

ZEROS OF THE ESTERMANN ZETA FUNCTION

A. DUBICKAS , R. GARUNKŠTIS, J. STEUDING and R. STEUDING

(Received 30 April 2010; accepted 11 October 2012; first published online 1 March 2013)

Communicated by I. Shparlinski

Abstract

In this paper we investigate the zeros of the Estermann zeta function $E(s; k/\ell, \alpha) = \sum_{n=1}^{\infty} \sigma_{\alpha}(n) \exp(2\pi i n k/\ell) n^{-s}$ as a function of a complex variable s , where k and ℓ are coprime integers and $\sigma_{\alpha}(n) = \sum_{d|n} d^{\alpha}$ is the generalized divisor function with a fixed complex number α . In particular, we study the question on how the zeros of $E(s; k/\ell, \alpha)$ depend on the parameters k/ℓ and α . It turns out that for some zeros there is a continuous dependency whereas for other zeros there is not.

2010 *Mathematics subject classification*: primary 11M41.

Keywords and phrases: Estermann zeta function, distribution of zeros, trajectories of zeros.

1. Introduction

Let $s = \sigma + it \in \mathbb{C}$. For $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}$, we define the generalized divisor function by

$$\sigma_{\alpha}(n) = \sum_{d|n} d^{\alpha}.$$

Then, for $\sigma > \max\{1 + \Re \alpha, 1\}$ and $\lambda \in \mathbb{R}$, the Estermann zeta function is given by the Dirichlet series

$$E(s; \lambda, \alpha) = \sum_{n=1}^{\infty} \frac{\sigma_{\alpha}(n)}{n^s} \exp(2\pi i n \lambda). \quad (1.1)$$

In some respects the Estermann zeta function behaves pretty much like the classical Lerch zeta function which, for $\sigma > 1$, is given by

$$L(\lambda, \beta, s) = \sum_{n=0}^{\infty} \frac{\exp(2\pi i n \lambda)}{(n + \beta)^s}, \quad (1.2)$$

where λ and β are real parameters satisfying $0 < \beta \leq 1$. Illustrating this relationship, in Section 2 we shall recall some well-known results on the distribution of zeros of the Estermann zeta function. However, our main aim in this note is to investigate the

The second author is supported by grant no. MIP-94 from the Research Council of Lithuania.

© 2013 Australian Mathematical Publishing Association Inc. 1446-7887/2013 \$16.00

dependency of zeros of the Estermann zeta function with respect to its parameters λ and α . It will turn out that for some zeros the dependency is not ‘continuous’ in a certain sense which will be specified below. In contrast, for the Lerch zeta function, it is known that all the zeros of $L(\lambda, \beta, s)$ depend continuously on $0 < \lambda < 1$ and on $0 < \beta < 1$ (see [4, Lemma 7]). It should be noted that Balanzario and Sánchez-Ortiz [1] used the trajectories of the zeros of some special Dirichlet series depending on a certain parameter for calculating *new* zeros of the Davenport–Heilbronn zeta function. Moreover, in [6] the trajectories of zeros of the Hurwitz zeta function $\zeta(s, \beta) = L(1, \beta, s)$ have been used for a classification of the nontrivial zeros of the Riemann zeta function.

Let us first explain the meaning of *continuous dependency* of a zero with respect to some parameter which runs through an interval or a dense subset of an interval. Let I be an interval in \mathbb{R} , and let S be a dense subset of I . Suppose that a function $f_1(s) = f(s, \alpha)$ is analytic in s for each $\alpha \in S$ and assume that $s = \rho_0$ is a zero of multiplicity m of $f(s, \alpha_0)$, where $\alpha_0 \in S$. We say that *the zero ρ_0 is S -continuous at α_0* if, for every sufficiently small open disk D with center at ρ_0 in which the function $f(s, \alpha_0)$ has no other zeros except for ρ_0 , there exists a $\delta = \delta(D) > 0$ such that each function $f(s, \alpha')$, where $\alpha' \in (\alpha_0 - \delta, \alpha_0 + \delta) \cap S$, has exactly m zeros (counted with multiplicities) in the disk D .

Note that if the zero ρ_0 of multiplicity $m = 1$ is S -continuous at α_0 , then there exists a neighborhood of α_0 and some function $\rho = \rho(\alpha)$, $\alpha \in S$, which is continuous at α_0 and, in addition, satisfies the relation $f(\rho(\alpha), \alpha) = 0$.

Our first theorem deals with the dependency of zeros of $E(s; \lambda, \alpha)$ with respect to α when the parameter λ is fixed.

THEOREM 1.1. *Each zero of $E(s; \lambda, \alpha)$ is S -continuous at every value of the parameter $\alpha \in S = (-1, 0)$.*

Furthermore, we shall also consider the continuity properties of $E(s; \lambda, \alpha)$ with respect to λ . Here we may assume that $a := \Re \alpha \leq 0$ which implies the absolute convergence of the Dirichlet series for $E(s; \lambda, \alpha)$ in the half-plane $\sigma > 1$ and, in particular, the uniform convergence for $\sigma \geq 1 + \varepsilon$ independent of λ , where ε is an arbitrary positive real number. Hence $E(s; \lambda, \alpha)$ is continuous with respect to $\lambda \in \mathbb{R}$ whenever $\sigma > 1$.

