma \leqq 1\}$.

(3) By (1), $\mathscr{S}^s_\phi$ is empty if and only if $\mathscr{S}_\phi = \mathscr{S}^a_\phi$. This and Lemma 1 imply (3).

It is interesting to find the condition on ϕ which implies that $\mathscr{S}_\phi = \mathscr{S}^s_\phi$. For $\mathscr{S}_\phi = \mathscr{S}^s_\phi$ if and only if $\mathscr{S}^a_\phi$ is empty, by Proposition 1. The following is the first result about $\rho(\phi)$.

Proposition 2. *Let* $\phi \in L^\infty$. *Then the following* (1) *and* (2) *are valid.*

(1) *If* $\|\phi + zH^\infty\| \gneqq \|\phi + H^\infty + C\|$ *then* $\rho(\phi) \ni 0$.

(2) *If* $\|\phi + H^\infty\| \gneqq \|\phi + H^\infty + C\|$ *then* $\rho(\phi) = \mathbb{C}$.

Proof. (1) is clear by (3) of Proposition 1 because $\mathscr{S}^a_\phi \cong \mathscr{S}_a$. (2) For any $s \in \mathbb{C}$, $\|\phi - s + zH^\infty\| \geqq \|\phi + H^\infty\| \gneqq \|\phi + H^\infty + C\|$ and hence (1) implies that $s \in \rho(\phi)$.

Proposition 2 is well known and it implies that if $\phi \in H^\infty + C$ then $\rho(\phi) = \mathbb{C}$ (see **[1]**).

3. $\rho(\phi) \supset \mathbb{C} \backslash E(\phi)$

Recall that $\rho(\phi)$ and $E(\phi)$ were defined in the Introduction.

Lemma 2. *If* $\phi \in L^\infty$, *then for any* $f \in H^\infty$ *and any* $a \in \mathbb{C}$

$$\rho(a\phi + f) = f(0) + a\rho(\phi).$$

Proof. $S_{a\phi+f} = S_{a\phi+f(0)}$, and when $a \neq 0$, $s \in \rho(a\phi + f(0))$ if and only if $(s - f(0))/a \in \rho(\phi)$. This implies the lemma.

Theorem 3. *Let* $\phi \in L^\infty$. *Then the following* (1)–(3) *are valid.*

(1) $\rho(\phi) \supseteq \int \phi\, d\theta/2\pi + \{s \in \mathbb{C} : |s| > \|\phi + H^\infty\|\}$.

(2) $\rho(\phi) \supset \mathbb{C} \backslash E(\phi)$.

(3) *If* $E(\phi)$ *is a single point* s *then* $\rho(\phi) = \mathbb{C} \backslash \{s\}$ *or* $\rho(\phi) = \mathbb{C}$.

Proof. (1) If $|s| \gneqq |\int \phi\, d\theta/2\pi| + \|\phi + H^\infty\|$ then

$$\|\phi - s + zH^\infty\| \geqq \left|\int \phi - s\, d\theta/2\pi\right| \geqq |s| - \left|\int \phi\, d\theta/2\pi\right|$$

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$$\gneqq \|\phi + H^\infty\|$$

and hence by (1) of Proposition 2 $0 \in \rho(\phi - s)$. Thus

$$\rho(\phi) \supseteq \{s \in \mathbb{C}: |s| > |\int \phi \, d\theta/2\pi| + \|\phi + H^\infty\|$$

and hence

$$\rho(\phi - \int \phi \, d\theta/2\pi) \supseteq \{s \in \mathbb{C}: |s| > \|\phi + H^\infty\|\}$$

because $\int (\phi - \int \phi \, d\theta/2\pi) \, d\theta/2\pi = 0$. Now Lemma 2 implies (1).

(2) If $s \in \mathbb{C} \backslash E(\phi)$ then there exists $g \in H^\infty$ such that $\|\phi - s + zH^\infty\| = \|\phi - s + zg\|_\infty$ and

$$\|\phi - s + zg\|_\infty \gneqq \|\phi + H^\infty\|.$$

By (1) of Proposition 2, $s \in \rho(\phi)$ and hence $\rho(\phi) \supset \mathbb{C} \backslash E(\phi)$. (3) is clear by (2).

(2) of Theorem 3 is essential in this paper. The following theorem, which is its corollary, is a little surprising. For if $\rho(\phi) \neq \mathbb{C}$ then for any $s \in \rho(\phi)$, $S_{\phi - s}$ consists of one element.

