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NON DECAY OF THE TOTAL ENERGY FOR THE WAVE EQUATION WITH THE DISSIPATIVE TERM OF SPATIAL ANISOTROPY

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Abstract. We consider the behavior of the total energy for the wave equation with the dissipative term. When the dissipative term works well uniformly in every direction, several authors obtain uniform decay estimates of the total energy. On the other hand, if the dissipative term is small enough uniformly in every direction, it is known that there exists a solution whose total energy does not decay. We examine the case that the dissipative term vanishes only in a neighborhood of a half-line. We introduce a uniform decay property, which is a natural generalization of the uniform decay estimates, and show that this property does not hold in our case. We prove this by constructing asymptotic solutions supported in the place where the dissipative term vanishes.

§1. Introduction

In the present paper we are concerned with the wave equation of the form

$$\partial_t^2 w - \Delta w + b(x)\partial_t w = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^N, \tag{1.1}$$

$$w(0,x) = w_0(x), \quad \partial_t w(0,x) = w_1(x) \quad \text{on } \mathbb{R}^N, \tag{1.2}$$

where $N \geq 1$ and $b(\,\cdot\,) \in \mathcal{B}^\infty(\mathbb{R}^N) = \{f \in C^\infty(\mathbb{R}^N)\ ;\ \sup|\partial_x^\alpha f(x)| < +\infty$ for any multi-index $\alpha\}$. As is well known, for any data $\{w_0, w_1\} \in H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ there exists a unique solution $w(t,\,\cdot\,)$ of (1.1)–(1.2) in the class

$$w(t,\,\cdot\,) \in C^0([0,\infty); H^2(\mathbb{R}^N)) \cap C^1([0,\infty); H^1(\mathbb{R}^N)) \cap C^2([0,\infty); L^2(\mathbb{R}^N))$$

satisfying

$$\|w(t)\|_E^2 + \int_0^t \int_{\mathbb{R}^N} b(x)|\partial_t w(\tau,x)|^2\,dxd\tau = \|w(0)\|_E^2 \tag{1.3}$$

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for any $t \geq 0$. Here $H^m(\mathbb{R}^N)$ denotes the Sobolev space of order m and $\|\cdot\|_E$ denotes the total energy

$$\|u(t)\|_E^2 = \frac{1}{2}\Big(\|\partial_t u(t,\,\cdot\,)\|_{L^2(\mathbb{R}^N)}^2 + \|\nabla u(t,\,\cdot\,)\|_{L^2(\mathbb{R}^N)}^2\Big).$$

If $b(x) \geq 0$, then $b(x)\partial_t w$ is dissipative and we find from (1.3) that $\|w(t)\|_E^2 \leq \|w(0)\|_E^2$ for any $t \geq 0$. Therefore a question naturally arises whether the total energy decays or not as t tends to infinity. The decay and non-decay problems have been studied by many authors, e.g., Matsumura [2], Rauch-Taylor [9], Mochizuki [3], [4], [5] and Mochizuki-Nakazawa [6], [7], Nakazawa [8] etc. These studies have clarified precisely relation between the decay property of the solutions and the decreasing condition of the coefficient $b(x)$ when the condition is uniform with respect to spatial directions. Let us explain some of the results.

Mochizuki-Nakazawa [7] considers the equation in an exterior domain Ω $(\subset \mathbb{R}^N, N \geq 3)$ with a smooth boundary star-shaped with respect to the origin $x = 0$ (where the boundary condition is the Dirichlet one). They assume that there exist positive constants R, b_0 and b_1 $(b_1 \geq b_0 > 1)$ such that

$$b_0(1 + t + |x|)^{-1} \leq b(t,x) \leq b_1 \quad \text{for } t \geq 0,\ |x| \geq R,$$

and show that the total energy decays uniformly as t tends to infinity, i.e.,

(1.4)
$$\|w(t)\|_E^2 \leq C_0(1+t)^{-\mu}\Big\{\|w_0\|_{H^1(\Omega)}^2 + \|w_1\|_{L^2(\Omega)}^2 + \|(w_0,w_1)\|_{E_\varphi(\Omega)}^2\Big\},\ \ t \geq 0$$

for positive constants C_0, μ with $1/2 < \mu \leq 1$ and $\mu < b_0/2$ and the data $\{w_0, w_1\} \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$ satisfying $\|(w_0,w_1)\|_{E_\varphi(\Omega)}^2 < \infty$, where

$$\|(w_0,w_1)\|_{E_\varphi(\Omega)}^2 = \int_\Omega (1+|x|)\big\{|w_1(x)|^2 + |\nabla w_0(x)|^2\big\}\,dx.$$