THEOREM 1.2. *Each zero of $E(s; \lambda, \alpha)$ in the half plane $\Re(s) > 1$ is \mathbb{R} -continuous at every $\lambda \in \mathbb{R}$.*

For positive integers m and n , we define

$$\mathbb{Q}_{m,n} = \left\{ \frac{m^2 q}{r} : (mnq)^2 \equiv 1 \pmod{r}, q \in \mathbb{Z}, r \in \mathbb{N} \right\}. \quad (1.3)$$

In particular,

$$\mathbb{Q}_{1,1} = \left\{ \frac{q}{r} : q^2 \equiv 1 \pmod{r}, q \in \mathbb{Z}, r \in \mathbb{N} \right\}.$$

By Proposition 3.2 below, each set $\mathbb{Q}_{m,n}$ is a dense subset of \mathbb{Q} . In the next section we will see that for $\lambda \in \mathbb{Q}$ and $\alpha \in \mathbb{C}$ the function $E(s; \lambda, \alpha)$ has an analytic continuation to a meromorphic function on \mathbb{C} . We now state the main theorem of this paper.

THEOREM 1.3. *Let m and n be positive integers. Suppose that $E(\rho; \lambda, \alpha) = 0$, where $\lambda \in \mathbb{Q}$, $-1 < \alpha \leq 0$ and $\Re \rho < \alpha$. Then the zero ρ is not \mathbb{Q} -continuous at $\lambda \in \mathbb{Q}$. However, for $\lambda \in \mathbb{Q}_{m,n}$, the zero ρ is $\mathbb{Q}_{m,n}$ -continuous at λ .*

The motivation behind the set $\mathbb{Q}_{m,n}$ is explained in Section 3 (see also Remark 4). Direct calculations with Mathematica, using formula (2.1), show that there exist zeros which satisfy the assumptions of Theorems 1.2 and 1.3. The question whether the zeros located in the strip $\alpha \leq \sigma \leq 1$ depend continuously on λ remains open.

Further, we are interested in a horizontal distribution of zeros of the Estermann zeta function. Assuming the Riemann hypothesis, the nontrivial zeros of the function

$$E(s; 1, \alpha) = \zeta(s)\zeta(s - \alpha) \tag{1.4}$$

lie on the lines $\sigma = \frac{1}{2}$ and $\sigma = \frac{1}{2} + a$. We conjecture that the example (1.4) is typical in the sense that a positive proportion of the nontrivial zeros is clustered around the two lines mentioned above. Towards this hypothesis we prove Theorem 1.4 below.

Throughout this paper, $k \in \mathbb{Z}$ and $\ell \in \mathbb{N}$ are two relatively prime integers. Denote by $N(T; k/\ell, \alpha)$ the number of nontrivial zeros $\rho = \beta + i\gamma$ of $E(s; k/\ell, \alpha)$ (counted with multiplicities) satisfying $|\gamma| \leq T$. Let $N(\sigma, T; k/\ell, \alpha)$ be the number of nontrivial zeros ρ of $E(s; k/\ell, \alpha)$ with $\beta > \sigma$ and $|\gamma| \leq T$. The next result shows that at most $\frac{1}{2} + \varepsilon$ of the nontrivial zeros of $E(s; k/\ell, \alpha)$ lie to the right of the line $\sigma = \frac{1}{2} + \varepsilon a$.

THEOREM 1.4. *Let $a = \Re \alpha < 0$ and let $0 < \varepsilon < \frac{1}{2}$. If $\frac{1}{2} + \varepsilon a \leq \sigma < \frac{1}{2}$ then*

$$\lim_{T \rightarrow \infty} \frac{N(\sigma, T; \frac{k}{\ell}, \alpha)}{N(T; \frac{k}{\ell}, \alpha)} \leq \frac{1}{2} + \varepsilon.$$

This paper is organized as follows. In the following section we will recall some basic facts about the analytic properties of the Estermann zeta function and the distribution of its zeros. Section 3 is devoted to the properties of the set $\mathbb{Q}_{m,n}$. In Section 4 the proofs of Theorems 1.1, 1.2 and 1.3 will be presented. Finally, in Section 5 we give the proof of Theorem 1.4.