Theorem 4. *If $\phi \in L^\infty$ and S_ϕ contains at least two functions then $\rho(\phi) = \mathbb{C}$.*

Proof. Since $0 \in \rho(\phi)$, by (3) of Theorem 3, it is sufficient to show that $E(\phi)$ is a single point 0. Suppose $f \in H^\infty$ and $\|\phi + f\|_\infty = \|\phi + H^\infty\|$. We will show that if $\|\phi + zH^\infty\| = \|\phi\|_\infty$ then $f = 0$ a.e.. By hypothesis and Theorem 9 in [1], $S_\phi \ni zh$ for some $h \in H^1$. Therefore $\|T_\phi\| = \|T_{z\phi}\|$ and hence $\|\phi + zH^\infty\| = \|\phi + H^\infty\|$. Since $S_\phi \ni zh$, $S_{z\phi}$ is nonempty and hence there exists a unique $g \in H^\infty$ such that

$$\|z\phi + zg\|_\infty = \|z\phi + zH^\infty\| = \|\phi + H^\infty\|$$
$$= \|\phi + zH^\infty\| = \|\phi\|_\infty = \|z\phi\|_\infty$$

and hence $g = 0$ a.e.. Now $\|z\phi + zf\|_\infty = \|z\phi + zH^\infty\|$ and hence $f = 0$ a.e.. If $\|\phi + zH^\infty\| \neq \|\phi\|_\infty$ then by Theorem 8.1 in [2] there exists $\psi \in L^\infty$ such that

$$\|\psi + zH^\infty\| = \|\psi\|_\infty \quad \text{and} \quad \psi = \phi + zk$$

for some nonzero $k \in H^\infty$. By Lemma 2, $E(\psi) = E(\phi)$ and hence from what was shown above $E(\phi) = \{0\}$ follows.

The following lemma due to P. Koosis (cf. [3, Chapter IV, Lemma 5.4]) will be used several times in this paper.

Lemma 3. *If $\phi \in L^\infty$ with $|\phi| = 1$ a.e. and there is $k \in H^\infty$, $k \not\equiv 0$, such that $\|\phi - k\|_\infty \leqq 1$, then there exists an outer function $g \in H^1$, $\|g\|_1 = 1$, such that $\phi = g/|g|$ a.e..*

Corollary 1. *Let $\phi \in L^\infty$. Then the following* (1)–(4) *are valid.*

(1) *If $\phi + H^\infty$ is an extreme point of the ball (L^∞/H^∞) then $\rho(\phi) = \mathbb{C}\backslash\{0\}$ or $\rho(\phi) = \mathbb{C}$.*

(2) *If $\bar{\phi}$ is an inner function then $\rho(\phi) = \mathbb{C}$.*

(3) *If $\phi = 2\chi_F - 1$ and $0 < d\theta(F) < 2\pi$* then *$\rho(\phi) = \mathbb{C}\backslash\{0\}$.*

(4) *If $\phi = |f|/f$ for some nonzero $f \in H^1$ with $f^{-1} \notin H^1$ then $\rho(\phi) = \mathbb{C}$.*

Proof. (1) By Exercise 17 in [3, Chapter IV], if $\phi + H^\infty$ is an extreme point then $\|\phi + f\|_\infty > 1$ for all $f \in H^\infty$ with $f \neq 0$. Hence $E(\phi) = \{0\}$. (3) of Theorem 3 implies (1). (2) If $\bar{\phi}$ is a finite Blaschke product then by (2) of Theorem 2 $\rho(\phi) = \mathbb{C}$. If ϕ is not so then S_ϕ contains at least two functions (see Lemma 2 in [10]) and hence by Theorem 4 $\rho(\phi) = \mathbb{C}$. (3) follows immediately from Example in [7, p. 198]. (4) By Lemma 3 if there exists a nonzero $g \in H^\infty$ such that $\|\phi + g\|_\infty \leqq 1$ then there exists a nonzero $h \in H^1$ and $\phi = h/|h|$. Therefore hf is nonnegative and hence constant because $H^{1/2}$ does not contain nonconstant nonnegative functions (cf. [3, Chapter II, Exercise 13]). This contradicts the fact that $f^{-1} \notin H^1$ and hence $E(\phi) = \{0\}$. (3) of Theorem 3 implies $\rho(\phi) = \mathbb{C}$ because S_ϕ is nonempty.

4. $\rho(\phi) \subset \mathbb{C}\backslash E(\phi)^0$

In Section 3 we showed that if $E(\phi)$ is a single point and $\rho(\phi) \neq \mathbb{C}$ then $\rho(\phi) = \mathbb{C}\backslash E(\phi)$. We can ask whether or not this is true for arbitrary $E(\phi)$. However we can show that if $\rho(\phi) \neq \mathbb{C}$ then $\rho(\phi) \subset \mathbb{C}\backslash E(\phi)^0$ where $E(\phi)^0$ denotes the interior of $E(\phi)$.