On the other hand, we can know from Nakazawa [8] that the above condition of the coefficient b is nearly best possible: If there exist a non-increasing and non-negative function $a(r) \in L^1(0,\infty)$ such that $b(x) \leq a(|x|)$ in $\mathbb{R}^N$, then the total energy of (1.1)–(1.2) does not decay in general (i.e., there exist non-trivial data such that the total energy of the corresponding solution for (1.1) does not decay), and there exists a free solution $w_0(t,\,\cdot\,)$ (which is the solution of (1.1)–(1.2) with $b(x) \equiv 0$) such that $\|w(t,\,\cdot\,) - w_0(t,\,\cdot\,)\|_E \to 0$ as t tends to infinity.

As is described above, the precise results are obtained in the case where decreasing conditions on $b(x)$ are assumed uniformly in spatial directions. However, we do not know anything when the conditions are not uniform in the directions. Our problem is as follows:

Does the total energy decay when the decay situation of the coefficient $b(x)$ is not uniform with respect to spatial direction?

We treat this decay problem by considering whether the uniform decay estimate like (1.4) holds or not. To formulate the problem, we introduce the following concept of the uniform decay which is a weak version of the estimate (1.4):

Definition 1.1. (U. D. P. = Uniform Decay Property) We say that the equations (1.1)–(1.2) have the uniform decay property if and only if for any $\varepsilon > 0$, there exists $T(\varepsilon) > 0$ independent of data $\{w_0, w_1\}$ such that the inequality

$$\|w(t)\|_E^2 \le \varepsilon\Big(\|w_0\|_{H^1(\mathbb{R}^N)}^2 + \|w_1\|_{L^2(\mathbb{R}^N)}^2 + \|(w_0, w_1)\|_{E_\varphi(\mathbb{R}^N)}^2\Big) \tag{1.5}$$

holds for any $t \ge T(\varepsilon)$ and any solutions of (1.1)–(1.2) with data $\{w_0, w_1\}$ satisfying the finite norms in the right-hand side of (1.5).

Our main result is

Theorem 1.2. *Let G be an unbounded domain defined as*

$$G = \big\{x \in \mathbb{R}^N \mid |x - x_0 - ((x - x_0)\cdot\omega)\omega| < \delta,\ \ |x - x_0| > R\big\}$$

for $x_0 \in \mathbb{R}^N$, $\omega \in S^{N-1}$, $\delta > 0$ and $R > 0$. Assume that $b(x)$ satisfies

$$\operatorname{supp} b(x) \subset \mathbb{R}^N \setminus G. \tag{1.6}$$

Then the equations (1.1)–(1.2) *do not have* (U. D. P.).

Remark 1.3. It is possible to weaken the assumption (1.6) in Theorem 1.2, i.e., if integration of $b(x)$ on any ray (half line) in G is uniformly finite, then the conclusion of the theorem is valid.

It is expected that the equations (1.1)–(1.2) have neither the uniform decay property nor the decay one if the assumption in Theorem 1.2 is satisfied. Unfortunately we cannot prove this expectation by our methods. Using asymptotic solutions, we can construct solutions to the equations (1.1)–(1.2) whose energy propagates along the direction ω and hence remaing in

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the region G. These solutions are sufficient to obtain Theorem 1.2; however, it seems difficult for us to verify the expectation using our solutions.

The content of the paper is as follows: In Section 2, we construct the asymptotic solution for (1.1) with some specified data. Using the L^∞-estimate of this solution, we derive the estimate of the total energy for the genuine solution of (1.1) in Section 3. In the final section we prove Theorem 1.2.