2. Background on Estermann’s zeta function

For $\lambda = k/\ell$, where k and ℓ are coprime integers, one finds the representation (see Kiuchi [7, formula (2.4)])

$$E\left(s; \frac{k}{\ell}, \alpha\right) = \ell^{\alpha-s} \sum_{h=1}^{\ell} \exp\left(2\pi i \frac{hk}{\ell}\right) L\left(1, \frac{h}{\ell}, s - \alpha\right) L\left(\frac{hk}{\ell}, 1, s\right), \tag{2.1}$$

where $L(\lambda, \beta, s)$ stands for the Lerch zeta function given above by (1.2). Thus $E(s; k/\ell, \alpha)$ can be analytically continued to a meromorphic function, which is regular in the whole complex plane up to two simple poles at $s = 1$ and at $s = 1 + \alpha$ if $\alpha \neq 0$, or up to one double pole at $s = 1$ if $\alpha = 0$ (see [8]). In view of the functional equation for the Lerch zeta function using (2.1) one easily finds that

$$\begin{aligned}
 E\left(s; \frac{k}{\ell}, \alpha\right) &= \frac{1}{\pi} \left(\frac{\ell}{2\pi}\right)^{1+\alpha-2s} \Gamma(1-s)\Gamma(1+\alpha-s) \\
 &\quad \times \left(\cos\left(\frac{\pi\alpha}{2}\right)E\left(1+\alpha-s; \frac{\bar{k}}{\ell}, \alpha\right) \right. \\
 &\quad \left. - \cos\left(\pi s - \frac{\pi\alpha}{2}\right)E\left(1+\alpha-s; -\frac{\bar{k}}{\ell}, \alpha\right)\right), \tag{2.2}
 \end{aligned}$$

where \bar{k} is defined by the congruence $k\bar{k} \equiv 1 \pmod{\ell}$ and the inequality $0 < \bar{k} \leq \ell$. For an irrational number λ , analytic continuation of $E(s; \lambda, \alpha)$ is an open problem. For $\alpha = 0$, the zeta function $E(s; k/\ell, \alpha)$ was introduced by Estermann [3] and for $\alpha \in (-1, 0]$ by Kiuchi [7]. Since $\sigma_\alpha(n)$ is an analytic function in α , the results above hold by analytic continuation for any complex number α .

Before we study the zeros of the Estermann zeta function with respect to its parameters α and λ , let us recall some results obtained in [10]. Since $\sigma_\alpha(n) = n^\alpha \sigma_{-\alpha}(n)$,

$$E\left(s, \frac{k}{\ell}, \alpha\right) = E\left(s - \alpha, \frac{k}{\ell}, -\alpha\right).$$

Therefore, in the following we may assume that $a = \Re\alpha \leq 0$. Let us denote the zeros of $E(s; k/\ell, \alpha)$ by $\rho = \beta + i\gamma$. As in the case of the Lerch zeta function, we have to distinguish between trivial and nontrivial zeros of $E(s; k/\ell, \alpha)$. It is easy to show that

$$E\left(s; \frac{k}{\ell}, \alpha\right) \neq 0 \quad \text{for } \sigma > 3. \tag{2.3}$$

By the functional equation (2.2) and the nonvanishing of the gamma function, $E(s; k/\ell, \alpha)$ vanishes if and only if

$$\cos\left(\frac{\pi\alpha}{2}\right)E\left(1+\alpha-s; \frac{\bar{k}}{\ell}, \alpha\right) = \cos\left(\pi s - \frac{\pi\alpha}{2}\right)E\left(1+\alpha-s; -\frac{\bar{k}}{\ell}, \alpha\right).$$

In view of (2.3) it follows that for $\sigma < -2 + a$ the function $E(s; k/\ell, \alpha)$ can only have zeros close to the negative real axis. Thus we call the zeros ρ of $E(s; k/\ell, \alpha)$ with $\beta < -2 + a$ trivial. However, we are interested in the nontrivial zeros of $E(s; k/\ell, \alpha)$. By the above and the zero-free region (2.3) the nontrivial zeros must lie in the vertical strip

$$-2 + a \leq \sigma \leq 3.$$

In [10] the asymptotic formula for the number of nontrivial zeros

$$N\left(T; \frac{k}{\ell}, \alpha\right) = 2\frac{T}{\pi} \log \frac{\ell T}{2\pi e} + O(\log T) \tag{2.4}$$

was proved. Since the main term does not depend on k and α , this already indicates the existence of zeros which do not depend on k/ℓ continuously. Furthermore, it was shown that

$$\frac{1}{N(T; \frac{k}{\ell}, \alpha)} \sum_{\substack{\rho \text{ nontrivial} \\ |\gamma| \leq T}} \beta = \frac{a+1}{2} + O(T^{-1}).$$

Therefore, the mean value of the real parts of the nontrivial zeros of $E(s; k/\ell, \alpha)$ exists and equals $(a + 1)/2$. This result can be compared to our Theorem 1.4.

3. Properties of the set $\mathbb{Q}_{m,n}$

To prove our main result (Theorem 1.3) we will use the functional equation (2.2). In order to apply this functional equation we shall investigate the properties of the function $g : \mathbb{Q} \rightarrow (0, 1)$, defined by

$$g\left(\frac{k}{\ell}\right) = \frac{\bar{k}}{\ell}.$$

(Recall that \bar{k} is defined by the congruence $k\bar{k} \equiv 1 \pmod{\ell}$, where $0 < \bar{k} \leq \ell$.) As we will see later, the function g behaves very chaotically. This implies the discontinuity of the zero trajectories. On the other hand, we will show that the function g , if restricted to the set $(0, m^2/n^2) \cap \mathbb{Q}_{m,n}$, is continuous. This will lead to continuous zero trajectories. More precisely, let us define

$$g_{m,n} : \mathbb{Q}_{m,n} \rightarrow (0, 1) \quad \text{where } g_{m,n}\left(\frac{k}{\ell}\right) = \frac{\bar{k}}{\ell}.$$

The aim of this section is to prove the following proposition.