For any nonzero $h \in H^1$, define $Q_h \in H^\infty$ by

$$\frac{1 + Q_h(z)}{1 - Q_h(z)} = \frac{1}{2\pi} \int \frac{e^{it} + z}{e^{it} - z} |h(e^{it})| \, dt.$$

For any $\phi \in L^\infty$, put

$$K_\phi = \{f \in H^\infty : \|\phi - f\|_\infty \leqq 1\}.$$

The following Lemma 4 is Exercise 18 in [3, Chapter IV] which is essentially due to Adamyan, Arov and Krein (cf. [3, Chapter IV, Theorem 5.3]).

Lemma 4. *Let $\phi = h/|h|$ for some nonzero $h \in H^1$. h is an exposed point of the ball (H^1) if and only if*

$$K_\phi = \left\{ \frac{h(1 - Q_h)(1 - w)}{1 - Q_h w} : w \in H^\infty \quad and \quad \|w\|_\infty \leqq 1 \right\}.$$

Lemma 5. *If $\phi = h/|h|$ and h is an exposed point of the ball (H^1) then*

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$$K_\phi(0)=\{z\in\mathbb{C}:|z-h(0)|\leqq|h(0)|\}.$$

The proof is clear.

Theorem 5. *Let* $\phi\in L^\infty$. *If* $\rho(\phi)\neq\mathbb{C}$ *then* $\rho(\phi)\subset\mathbb{C}\backslash E(\phi)^0$.

Proof. We can assume that $E(\phi)$ has a nonempty interior. Moreover we may assume $\|\phi+H^\infty\|=1$. If $s\in E(\phi)^0\cap\rho(\phi)$ then by [**1**, p. 479] there exist $f\in H^1$ with $\|f\|_1=1$ and $k\in H^\infty$ such that

$$\phi-s-zk=|f|/f.$$

If k is a nonzero function or $s\neq 0$ then by Lemma 3, f^{-1} belongs to H^1. If $k=0$ a.e. and $s=0$ then $\phi=|f|/f$. If $f^{-1}\notin H^1$, this contradicts the hypothesis by (4) of Corollary 1 and hence f^{-1} belongs to H^1. Then $f^{-1}/\|f^{-1}\|$ is an exposed point of ball (H^1) (see [**9**, Proposition 5]) and hence by Lemma 5

$$E\left(\frac{|f|}{f}\right)=\{z\in\mathbb{C}:|z-f^{-1}(0)|\leqq|f^{-1}(0)|\}$$

because $\|\phi+H^\infty\|=1$. But

$$E\left(\frac{|f|}{f}\right)=E(\phi-s)=E(\phi)-s$$

and hence $E(|f|/f)$ contains 0 as an interior because $s\in E(\phi)^0$. This contradiction implies that $E(\phi)^0\cap\rho(\phi)=\emptyset$.

5. Description of $E(\phi)$

In the previous sections, we showed that

$$\mathbb{C}\backslash E(\phi)\subset\rho(\phi)\subset\mathbb{C}\backslash E(\phi)^0.$$

Therefore it will be useful to describe $E(\phi)$. These are corollaries of a powerful result of Adamyan, Arov and Krein (cf. [**3**, Chapter IV, Theorem 5.3]).

Let $\phi\in L^\infty$ and $\phi\notin H^\infty$. If $E(\phi)$ is not a single point there exists a unique outer function $F\in H^1$ with $F/|F|\in\phi/a+H^\infty$, $\|F\|_1=1$ and

$$\operatorname{Re}\int\frac{F}{|F|}d\theta/2\pi=\sup\left\{\operatorname{Re}\int\left(\frac{\phi}{a}-k\right)d\theta/2\pi:\left\|\frac{\phi}{a}-k\right\|_\infty=1\right\}$$

where $a=\|\phi+H^\infty\|$ (see [**3**, Chapter IV, Theorem 5.3]).

Proposition 6. *Let $\phi \in L^\infty$ with $\phi \notin H^\infty$ and $a = \|\phi + H^\infty\|$. If $E(\phi)$ is not a single point then for the F defined above*

$$E(\phi) = a\{z \in \mathbb{C}: |z - z_0| \leqq |F(0)|\}$$

where

$$z_0 = \int \frac{\phi}{a} d\theta/2\pi - \int \frac{F}{|F|} d\theta/2\pi + F(0).$$

In particular $E(\phi)$ is a closed disc.

Proof. Since $\|\phi/a + H^\infty\| = 1$, $E(\phi/a) = K_{\phi/a}(0)$. By Theorem 5.3 in [3, Chapter IV],

$$K_{\phi/a} = \left\{ f = \frac{\phi}{a} - \frac{F}{|F|} + \frac{F(1-Q_F)(1-w)}{1-Q_F w}: w \in H^\infty, \|w\|_\infty \leqq 1 \right\}.$$

Hence

$$E(\phi/a) = \left\{ \int \frac{\phi}{a} d\theta/2\pi - \int \frac{F}{|F|} d\theta/2\pi + F(0)(1-w(0)): w \in H^\infty, \|w\|_\infty \leqq 1 \right\}.$$

This implies the proposition.