§2. Construction of asymptotic solutions

Let $f(x)$ be a function satisfying

$$f(\,\cdot\,) \in C_0^\infty(\mathbb{R}^N), \quad \operatorname{supp} f(x) \subset G$$

and let $k > 0$ be a parameter. We shall construct the solution of (1.1) with data $\{0, f(x)e^{ik\omega\cdot x}\}$. For this purpose, we construct the asymptotic solution $v(t, x; k)$ satisfying

$$\big(\partial_t^2 - \Delta + b(x)\partial_t\big)v(t, x; k) = O\big((t^\beta k^{-1})^n\big), \tag{2.1}$$

$$v(0, x; k) = O\big(k^{-n}\big), \tag{2.2}$$

$$\partial_t v(0, x; k) - f(x)e^{ik\omega\cdot x} = O\big(k^{-n}\big), \tag{2.3}$$

for a constant $\beta > 0$ and an integer n large enough, where $O(\mu^{-m})$ means that the supremum norm in x is estimated by the parameter μ^{-m} (as $\mu \to \infty$). Ikawa [1] explains the procedures of the construction. For precise description of them, see Section 3 of [1].

We set

$$v(t, x; k) = e^{ik\varphi^+(t,x)} \sum_{j=1}^{n} v_j^+(t, x)(ik)^{-j} - e^{ik\varphi^-(t,x)} \sum_{j=1}^{n} v_j^-(t, x)(ik)^{-j}, \tag{2.4}$$

where

$$\varphi^\pm(t, x) = \pm t + \omega \cdot x,$$

and determine $v_j^\pm(t, x)$ $(j = 1, 2, \dots, n)$ as is described later. We rewrite

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$\big(\partial_t^2 - \Delta + b(x)\partial_t\big)v$ in the following way:

$$
\begin{aligned}
(2.5)\qquad Pv(t,x;k) = e^{ik\varphi^+(t,x)}\Big\{&(ik)^0 K^+ v_1^+(t,x) \\
&+ \sum_{j=1}^{n-1}(ik)^{-j}\Big(K^+ v_{j+1}^+(t,x) + Pv_j^+(t,x)\Big) + (ik)^{-n} Pv_n^+(t,x)\Big\} \\
&- e^{ik\varphi^-(t,x)}\Big\{(ik)^0 K^- v_1^-(t,x) \\
&+ \sum_{j=1}^{n-1}(ik)^{-j}\Big(K^- v_{j+1}^-(t,x) + Pv_j^-(t,x)\Big) + (ik)^{-n} Pv_n^-(t,x)\Big\},
\end{aligned}
$$

where

$$
P = \partial_t^2 - \Delta + b(x)\partial_t, \quad K^{\pm} = \pm 2\partial_t - 2\omega\cdot\nabla \pm b(x).
$$

If we assume

$$
(2.6)\qquad \begin{cases} K^{\pm} v_1^{\pm}(t,x) = 0, \\ K^{\pm} v_j^{\pm}(t,x) = -Pv_{j-1}^{\pm}(t,x) \quad (j = 2,3,\dots,n), \end{cases}
$$

then

$$
\begin{aligned}
(2.7)\qquad Pv(t,x;k) &= g(t,x;k) \\
&\equiv (ik)^{-n}\big\{e^{ik\varphi^+(t,x)} Pv_n^+(t,x) - e^{ik\varphi^-(t,x)} Pv_n^-(t,x)\big\}.
\end{aligned}
$$

For construction of the genuine solution of (1.1) with data $\{0, f(x)e^{ik\omega\cdot x}\}$, we define the function $u(t,x;k)$ by

$$
(2.8)\qquad \begin{cases} Pu(t,x;k) = -g(t,x;k), \\ u(0,x;k) = u_0(x;k), \\ \partial_t u(0,x;k) = u_1(x;k), \end{cases}
$$

for appropriate functions u_0 and u_1. Then the function $w(t,x;k)$ defined by

$$
(2.9)\qquad w(t,x;k) = v(t,x;k) + u(t,x;k)
$$

satisfies $Pw(t,x;k) = 0$. We choose the data u_0 and u_1 as follows: Let $v_j^{\pm}(0,x;k)$ $(j = 1,2,\dots,n)$ be the solution defined by

$$\begin{cases} v_1^{+}(0,x) - v_1^{-}(0,x) = 0, \\ v_1^{+}(0,x) + v_1^{-}(0,x) = f(x), \end{cases} \tag{2.10}$$

$$\begin{cases} v_j^{+}(0,x) - v_j^{-}(0,x) = 0, \\ v_j^{+}(0,x) + v_j^{-}(0,x) = -\partial_t v_{j-1}^{+}(0,x) + \partial_t v_{j-1}^{-}(0,x), \end{cases} \tag{2.11}$$

for any $j = 2,3,\dots,n$. Then (2.2) and (2.3) hold. Set

$$\begin{cases} u_0(x;k) = 0, \\ u_1(x;k) = -(ik)^{-n} e^{ik\omega\cdot x}\{\partial_t v_n^{+}(0,x) - \partial_t v_n^{-}(0,x)\} \end{cases} \tag{2.12}$$

and solve (2.8). Then the function $w(t,x;k)$ defined by (2.9) is the genuine solution of (1.1) with data $\{0, f(x)e^{ik\omega\cdot x}\}$.