PROPOSITION 3.1. *The function g is everywhere discontinuous. Moreover, for any neighborhood V_a of a rational number a , the image $g(V_a \cap \mathbb{Q})$ is everywhere dense in the interval $(0, 1)$. Furthermore,*

$$g_{m,n}(x) = \frac{n^2}{m^2}x \tag{3.1}$$

for each $x \in (0, m^2/n^2) \cap \mathbb{Q}_{m,n}$ and

$$g_{m,n}(x) - \frac{n^2}{m^2}x \in \mathbb{Z}$$

for $x \in \mathbb{Q}_{m,n}$.

For us it will be important that the function $\exp(2\pi i k g_{m,n}(x))$, where k is a fixed integer, is continuous on $\mathbb{Q}_{m,n}$. The question whether there are other dense subsets of rational numbers on which the function $\exp(2\pi i k g(x))$ is continuous remains open. Proposition 3.1 will be derived from the following proposition.

PROPOSITION 3.2. *For any $m, n \in \mathbb{N}$ the set $\mathbb{Q}_{m,n}$ defined in (1.3) is everywhere dense in \mathbb{R} .*

To prove Proposition 3.2 the following lemma will be crucial.

LEMMA 3.3. *Let u and v be two positive integers satisfying $\gcd(u, v) = 1$. Then the set of rational numbers k/ℓ , where k is an integer and ℓ is a positive integer satisfying $u|k$, $v|\ell$ and $k^2 \equiv 1 \pmod{\ell}$, is everywhere dense in \mathbb{R} .*

PROOF. Clearly, the set of rational numbers us/vr , where $s \in \mathbb{Z}$, $r \in \mathbb{N}$ are such that $\gcd(us, vr) = 1$, is everywhere dense in \mathbb{R} . Fix any pair s, r satisfying $\gcd(us, vr) = 1$. We will show that $k \in \mathbb{Z}$ and $\ell \in \mathbb{N}$, where $u|k$, $v|\ell$, $k^2 \equiv 1 \pmod{\ell}$, can be chosen such that the quotient k/ℓ is arbitrarily close to us/vr . Since us and vr are relatively prime, there is an integer g satisfying

$$us(ug + 1) \equiv -1 \pmod{vr}. \tag{3.2}$$

We select

$$\ell := vr(N - 2)N \tag{3.3}$$

and

$$k := (usN + 1)(N - 2) + 1, \tag{3.4}$$

where

$$N := vrun + ug + 1$$

with some $n \in \mathbb{N}$. Clearly, $N \rightarrow \infty$ as $n \rightarrow \infty$. By (3.3) and (3.4), $k/\ell \rightarrow us/vr$ as $n \rightarrow \infty$.

In order to prove the congruence $k^2 \equiv 1 \pmod{\ell}$ we first observe that

$$usN + 1 = us(vrun + ug + 1) + 1 = u^2svrn + us(ug + 1) + 1$$

is divisible by vr , by (3.2). Hence, by (3.4), $k - 1$ is divisible by $vr(N - 2)$. Also,

$$k + 1 = (usN + 1)(N - 2) + 2 = usN^2 + N(1 - 2us)$$

is divisible by N . From (3.3), we conclude that ℓ divides the product $(k + 1)(k - 1) = k^2 - 1$.

Finally, by (3.3), we have $v|\ell$. Since $N \equiv 1 \pmod{u}$, by (3.4),

$$k \equiv N - 1 \equiv 0 \pmod{u}.$$

Thus $u|k$. This completes the proof of the lemma. □

PROOF OF PROPOSITION 3.2. Note that

$$\frac{n}{m}\mathbb{Q}_{m,n} = \left\{ \frac{n}{m}a : a \in \mathbb{Q}_{m,n} \right\} = \left\{ \frac{mnq}{r} : (mnq)^2 \equiv 1 \pmod{r}, q \in \mathbb{Z}, r \in \mathbb{N} \right\}.$$

By Lemma 3.3 with $v = 1$, $\ell = r$, $u = mn$, $k = mnq$, the set $n/m\mathbb{Q}_{m,n}$ is everywhere dense in \mathbb{R} , hence so is $\mathbb{Q}_{m,n}$. □

PROOF OF PROPOSITION 3.1. Fix $k/\ell \in \mathbb{Q}_{m,n}$. We will first show that

$$g_{m,n}\left(\frac{k}{\ell}\right) - \frac{n^2}{m^2} \frac{k}{\ell} \in \mathbb{Z}. \tag{3.5}$$

By the definition of the set $\mathbb{Q}_{m,n}$, the number k/ℓ can be written as

$$\frac{k}{\ell} = \frac{m^2q}{r},$$

where $\overline{m^2q} \equiv n^2q \pmod{r}$. Hence r divides the difference $\overline{m^2q} - n^2q$. Using this and

$$g_{m,n}\left(\frac{k}{\ell}\right) - \frac{n^2}{m^2} \frac{k}{\ell} = g_{m,n}\left(\frac{m^2q}{r}\right) - \frac{n^2q}{r} = \frac{\overline{m^2q}}{r} - \frac{n^2q}{r}$$

we obtain (3.5).