We will concentrate on unimodular functions, that is, $\phi \in L^\infty$ and $|\phi| = 1$ a.e.. Then we can describe $E(\phi)$ more exactly than Proposition 6.

Lemma 6. *Let $\phi = f/|f|$ for some nonzero $f \in H^1$. Then*

$$K_\phi = \left\{ \frac{g(1-Q_g)(1-w)}{1-Q_g w}: w \in H^\infty, \|w\|_\infty \leqq 1 \quad \textit{and} \quad g \in S_{\bar{\phi}} \right\}.$$

Proof. By Lemma 5.5 in [3, Chapter IV],

$$K_\phi \supseteq \left\{ \frac{g(1-Q_g)(1-w)}{1-Q_g w}: w \in H^\infty, \|w\|_\infty \leqq 1 \quad \text{and} \quad g \in S_{\bar{\phi}} \right\}.$$

For the reverse inclusion, if $\|\phi - k\|_\infty \leqq 1$, set $\alpha = \arg \bar{\phi} k$ and $\psi = e^{\tilde{\alpha} - i\alpha}$; then $\psi k \in H^1$ and

$$\phi = g/|g| \quad \text{and} \quad g = \psi k / \|\psi k\|_1$$

(see [3, Chapter IV, Lemma 5.4]). This implies $g \in S_{\bar{\phi}}$ and by the proof of Theorem 5.3 in [3, Chapter IV],

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$$k=\frac{g(1-Q_g)(1-w)}{1-Q_g w}$$

for some $w\in H^\infty$.

Proposition 7. *Let $\phi\in L^\infty$ and $|\phi|=1$ a.e..*

(1) If ϕ is not of the form $\phi=f/|f|$ for some nonzero $f\in H^1$, then $E(\phi)=\{0\}$.

(2) If $\phi=f/|f|$ for some nonzero $f\in H^1$ with $\|f\|_1=1$, then

$$E(\phi)\subseteq\{z\in\mathbb{C}:|z-g(0)|\leqq|g(0)|,\ g\in S_{\bar\phi}\}.$$

If $\|\phi+H^\infty\|=1$ then

$$E(\phi)=\{z\in\mathbb{C}:|z-g(0)|\leqq|g(0)|, g\in S_{\bar\phi}\}$$

and hence $E(\phi)$ is not a single point.

Proof. (1) From Lemma 3, $E(\phi)=\{0\}$ obviously follows. (2) Evaluate K_ϕ in Lemma 6 at $z=0$; then it contains $E(\phi)$ and if $\|\phi+H^\infty\|=1$ then it coincides with $E(\phi)$. This implies (2).

6. $\rho(\phi)$ for special symbols ϕ

Let q be an inner function and k be in H^∞. In this section for a special $\phi\in L^\infty$ such that $\phi=\bar q k$ we will study $\rho(\phi)$.

Proposition 8. *Let q be an inner function and $k\in H^\infty$. If $\phi=\bar q k$, $\|T_\phi\|=\alpha>0$ and $0\in\rho(\phi)$ then there exists an inner function q_0 such that $\bar q k-\alpha\bar q q_0$ is in zH^∞. Then $\alpha\bar q q_0$ is an extremal kernel, that is, $\|\alpha\bar q q_0+zH^\infty\|=\alpha$. In particular, $\rho(\bar q k)=\alpha\rho(\bar q k)$.*

Proof. If $f\in S_\phi$ then $f\in S_{\phi/\alpha}$ and $\|T_{\phi/\alpha}\|=1$. Hence there exists a function $g\in H^\infty$ such that $\bar q k/\alpha+zg=|f|/f$. Let $q_0=k/\alpha+zqg$; then q_0 is an inner function and $\alpha\bar q q_0=\bar q k+z\alpha g$. This implies the proposition.

For each function f in H^1, sing f denotes the set of the unit circle on which f cannot be analytically extended. Let q and q_0 be inner functions. q_0 is called the maximum multiplier of a nonzero function h in $H^2\ominus qzH^2$ if $q_0h\in H^2\ominus qzH^2$ and $q_1h\in H^2\ominus qzH^2$ for some inner function q_1 implies that $q_0\bar q_1\in H^\infty$. Since $q\bar h$ is in H^2, q_0 can be obtained as the inner part of $q\bar h$.

Theorem 9. *Let q and q_0 be inner functions, and suppose $\phi=\bar q q_0$.*

(1) $0\in\rho(\phi)$ and ϕ is an extremal kernel if and only if there exists a nonzero function f in $H^2\ominus zqH^2$ such that $\bar q_0 f$ is in H^2.