The solutions $v_j^{\pm}(t,x)$ $(j = 1,2,\dots,n)$ are given by solving (2.6) with data determined by (2.10) and (2.11). We can give the precise form of $v_j^{\pm}(t,x)$:

Lemma 2.1. *Let $v_j^{\pm}(t,x)$ be the solutions of* (2.6) *with the data* (2.10) *and* (2.11). *Then we have*

$$v_1^{\pm}(t,x) = \frac{f(x \pm \omega t)}{2} \exp\left\{-\frac{1}{2}\int_0^t b(x \pm (t-s)\omega)\,ds\right\}, \tag{2.13}$$

$$\begin{aligned} v_j^{\pm}(t,x) = v_j^{\pm}(0,x \pm \omega t) &\exp\left\{-\frac{1}{2}\int_0^t b(x \pm (t-s)\omega)\,ds\right\} \\ &\mp \frac{1}{2}\int_0^t P v_{j-1}^{\pm}(\tau, x \pm (t-\tau)\omega) \\ &\qquad \times \exp\left\{-\frac{1}{2}\int_\tau^t b(x \pm (t-s)\omega)\,ds\right\} d\tau \end{aligned} \tag{2.14}$$

for any $j = 2,3,\dots,n$.

§3. L^∞-estimates of asymptotic solutions

In this section, we derive L^∞-estimates of (2.13) and (2.14):

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LEMMA 3.1. *Assume that* $b(\,\cdot\,) \in \mathcal{B}^\infty(\mathbb{R}^N)$ *is non-negative and satisfies* (1.6). *Then for each multi-index* α *there exists a positive constant* C_α *such that the estimate*

$$\left|\partial^\alpha_{(t,x)} v_j^\pm(t,x)\right| \le C_\alpha (1+t)^{3(j-1)+|\alpha|} \tag{3.1}$$

holds for any $t \ge 0$ *and* $x \in \mathbb{R}^N$.

Proof. Let $T_\tau(t,x)$ $(0 \le \tau \le t)$ be the function defined by

$$T_\tau(t,x) = \frac{1}{2}\int_\tau^t b(x \pm (t-s)\omega)\,ds.$$

Then direct computation gives

$$\left|\partial^\alpha_{t,x} T_\tau(t,x)\right| \le C_\alpha (1+t)$$

for some $C_\alpha > 0$. Thus we have

$$\left|\partial^\alpha_{t,x} e^{-T_\tau(t,x)}\right| \le C_\alpha (1+t)^{|\alpha|} \tag{3.2}$$

for another positive constant C_α. Note that

$$\begin{aligned}(3.3)\quad & \partial^\alpha_{t,x} v_1^\pm(t,x) \\ &= \left\{(-1)^{|\alpha|} v_1^\pm(0,x)\{\partial_t T_0(t,x)\}^{\alpha_0} \prod_{l=1}^N \{\partial_{x_l} T_0(t,x)\}^{\alpha_l} + R_{|\alpha|}(t,x)\right\} e^{-T_0(t,x)},\end{aligned}$$

where $\alpha = (\alpha_0, \alpha') = (\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_N)$ and $R_{|\alpha|}(t,x)$ is the function containing multiplications of the derivatives of $T_0(t,x)$ at most of order m with $m \le |\alpha'| - 1$. Therefore we obtain (3.1) with $j = 1$ from (3.2). Next we shall show (3.1) with $j = 2$. Putting

$$\begin{aligned} v_{2,1}(t,x) &= v_2^\pm(0,x) e^{-T_0(t,x)}, \\ v_{2,2}(t,x) &= \int_0^t P v_1^\pm(\tau, x \pm (t-\tau)\omega) e^{-T_\tau(t,x)}\,d\tau, \end{aligned}$$

we have $v_2^\pm(t,x) = v_{2,1}(t,x) \mp \frac{1}{2} v_{2,2}(t,x)$. Thus we find from (3.2)

$$\left|\partial^\alpha_{(t,x)} v_{2,1}(t,x)\right| \le C_{|\alpha|}(1+t)^{|\alpha|}.$$