Note that if $x \in (0, (m^2/n^2)) \cap \mathbb{Q}_{m,n}$ then $0 < n^2/m^2x \leq 1$. Thus, from the formula (3.5) and the definition of $g_{m,n}$, we derive (3.1).

The remaining statements of Proposition 3.1 follow from Proposition 3.2 in view of the equality $g_{m,n}(k/\ell) = g(k/\ell)$, where $k/\ell \in \mathbb{Q}_{m,n}$. □

4. Proofs of Theorems 1.1–1.3

The proofs below will use the following lemma.

LEMMA 4.1. *Let $f(\rho, \alpha_0) = 0$, where the function $f_1(s) = f(s, \alpha)$ is analytic for each $\alpha \in S$ (S is a dense subset of an interval in \mathbb{R}) and does not vanish identically at $\alpha = \alpha_0$. Suppose that there is a neighborhood V_ρ of ρ such that the function $f_2(\alpha) = f(s, \alpha)$, $\alpha \in S$, is continuous at $\alpha = \alpha_0$ for each $s \in V_\rho$. Then the zero ρ is S -continuous at α_0 .*

PROOF. Let D be a disk with center at ρ such that $D \subset V_\rho$ and $f_1(s) \neq 0$ on the boundary ∂D for $\alpha = \alpha_0$. The function $f(s, \alpha)$ is continuous at $\alpha = \alpha_0$ uniformly in $s \in D$. Thus there is a constant $\delta = \delta(D) > 0$ such that

$$|f(s, \alpha_0) - f(s, \alpha)| < |f(s, \alpha_0)|$$

for $s \in \partial D$ and $\alpha \in (\alpha_0 - \delta, \alpha_0 + \delta) \cap S$. Now Lemma 4.1 follows by Rouché’s theorem and the definition of the continuity of zero with respect to the parameter α . □

PROOF OF THEOREM 1.1. From the representation (2.1) of the Estermann zeta function in terms of the Lerch zeta function it follows that $E(s; \lambda, \alpha)$ is continuous in α . Hence, the theorem follows immediately from Lemma 4.1. □

PROOF OF THEOREM 1.2. The proof follows immediately from the comment before the statement of this theorem combined with Lemma 4.1. □

The following bound is useful is what follows.

LEMMA 4.2. For $x \geq 1$,

$$\left| \sum_{n \leq x} \sigma_0(n) - x \log x \right| \leq \gamma x + 2,$$

where γ is Euler's constant.

PROOF. Note that

$$\left| \sum_{1 \leq n \leq X} \frac{1}{n} - \log x - \gamma \right| \leq \frac{2}{X}$$

(see, for example, [2, Section 6.3]). Then Lemma 4.2 follows from the identity

$$\sum_{n \leq x} \sigma_0(n) = \sum_{n \leq x} \left[\frac{[x]}{n} \right] = \sum_{n \leq x} \left[\frac{x}{n} \right] = \sum_{n \leq x} \frac{x}{n} - \sum_{n \leq x} \left\{ \frac{x}{n} \right\}. \quad \square$$

PROOF OF THEOREM 1.3. In this proof the parameter α is fixed. Let $\lambda_0 \in \mathbb{Q}, \rho_0 = \beta_0 + i\gamma_0, \beta_0 < \alpha$, and let $E(\rho_0; \lambda_0, \alpha) = 0$. Actually we will prove a stronger result, namely, that the zero ρ_0 is not $\bigcup_{(m,n) \in \mathbb{N}^2} \mathbb{Q}_{m,n}$ -continuous at λ_0 . For this it suffices to show that there exists an open disk $D \subset \mathbb{C}$ with center at ρ_0 such that for every open interval I containing λ_0 there is a rational number $\frac{k_j}{\ell_j} \in \bigcup_{(m,n) \in \mathbb{N}^2} (\mathbb{Q}_{m,n} \cap I)$ satisfying

$$E\left(s; \frac{k_j}{\ell_j}, \alpha\right) \neq 0$$

for $s \in D$. Using the functional equation (2.2) and the properties of function g , described in Section 3, we will show that this is indeed the case.