(2) *If q_0 is not the maximum multiplier of a nonzero function f in $H^2 \ominus zqH^2$ but $\bar{q}_0 f \in H^2$, then $\rho(\phi) = \mathbb{C}$.*

(3) *If $0 \in \rho(\phi)$ and ϕ is an extremal function then sing $q \supset$ sing q_0.*

(4) *If (sing q) $\cap$ $\{T \backslash$sing $q_0\}$ is nonempty then $\rho(\phi) = \mathbb{C}\backslash\{0\}$ or $\rho(\phi) = \mathbb{C}$.*

(5) *Let (sing q) $\cap$ (sing q_0) be empty. If q_0 is a finite Blaschke product then $\rho(\phi) = \mathbb{C}$ and if q_0 is not so then $\rho(\phi) = \mathbb{C}\backslash\{0\}$.*

Proof. (1) If $f \in H^2 \ominus qzH^2$ and $f = q_0 h$ for some $h \in H^2$ then $q\bar{q}_0\bar{h} \in H^2$. Hence $q\bar{q}_0|h|^2 \in H^1$ and this implies that $0 \in \rho(\phi)$ and ϕ is an extremal kernel. Conversely if $0 \in \rho(\phi)$ and ϕ is an extremal kernel then there exists an outer function h such that $\bar{z}\bar{q}q_0 h = \bar{z}\bar{h}$. Hence $q_0 h$ is orthogonal to zqH^2. $f = q_0 h$ is a desired function.

(2) If $\bar{q}_0 f \in H^2$ but q_0 is not the maximum multiplier then there exists an inner function q_1 such that $\bar{q}_1\bar{q}_0 f \in H^2$. By (1) $\bar{q}q_0 q_1$ is the extremal kernel and $S_{\bar{q}q_0q_1}$ is nonempty. Hence S_ϕ has at least two functions. For $S_\phi \supsetneqq q_1 S_{\bar{q}q_0q_1}$ because S_ϕ contains always an outer function. By Theorem 4 $\rho(\phi) = \mathbb{C}$.

(3) By (1) there exists a nonzero function $f \in H^2 \ominus qzH^2$ and $f = q_0 h$ for some $h \in H^2$. It is known that sing $f \subset$ sing q. By Lemma 4 in [7] sing $q_0 \subset$ sing q.

(4) If there exists a nonzero function $k \in H^\infty$ such that $\|\phi - k\|_\infty \leqq 1$, by Lemma 3 there exists a nonzero function $f \in H^1$ such that $q\bar{q}_0 = |f|/f$. Hence (3) implies that sing $q_0 \supset$ sing q, and this contradiction implies that $k = 0$ a.e. and so $E(\phi) = \{0\}$. By (3) of Theorem 3, $\rho(\phi) = \mathbb{C}\backslash\{0\}$ or $\rho(\phi) = \mathbb{C}$.

(5) If q is a finite Blaschke product then by (2) of Theorem 2 $\rho(\phi) = \mathbb{C}$. Suppose q is not a finite Blaschke product. If q_0 is a finite Blaschke product then $S_{\bar{q}} \ni zq_0 h$ for some $h \in H^1$ by Theorem 9 in [1] and hence S_ϕ contains at least two functions. Thus by Theorem 4 $\rho(\phi) = \mathbb{C}$. If q_0 is not a finite Blaschke product then by the hypothesis $E(\phi) = \{0\}$ and $0 \notin \rho(\phi)$. $E(\phi) = \{0\}$ by the same reason as in (4), and $0 \notin \rho(\phi)$ follows from (3) because ϕ is an extremal kernel by the proof of (4). Thus by (3) of Theorem 3, $\rho(\phi) = \mathbb{C}\backslash\{0\}$.

By (3)–(5) of Theorem 9 we are interested in the case sing $q =$ sing q_0.

Corollary 2. *Put $q = \prod_{j=1}^n q_j$ where q_j is a non-constant inner function for each j. If*

$$\phi = \bar{q} \prod_{j=1}^m \frac{q_j - a_j}{1 - \bar{a}_j q_j}$$

where $|a_j| < 1$ for each j and $m \leqq n$, then $0 \in \rho(\phi)$ and ϕ is an extremal kernel. If $m < n$ then $\rho(\phi) = \mathbb{C}$.