Note that

$$\begin{aligned}
&\partial_t^m T_\tau(t,x)\big|_{\tau=t} = \begin{cases} 0, & (m=0), \\ \frac{b_{m-1}(x)}{2}, & (m\in\mathbb{N}), \end{cases} \\
&\partial_x^{\alpha'} T_\tau(t,x)\big|_{\tau=t} = 0 \quad \text{for any } \alpha',
\end{aligned}$$

where

$$\begin{aligned}
&b_0(x) = b(x), \\
&b_m(x\pm(t-s)\omega) = \pm\omega\cdot\nabla b_{m-1}(x\pm(t-s)\omega) \quad (m\in\mathbb{N}).
\end{aligned}$$

Then we obtain (3.1) with $j=2$ by direct computation and induction. By an argument similar to this we have (3.1) for all $j=1,2,\dots,n$. ∎

As is easily seen (e.g., cf. Section 3 in Ikawa [1]), we have

LEMMA 3.2. *Assume that $b(\,\cdot\,)\in\mathcal{B}^\infty(\mathbb{R}^N)$ is non-negative and satisfies* (1.6)*. Then we have*

$$\operatorname{supp}_x v_j^\pm(t,x) \subset \bigcup_{0\le\tau\le t} \{y\mp\tau\omega \mid y\in\operatorname{supp}_x f(x)\}, \tag{3.4}$$

$$\operatorname{diam}\big(\operatorname{supp}_x Pv_j^\pm(t,x)\big) \le C\big\{\operatorname{diam}\big(\operatorname{supp}_x f(x)\big)+t\big\} \tag{3.5}$$

for some positive constant C, any $t\ge 0$ and any $j=1,2,\dots,n$, where $\operatorname{diam} D = \max_{x,y\in D}|x-y|$.

Let us give a lower bound to the asymptotic solutions $v(t)=v(t,x;k)$:

PROPOSITION 3.3. *Assume that $b(\,\cdot\,)\in\mathcal{B}^\infty(\mathbb{R}^N)$ is non-negative and satisfies* (1.6)*. Then there exists some positive constant C_n depending only on n such that*

$$\|v(t)\|_E^2 \ge \frac{1}{8}\|f\|_E^2 - C_n\left\{\sum_{j=1}^{n-1} k^{-2j}(1+t)^{6j} + k^{-2n}(1+t)^{6n-4}\right\} \tag{3.6}$$

holds for any $t>\operatorname{diam}\big(\operatorname{supp}_x f(x)\big)$.

Proof. Note that

$$\partial_t v(t,x) = \sum_{j=0}^{n}(ik)^{-j}p_j(t,x)$$

where

$$p_j(t,x) = \begin{cases} e^{ik\varphi^+(t,x)}v_1^+(t,x) + e^{ik\varphi^-(t,x)}v_1^-(t,x) & (j=0), \\ e^{ik\varphi^+(t,x)}\{\partial_t v_j^+(t,x) + v_{j+1}^+(t,x)\} & \\ \qquad + e^{ik\varphi^-(t,x)}\{\partial_t v_j^-(t,x) + v_{j+1}^-(t,x)\} & (j=1,2,\dots,n-1), \\ e^{ik\varphi^+(t,x)}\partial_t v_n^+(t,x) - e^{ik\varphi^-(t,x)}\partial_t v_n^-(t,x) & (j=n). \end{cases}$$

Since

$$\operatorname{supp} f(x) \cap \operatorname{supp} b(x) = \emptyset,$$

we have

$$v_1^-(t,x) = \frac{f(x-\omega t)}{2}$$

by (2.10)–(2.14). This gives

$$\|p_0(t,\,\cdot\,)\|_{L^2}^2 \ge \frac{1}{4}\|f\|_{L^2}^2.$$

Moreover, by Lemma 3.1 we find

$$\begin{aligned} &|p_j(t,x)| \le Ck^{-j}(1+t)^{3j} \qquad (j=1,2,\dots,n-1), \\ &|p_n(t,x)| \le C'k^{-n}(1+t)^{3n-2} \end{aligned}$$

for some positive constants C and C'. Thus we obtain the desired result. $\square$