In view of (2.2) let us define $f : \mathbb{R}^3 \rightarrow \mathbb{C}$ by

$$\begin{aligned} f(\sigma, t, x) &= \cos\left(\frac{\pi\alpha}{2}\right)E(1 + \alpha - (\sigma + it); x, \alpha) \\ &\quad - \cos\left(\pi(\sigma + it) - \frac{\pi\alpha}{2}\right)E(1 + \alpha - (\sigma + it); -x, \alpha). \end{aligned}$$

For $\sigma < \alpha$, the function f is continuous at any point (σ, t, x) . Therefore if $f(\sigma', t', x') \neq 0$, then there is an open set $V \subset \mathbb{R}^3$ containing (σ', t', x') such that $f(\sigma, t, x) \neq 0$ for $(\sigma, t, x) \in V$.

Suppose that there exists $\lambda_1 \in \mathbb{Q}$ for which

$$f(\beta_0, \gamma_0, \lambda_1) \neq 0. \tag{4.1}$$

We will show that that the zero ρ_0 is not \mathbb{Q} -continuous at λ_0 if condition (4.1) is satisfied. It follows from the functional equation (2.2) that $E(\rho; k/\ell, \alpha) = 0$ if and only if $f(\sigma, t, \bar{k}/\ell) = 0$. By Proposition 3.1, there is a sequence $\{\frac{k_j}{\ell_j}\}, j = 1, 2, \dots$, tending to λ_0 , such that $\{\bar{k}_j/\ell_j\}$ tends to λ_1 . Hence, for the zero ρ_0 of the Estermann zeta function,

there exists an open disk $D \subset \mathbb{C}$ with center at ρ_0 such that for every open interval I containing λ_0 there is a rational number $k_j/\ell_j \in I$ satisfying

$$E\left(s; \frac{k_j}{\ell_j}, \alpha\right) = \frac{1}{\pi} \left(\frac{\ell_n}{2\pi}\right)^{1+\alpha-2s} \Gamma(1-s)\Gamma(1+\alpha-s) f\left(\sigma, t, \overline{\frac{k_j}{\ell_j}}\right) \neq 0$$

for $s \in D$.

To finish the proof of the first part of the theorem we will verify condition (4.1). For this we will show that, for any $\sigma < \alpha$ and any t , the function $f_3(x) = f(\sigma, t, x)$ is not identically zero. If $s \neq -2m$ and $s \neq -2m + \alpha$, where $m = 1, 2, \dots$, then

$$f(0) = \zeta(1 + \alpha - s)\zeta(1 - s) \left(\cos\left(\frac{\pi\alpha}{2}\right) - \cos\left(\pi s - \frac{\pi\alpha}{2}\right) \right) \neq 0.$$

On the other hand, for $s = -2m$ or $s = -2m + \alpha$,

$$f(x) = 2i \cos\left(\frac{\pi\alpha}{2}\right) \mathfrak{I}(E(1 + \alpha - s; x, \alpha)).$$

We next prove that $\mathfrak{I}(E(1 + \alpha - \sigma; 1/4, \alpha)) \neq 0$ for $\sigma \leq -2$. Indeed, observe that

$$\begin{aligned} \mathfrak{I}(E(1 + \alpha - \sigma; 1/4, \alpha)) &= \sum_{n=1}^{\infty} \frac{\sigma_\alpha(n) \sin\left(\frac{\pi n}{2}\right)}{n^{1+\alpha-\sigma}} \\ &\geq 1 - \frac{2}{3^2} - \frac{2}{7^2} - \frac{2}{11^2} - \sum_{n=13}^{\infty} \frac{\sigma_0(n)}{n^2} \geq 0.07, \end{aligned}$$

because

$$1 - \frac{2}{3^2} - \frac{2}{7^2} - \frac{2}{11^2} \geq 0.72$$

and, by Lemma 4.2,

$$\sum_{n=13}^{\infty} \frac{\sigma_0(n)}{n^2} = 2 \int_{13}^{\infty} \frac{D(x) dx}{x^3} \leq 2 \int_{13}^{\infty} \frac{x \log x + \gamma x + 2}{x^3} dx \leq 0.65,$$

where $D(x) := \sum_{n \leq x} \sigma_0(n)$. This proves the first part of Theorem 1.3.

The second part of Theorem 1.3 follows from Lemma 4.1 in view of the continuity (in λ) of the function $(2\pi/\ell)^{1+\alpha-2s} E(s; \lambda, \alpha)$ when λ runs through the set $\mathbb{Q}_{m,n}$ (see Proposition 3.1, the functional equation (2.2), and the expression of the Estermann zeta function by the Dirichlet series (1.1)). This completes the proof of the theorem. \square

REMARK. Evidently, $E(s; \lambda + 1, \alpha) = E(s; \lambda, \alpha)$ for $\lambda \in \mathbb{Q}$. Suppose that $\lambda \in \mathbb{Q}_{m,n}$. Using (1.3) one can verify that $\lambda + 1 \in \mathbb{Q}_{m,n}$ if and only if $m = 1$. From this point of view the sets $\mathbb{Q}_{1,n}$ are the most natural sets in the context of our problem.

5. Proof of Theorem 1.4

To prove Theorem 1.4 we will express the number of zeros of the Estermann zeta function by a mean value of the Estermann zeta function (see formula (5.3) below). Hence the following lemma will be useful.