Proof.

$$\prod_{j=1}^m (q_j - a_j) \in H^2 \ominus zqH^2$$

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because

$$\bar{q}\prod_{j=1}^{m}(q_j-a_j)=\prod_{j=m+1}^{n}\bar{q}_j\prod_{j=1}^{m}(1-a_j\bar{q}_j).$$

$$\overline{\prod_{j=1}^{m}(q_j-a_j)(1-\bar{a}_jq_j)^{-1}}\times\prod_{j=1}^{m}(q_j-a_j)$$

belongs to H^2 and hence by (1) of Theorem 9, $0\in\rho(\phi)$ and ϕ is an extremal kernel. Put

$$\phi_0=\prod_{j=1}^{m}\bar{q}_j\prod_{j=1}^{m}\frac{q_j-a_j}{1-\bar{a}_jq_j};$$

then, by what was shown just now, $0\in\rho(\phi_0)$ and ϕ_0 is an extremal kernel. If $m<n$ then $\phi=(\prod^{n}{}_{j=m+1}\bar{q}_j)\phi_0$ and hence S_ϕ contains $\{\gamma(q_{m+1}-a)(1-\bar{a}q_{m+1})f\}$ where γ is a positive constant, $f\in S_{\phi_0}$ and a is any complex number with $|a|\leqq 1$. By Theorem 4 $\rho(\phi)=\mathbb{C}$.

Corollary 3. *Let q and q_1 be nonconstant inner functions with $q\bar{q}_1\in H^\infty$. Suppose $\{z_j\}_{j=1}^{n}$ is a sequence in the unit disc such that $q_1(z_j)=a$ for some complex number a with $|a|<1$. Here n may be infinite or finite. If*

$$\phi=\bar{q}\prod_{j=1}^{n}\frac{|z_j|}{z_j}\frac{z-z_j}{1-\bar{z}_jz}$$

then $0\in\rho(\phi)$ and ϕ is an extremal kernel. If q is not a scalar multiple of q_1 then $\rho(\phi)=\mathbb{C}$.

Proof. There exists an inner function q_2 such that

$$\frac{q_1-q}{1-\bar{a}q_1}=q_2\times\prod_{j=1}^{n}\frac{|z_j|}{z_j}\frac{z-z_j}{1-\bar{z}_jz}.$$

By Corollary 2 there exists a function f in H^1 such that

$$\bar{q}\frac{q_1-a}{1-\bar{a}q_1}=\frac{|f|}{f}$$

and so

$$\bar{q}\prod_{j=1}^{n}\frac{|z_j|}{z_j}\frac{z-z_j}{1-\bar{z}_jz}=\frac{|q_2f|}{q_2f}.$$

Thus $0\in\rho(\phi)$ and ϕ is an extremal kernel. As in the proof of Corollary 2 we show $\rho(\phi)=\mathbb{C}$ if q is not a scalar multiple of q_1.

Corollary 4. *Let q be an inner function, $\phi_0(z)=\sum_{j=0}^{n}\alpha_j\bar{z}^j$ and $\phi=\phi_0(q)$.*

(1) *If $n=1$ then $\rho(\phi)=\mathbb{C}$.*

(2) *If $n=1$, $\alpha_0\neq 0$ and $\alpha_1\neq 0$ then $S_{\phi_0}=\{(1-\bar{a}z)^2/\|(1-\bar{a}z)^2\|_1\}$ for a nonzero a with $|a|\leqq 1$. However S_ϕ does not coincide with $\{(1-\bar{a}q)^2/\|(1-\bar{a}q)^2\|_1\}$ if $q(0)\neq 0$ or $q(0)\neq 1/\bar{a}+(a-1/\bar{a})\alpha$.*

(3) *For any n*

$$\rho(\phi)\supseteq\sum_{j=0}^{n}\alpha_j\overline{q(0)}^j+\{s\in\mathbb{C}:|s|>\|\phi+H^\infty+C\|\}.$$

(4) *For any n, if $q(0)=0$ then $\rho(\phi)=\mathbb{C}$.*

Proof. (1) follows from (2) of Corollary 1. (2) It is known that, if $\alpha_0\neq 0$ and $\alpha_1\neq 0$, then $\alpha_1=-a\alpha$ and

$$S_{\alpha_0+\alpha_1\bar{z}}=S_{\alpha\bar{z}(z-a/1-\bar{a}z)}=\{(1-\bar{a}z)^2/\|(1-\bar{a}z)^2\|_1\}.$$

$$(\alpha_0+\alpha_1\bar{q})-\alpha\bar{q}\frac{q-a}{1-\bar{a}q}\in qH^\infty$$

but if

$$q(0)\neq 0, \alpha_0 q+\alpha_1-\alpha\frac{q-a}{1-\bar{a}q}\notin zH^\infty.$$

Hence

$$S_{\alpha_0+\alpha_1\bar{q}}\neq S_{\alpha\bar{q}(q-a/1-\bar{a}q)}\ni(1-aq)^2/\|(1-\bar{a}q)^2\|_1.$$

(3) follows from (1) of Theorem 3. (4) By (2) of Theorem 2, S_{ϕ_0} is nonempty. Since there is $k\in zH^\infty$ such that $\phi_0=\bar{z}^{n+1}k$, by Proposition 8 there exists a finite Blaschke product b of degree at least $n+1$ such that $\psi_0=\alpha\bar{z}^{n+1}b$ is the extremal kernel of ϕ_0. Put $\psi=\psi_0(q)$ and $\phi=\phi_0(q)$; then ψ is the extremal kernel of ϕ because $q(0)=0$. By Corollary 2, $0\in\rho(\phi)$.