PROPOSITION 3.4. *Assume that $b(\,\cdot\,) \in \mathcal{B}^\infty(\mathbb{R}^N)$ is non-negative and satisfies* (1.6). *Then there exists some positive constant C independent of t and k such that the remainder term $u(t,x;k)$ in* (2.8) *satisfies*

$$\|u(t)\|_E^2 \le Ck^{-2n}(1+t)^{6n+N}. \tag{3.7}$$

Proof. Since $Pu = -g$ (see (2.8)), it follows from the usual energy estimate that

$$\|u(t)\|_E^2 \le C\Big(\|u(0)\|_E^2 + (1+t)\int_0^t \|g(\tau,\,\cdot\,)\|_{L^2}^2\,d\tau\Big)$$

for some $C > 0$. Noting (2.12), Lemma 3.1 with $t=0$ and Lemma 3.2, we find

$$\|u(0)\|^2 = \frac{1}{2}\|u_1(0,\,\cdot\,;k)\|_{L^2}^2 \le Ck^{-2n}$$

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for some $C > 0$. Moreover we have

$$\|g(\tau,\,\cdot\,)\|_{L^2}^2 \le Ck^{-2n}\big(\|Pv_n^+(\tau,\,\cdot\,)\|_{L^2}^2 + \|Pv_n^-(\tau,\,\cdot\,)\|_{L^2}^2\big)$$

for any $\tau \in [0,t]$ by (2.7). Finally we obtain from Lemma 3.1 and Lemma 3.2 that

$$\|Pv_n^\pm(\tau,\,\cdot\,)\|_{L^2}^2 = \int_{\operatorname{supp} Pv_n^\pm(\tau,\,\cdot\,)} |Pv_n^\pm(\tau,x)|^2\,dx \le C(1+t)^{6n-2+N}.$$

Combining these estimates we have the desired result. □

§4. Proof of Theorem 1.2

Assume that (1.1) has (U. D. P.), i.e., for any $\varepsilon > 0$, there exists $T(\varepsilon) > 0$ such that

$$\|w(t)\|_E^2 \le \varepsilon\big(\|w_0\|_{H^1}^2 + \|w_1\|_{L^2}^2 + \|(w_0,w_1)\|_{E_\varphi}^2\big) \tag{4.1}$$

for any $t \ge T(\varepsilon)$ and any solution $w(t,x)$ of (1.1)–(1.2). Insert $w(t,x) = w(t,x;k)$ into (4.1) (see (2.9)). Since $\{w(0,x), \partial_t w(0,x)\} = \{0, f(x)e^{ik\omega\cdot x}\}$, the right-hand side of (4.1) is estimated by $\varepsilon C\|f\|_{L^2}^2$ for some $C > 0$ where we have used

$$\|(0, f(x)e^{ik\omega\cdot x})\|_{E_\varphi}^2 \le C\|f\|_{L^2}^2$$

for some $C > 0$. On the other hand, using Propositions 3.3 and 3.4, we find in (4.1)

$$\begin{aligned} \|w(t)\|_E^2 &\ge \frac{1}{2}\|v(t,\,\cdot\,)\|_E^2 - \|u(t,\,\cdot\,)\|_E^2 \\ &\ge \frac{1}{16}\|f\|_{L^2}^2 - C_n\Big\{\sum_{j=1}^n k^{-2j}(1+t)^{6j} + k^{-2n}(1+t)^{6n-4} + k^{-2n}(1+t)^{6n+N}\Big\} \end{aligned} \tag{4.2}$$

for any $t > \operatorname{diam}(\operatorname{supp} f(x))$ and for some $C_n > 0$ depending only on n. Now choose t large enough as $t \ge T(\varepsilon)$ and $t > \operatorname{diam}(\operatorname{supp} f(x))$ and choose k large enough as $(1+t)^{n+\frac{N}{6}} \le k$. Then we have

$$[\text{right-hand side of (4.2)}] \ge \frac{1}{16}\|f\|_{L^2}^2 - C_n\sum_{j=1}^n k^{-j}.$$

Moreover choose k as

$$\frac{1}{16}\|f\|_{L^2}^2 - C_n\sum_{j=1}^n k^{-j} \ge \frac{1}{32}\|f\|_{L^2}^2.$$

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Thus we obtain

$$\|w(t)\|_E^2 \geq \frac{1}{32}\|f\|_{L^2}^2.$$

Therefore we have

$$\frac{1}{32C} \leq \varepsilon,$$

and this is the contradiction. □

Finally, we note that Remark 1.3 in Introduction can be verified only by making a little more precise estimation in Proposition 3.3.

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