LEMMA 5.1. *Let $a \leq 0$. For $\frac{1}{2} + a < \sigma < \frac{1}{2}$, as $T \rightarrow \infty$,*

$$\int_{-T}^T \left| E\left(\sigma + it; \frac{k}{\ell}, \alpha\right) \right|^2 dt \ll T^{2(1-\sigma)}.$$

PROOF. In [5] it was shown that

$$E\left(s; \frac{k}{\ell}, \alpha\right) = \Lambda(s; \alpha)L(s, \chi_0)L(s - \alpha, \chi_0) + \frac{1}{\varphi(\ell)} \sum_{\substack{\chi \pmod{\ell} \\ \chi \neq \chi_0}} \tau(\bar{\chi})\chi(k)L(s, \chi)L(s - \alpha, \chi),$$

where the summation is over all characters $\chi \pmod{\ell}$ different from the principle character χ_0 , the associated Gauss sum is denoted by $\tau(\chi)$, and

$$\Lambda(s; \alpha) := \begin{cases} \frac{2\ell - \ell^{1-s} - \ell^s}{\ell^s(\ell - 1)(1 - \ell^{-s})^2} & \text{if } \alpha = 0, \\ \frac{\ell - \ell^{1+\alpha+s} - \ell^{1+2\alpha} + \ell^{1+2\alpha-s} - \ell^s + \ell^{\alpha+s}}{\ell^s(\ell - 1)(1 - \ell^\alpha)(1 - \ell^{-s})(1 - \ell^{\alpha-s})} & \text{if } \alpha \neq 0. \end{cases}$$

Using this representation we find that

$$\int_{-T}^T \left| E\left(\sigma + it; \frac{k}{\ell}, \alpha\right) \right|^2 dt \ll \sum_{\psi, \chi \pmod{\ell}} \int_1^T |L(\sigma + it, \psi)L(\sigma - a + i(t - \Im\alpha), \chi)|^2 dt,$$

where the summation is over all characters modulo ℓ and the implied constant depends on ℓ and a . Applying the Cauchy–Schwarz inequality,

$$\begin{aligned} & \int_{-T}^T \left| E\left(\sigma + it; \frac{k}{\ell}, \alpha\right) \right|^2 dt \\ & \ll \sum_{\psi \pmod{\ell}} \left(\int_1^T |L(\sigma + it, \psi)|^4 dt \right)^{\frac{1}{2}} \\ & \quad \times \sum_{\chi \pmod{\ell}} \left(\int_1^T |L(\sigma - a + i(t - \Im\alpha), \chi)|^4 dt \right)^{\frac{1}{2}}. \end{aligned} \tag{5.1}$$

Recall the functional equation for Dirichlet L -functions associated to primitive characters,

$$L(s, \chi) = \Delta(s, \chi)L(1 - s, \bar{\chi})$$

with

$$\Delta(s, \chi) := \frac{\tau(\chi)}{i^\delta \sqrt{\ell}} \left(\frac{\ell}{\pi}\right)^{\frac{1}{2}-s} \frac{\Gamma\left(\frac{1+\delta-s}{2}\right)}{\Gamma\left(\frac{s+\delta}{2}\right)},$$

where $\delta := \frac{1}{2}(1 - \chi(-1))$ and $\tau(\chi)$ is the Gauss sum attached to χ . By Stirling's formula, $\Delta(\sigma + it, \chi) \ll t^{\frac{1}{2} - \sigma}$ as $t \rightarrow \infty$. Hence, for $\sigma \leq \frac{1}{2}$, we find that

$$\int_1^T |L(\sigma + it, \chi)|^4 dt \ll T^{4(\frac{1}{2} - \sigma)} \int_1^T |L(1 - \sigma, \chi)|^4 dt \tag{5.2}$$

for any primitive character $\chi \pmod{\ell}$. However, every nonprimitive character $\chi \pmod{\ell}$ is induced by some primitive character $\chi_1 \pmod{\ell_1}$ with $\ell_1 \mid \ell$ and

$$L(s, \chi) = L(s, \chi_1) \prod_{p \mid \ell} \left(1 - \frac{\chi_1(p)}{p^s}\right).$$

Hence (5.2) holds for nonprimitive characters too. Using the generalized Carlson mean-square estimate from [11, Theorem 2.4] applied to $L(s, \chi)^2$, we thus find that

$$\int_1^T |L(\sigma + it, \chi)|^4 dt \ll T^{1+4(\frac{1}{2} - \sigma)}$$

for $\sigma < \frac{1}{2}$.

By the same argument we can bound the other integral appearing in (5.1) by T . Thus the mean square of $E(\sigma + it; k/\ell, \alpha)$ is bounded from above by $T^{2(1-\sigma)}$. This proves the lemma. □

We are now in a position to prove Theorem 1.4.