7. Interpolation Blaschke product

Let $\{z_n\}$ be a sequence of distinct points in the open unit disc. Put

$$\rho_n=\prod_{m;m\neq n}\left|\frac{z_n-z_m}{1-\bar{z}_m z_n}\right|.$$

Let b be a Blaschke product with zeros $\{z_n\}$. We call b an interpolation Blaschke product when $\inf \rho_n > 0$, that is, $\{z_n\}$ is a uniformly separated sequence.

Proposition 10. *Let b be a Blaschke product with zeros $\{z_n\}$ which is the union of a finite number of uniformly separated sequences and let $k \in H^\infty$. Suppose $\phi = \bar{b}k$ and $\rho_n^{-1} k(z_n) \to 0$ as $n \to \infty$. Then $\rho(\phi) = \mathbb{C}$.*

Proof. By Lemma 3 in [11], $\bar{b}k \in H^\infty + C$ and hence by (2) of Theorem 2, $\rho(\phi) = \mathbb{C}$.

Theorem 11. *Let b be an interpolation Blaschke product with zeros $\{z_i\}$ and b_0 a Blaschke product with zeros $\{a_j\}$. Put $\phi = \bar{b}b_0$. Then $0 \in \rho(\phi)$ and ϕ is an extremal kernel if and only if an infinite matrix $\{1/1 - \bar{z}_i a_j\}_{i,j=1}^\infty$ has a nontrivial kernel in ℓ^2.*

Proof. Since $\{z_i\}$ is a uniformly separated sequence, $\{1/1 - \bar{z}_i z\}_{i=1}^\infty$ is an unconditional basis in $H^2 \ominus bzH^2$ (see [6]). If an infinite matrix $\{1/1 - \bar{z}_i a_j\}_{i,j=1}^\infty$ has a nontrivial kernel, then for some $\{c_i\} \in \ell^2$

$$\sum_{i=1}^\infty c_i \frac{1}{1 - \bar{z}_i a_j} = 0 \quad j = 1, 2, \ldots.$$

Put $f(z) = \sum_{i=1}^\infty c_i (1/1 - \bar{z}_i z)$; then $f \in H^2 \ominus bzH^2$ because $\{1/1 - \bar{z}_i z\}_{i=1}^\infty$ is an unconditional basis in $H^2 \ominus bzH^2$. Now $f(a_j) = 0$ for $j = 1, 2, \ldots$. Hence $\bar{b}_0 f \in H^2$. By (1) of Theorem 9, $0 \in \rho(\phi)$ and ϕ is an extremal kernel. Conversely if $0 \in \rho(\phi)$ and ϕ is an extremal kernel then by (1) of Theorem 9, there exists $f \in H^2 \ominus bzH^2$ such that $\bar{b}_0 f \in H^2$. Since $\{1/1 - \bar{z}_i z\}_{i=1}^\infty$ is an unconditional basis in $H^2 \ominus bzH^2$,

$$f = \sum_{i=1}^\infty c_i \frac{1}{1 - \bar{z}_i z} \quad \text{and} \quad \{c_i\} \in \ell^2. \text{ Then}$$

$$\sum_{i=1}^\infty c_i \frac{1}{1 - \bar{z}_i a_j} = 0 \quad j = 1, 2, \ldots.$$

This proves the theorem.

8. Applications

Let $\{z_n\}$ be a Blaschke sequence and let a bounded sequence $\{w_n\}$ be given. If we can find an f in H^∞ such that $f(z_n) = w_n$ we may assume that $\|f\|_\infty$ is minimal. Such an f need not be unique, but K. Øyma gave a sufficient condition for uniqueness. Let $\{z_n\}$ be a uniformly separated sequence in the unit disc and assume $w_n \to 0$. Then there exist a unique f in H^∞ of minimal norm such that $f(z_n) = w_n$ for all n [12, Theorem 2]. The author [11] gave a sufficient condition for uniqueness in the case of the union of a finite number of uniformly separated sequence $\{z_n\}$, that contains the result of K. Øyma. The following theorem gives a solution on this problem in the case of a Blaschke sequence $\{z_n\}$.

Theorem 12. *Let $\{z_n: n=0,1,2,\dots\}$ be a Blaschke sequence with $z_0=0$ and $\{s,w_1,w_2,\dots\}$ a bounded sequence. Let b be a Blaschke product with zeros $\{z_1,z_2,\dots\}$. Suppose there exists a function f in H^∞ such that $f(0)=0$ and $f(z_j)=w_j$ for $j=1,2,\dots$ If $\rho(\bar{b}f)\ni -sb(0)^{-1}$ then there exists a unique g in H^∞ of minimal norm such that $g(0)=s$ and $g(z_j)=w_j$ for $j=1,2,\dots$. This function is a complex constant times an inner function and has analytic continuation across $T\backslash\overline{\{z_n\}}$.*

Proof. If $\rho(\bar{b}f)\ni -sb(0)^{-1}$ then $S_{\bar{b}f+sb(0)^{-1}}$ is nonempty and hence there exists a unique function $k\in H^\infty$ such that $\|\bar{b}f+sb(0)^{-1}+zH^\infty\|=\|\bar{b}f+sb(0)^{-1}+zk\|_\infty$. This implies that there is a unique function k such that $\|f+sb(0)^{-1}b+zbH^\infty\|=\|f+sb(0)^{-1}b+zbk\|_\infty$. Let $g=f+sb(0)^{-1}b+zbk$; then $g(0)=s$ and $g(z_j)=f(z_j)$ for $j=1,2,\dots$, and it is of minimal norm.

In the theorem above, if $\rho(\bar{b}f)=\mathbb{C}$ then $\{s,w_1 w_2,\dots\}$ for any s has always a unique minimal interpolating function. Proposition 10 shows that if b is a Blaschke product whose zeros is the union of a finite number of uniformly separated sequence $\{z_n\}$ and if $\rho_n^{-1}f(z_n)\to 0$ then $\rho(\bar{b}f)=\mathbb{C}$.

Let P be the orthogonal projection from L^2 onto H^2 and ϕ a fixed function in L^∞. The Hankel operator with symbol ϕ is the operator H_ϕ from H^2 to $(H^2)^\perp$ is defined by $H_\phi f=(1-P)(\phi f)$, $f\in H^2$. Now we will study when H_ϕ has an accessible norm, that is, $\|H_\phi\|^2$ is an eigenvalue of $H_\phi^* H_\phi$. Put $\gamma(\phi)=\{s\in\mathbb{C}: H_{\phi-s\bar{z}}$ has an accessible norm$\}$.

Theorem 13. *For any $\phi\in L^\infty$,*

$$\gamma(\phi)=\rho(z\phi).$$

Proof. For any $f\in zH^1$, $\bar{z}f\in H^1$ and $\int f(\phi-s\bar{z})\,d\theta/2\pi=\int \bar{z}f(z\phi-s)\,d\theta/2\pi$. Since $\|H_{\phi-s\bar{z}}\|=\|T_{z\phi-s}\|$, $\gamma(\phi)=\rho(z\phi)$.

Several characterizations of exposed points of the ball (H^1) are known (cf. [**10**, Theorem 3], [**4**, Theorem 8]). Now we will give two more characterizations of such functions. Recall that $K_\phi=\{k\in H^\infty: \|\phi-k\|_\infty\leqq 1\}$ for $\phi\in L^\infty$ (see Section 5).

Proposition 14. *Let $\phi=f/|f|$ for some nonzero $f\in H^1$ with $\|f\|_1=1$. Then f is an exposed point of the ball (H^1) if and only if the interior of $K_\phi(0)$ does not contain 0.*

Proof. Lemma 5 implies the part of "only if". Conversely, if f is not an exposed point then by Lemma 6

$$K_\phi(0)=\{z: |z-g(0)|\leqq|g(0)|, g\in S_{\bar{\phi}}\}.$$

By Theorem 5.2 in [**3**, Chapter IV], $\{g(0): g\in S_{\bar{\phi}}\}$ contains a disc centred at the origin and the interior of $K_\phi(0)$ contains 0.

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Theorem 15. *Let $\phi = f/|f|$ for some nonzero $f \in H^1$ with $\|f\|_1 = 1$. Suppose $\|\phi + H^\infty\| = 1$ and $\rho(\phi) \neq \mathbb{C}$. f is an exposed point of the ball (H^1) if and only if the boundary of $\rho(\phi)$ contains* 0.

Proof. Since $\|\phi + H^\infty\| = 1$, by (2) of Proposition 7

$$E(\phi) = \{z \in \mathbb{C} : |z - g(0)| \leqq |g(0)|, g \in S_{\bar{\phi}}\}.$$

Hence $E(\phi)$ is not a single point, and by Theorems 3 and 5,

$$\mathbb{C} \backslash E(\phi) \subset \rho(\phi) \subset \mathbb{C} \backslash E(\phi)^0$$

because $\rho(\phi) \neq \mathbb{C}$. The result of the theorem now follows.

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