PROOF OF THEOREM 1.4. Let $\delta > 0$ and let $\frac{1}{2} + a < u < \frac{1}{2}$. By arguments similar to those in [10], an application of Littlewood's lemma (see [9] or [12]) yields

$$\begin{aligned} \delta N\left(u + \delta, T; \frac{k}{\ell}, \alpha\right) &\leq \sum_{\substack{\beta > u \\ |\gamma| \leq T}} (\beta - u) \\ &= \frac{1}{2\pi} \int_{-T}^T \log \left| E\left(u + it; \frac{k}{\ell}, \alpha\right) \right| dt + O(\log T). \end{aligned} \tag{5.3}$$

By Jensen's inequality,

$$\frac{1}{2\pi} \int_{-T}^T \log \left| E\left(u + it; \frac{k}{\ell}, \alpha\right) \right| dt \leq \frac{T}{2\pi} \log \left(\frac{1}{2T} \int_{-T}^T \left| E\left(u + it; \frac{k}{\ell}, \alpha\right) \right|^2 dt \right).$$

In view of Lemma 5.1

$$\frac{1}{2\pi} \int_{-T}^T \log \left| E\left(u + it; \frac{k}{\ell}, \alpha\right) \right| dt \leq \frac{1 - 2u}{2\pi} T \log T + O(T).$$

This, (5.3) and (2.4) give

$$\lim_{T \rightarrow \infty} \frac{N(u + \delta, T; \frac{k}{\ell}, \alpha)}{N(T; \frac{k}{\ell}, \alpha)} \leq \frac{1 - 2u}{4\delta}.$$

Note that the inequalities

$$\frac{1 - 2u}{4\delta} \leq \frac{1}{2} + \varepsilon \quad \text{and} \quad u + \delta \geq \frac{1}{2} - 2\varepsilon\delta$$

are equivalent. We choose $u > \frac{1}{2} + a$ and $\delta = -(1 - \varepsilon)a$. Then, for $0 < \varepsilon < \frac{1}{2}$,

$$u + \delta > \frac{1}{2} + \varepsilon a \geq \frac{1}{2} + 2\varepsilon(1 - \varepsilon)a = \frac{1}{2} - 2\varepsilon\delta.$$

Now with $\sigma = u + \delta$ the assertion of the theorem follows. \square

References

- [1] E. P. Balanzario and J. Sánchez-Ortiz, ‘Zeros of the Davenport–Heilbronn counterexample’, *Math. Comp.* **76** (2007), 2045–2049.
- [2] K. Chandrasekharan, *Introduction to Analytic Number Theory* (Springer, Berlin, 1968).
- [3] T. Estermann, ‘On the representation of a number as the sum of two products’, *Proc. Lond. Math. Soc.* **31** (1930), 123–133.
- [4] R. Garunkštis *et al.*, ‘On zeros of the Lerch zeta-function. II’, in: *Probability Theory and Mathematical Statistics: Proceedings of the Seventh Vilnius Conf. (1998)*, (ed. B. Grigelionis) (TEV/Vilnius, VSP/Utrecht., 1999), 267–276.
- [5] R. Garunkštis, A. Laurinčikas, R. Šleževičienė and J. Steuding, ‘On the universality of Estermann zeta-functions’, *Analysis* **22** (2002), 285–296.
- [6] R. Garunkštis and J. Steuding, ‘On the distribution of zeros of the Hurwitz zeta-function’, *Math. Comp.* **76** (2007), 323–337.
- [7] I. Kiuchi, ‘On an exponential sum involving the arithmetic function $\sigma_a(n)$ ’, *Math. J. Okayama Univ.* **29** (1987), 193–205.
- [8] A. Laurinčikas and R. Garunkštis, *The Lerch Zeta-function* (Kluwer Academic Publishers, Dordrecht, 2002).
- [9] J. E. Littlewood, ‘On the zeros of the Riemann zeta-function’, *Proc. Camb. Phil. Soc.* **22** (1924), 295–318.
- [10] R. Šleževičienė and J. Steuding, ‘On the zeros of the Estermann zeta-function’, *Int. Trans. Spec. Funct.* **13** (2002), 363–371.
- [11] J. Steuding, *Value-Distribution of L-functions*, Lecture Notes in Mathematics, 1877 (Springer, New York, 2007).
- [12] E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, 2nd edn (Oxford University Press, Oxford, 1986), revised by D. R. Heath-Brown.

A. DUBICKAS, Department of Mathematics and Informatics,
Vilnius University, Naugarduko 24, LT-03225 Vilnius, Lithuania
e-mail: arturas.dubickas@mif.vu.lt

R. GARUNKŠTIS, Department of Mathematics and Informatics,
Vilnius University, Naugarduko 24, LT-03225 Vilnius, Lithuania
e-mail: ramunas.garunkstis@mif.vu.lt

J. STEUDING, Department of Mathematics, University of Würzburg,
Am Hubland, Campus-Nord, Emil Fischer Str. 40, 97074 Würzburg, Germany
e-mail: steuding@mathematik.uni-wuerzburg.de

R. STEUDING, Department of Mathematics, University of Würzburg,
Am Hubland, Campus-Nord, Emil Fischer Str. 40, 97074 Würzburg, Germany
e-mail: rasa.steuding@mathematik.uni-wuerzburg.de