Canad. Math. Bull. Vol. 59 (3), 2016 pp. 592–598
http://dx.doi.org/10.4153/CMB-2016-029-5

The Dirichlet Divisor Problem of Arithmetic Progressions

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Abstract. We present an elementary method for studying the problem of getting an asymptotic formula that is better than Hooley's and Heath-Brown's results for certain cases.

1 Introduction

Let x be a large real number q and r be positive integers, and

$$D(x;q,r) := \sum_{\substack{n\le x\\ n\equiv r(\bmod q)}} d(n),$$

where $d(n)$ is the well-known Dirichlet divisor function. The classical result of Selberg and Hooley [3] is that for $(q,r)=1$ and any $\varepsilon>0$, there exists $\delta>0$ such that

$$D(x;q,r) = \frac{x}{\varphi(q)}P(\log x;q) + O\Big(\frac{x^{1-\delta}}{\varphi(q)}\Big), \tag{1.1}$$

for any $q < x^{2/3-\varepsilon}$. Here $P(\log x;q)$ is the residue at $s=1$ of $s^{-1}L^2(s,\chi_0)x^{s-1}$ and χ_0 is the principal character modulus q. The study of $D(x;q,r)$ is of special interest when $(q,r)>1$, for the result would have important applications to other problems (see [3–5]). But actually works by Hooley [5, Lemma C], Heath-Brown [4, Theorem 3], and Smith [11, Theorem 3] all used some deep and complicated tools of complex analysis. In this paper we shall give an elementary treatment that is similar to the well-known method for the original Dirichlet divisor problem (see [7, §6.12, Theorem 3]). Our result is as follows.

Theorem 1.1 *Given any small positive constant ε, for any* $q<\min(x^{1-\varepsilon}, x^{2/3-\varepsilon}u^{1/3})$, *we have*

$$D(x;q,r) = P(u,\rho)\frac{\varphi(q)}{q\rho}x \times\Big[P_1(\log x+2\gamma-1) - P_2 - 2P_1\frac{q}{\varphi(q)}\Big(\sum_{t\mid q}\frac{\mu(t)\log t}{t}\Big)\Big] + \Delta(x;q,r),$$

where

$$\Delta(x;q,r) = O\Big(\big(u^{1/4}q^{-1}x^{3/4} + x^{1/2}(uq)^{-1/4} + \Big(\frac{q}{u}\Big)^{1/2}\big)x^{\varepsilon}\Big),$$

Received by the editors June 7, 2014; revised March 8, 2016.
Published electronically May 30, 2016.
AMS subject classification: **11L07**, 11B83.
Keywords: Dirichlet divisor problem, arithmetic progression.

(the implied constant does not depend on r , but depends on ε), $u = (q,r)$, $\rho = q/(q,r)$, γ is the Euler constant, and

$$P_1 = \sum_{\substack{(a,q)=u \\ p|a \Rightarrow p|u}} \frac{d(a)}{a}, \quad P_2 = \sum_{\substack{(a,q)=u \\ p|a \Rightarrow p|u}} \frac{d(a)\log a}{a}, \quad P(u,\rho) = \prod_{\substack{p|u \\ (p,\rho)=1}} \Big(1 - \frac{1}{p}\Big).$$

Moreover if $(q,r) = 1$, then the x^{ε} factor above can be replaced by $(L \cdot d(q))^3$, $L = \log x$.

Theorem 1.1 certainly gives an asymptotic formula for $q \ll x^{2/3-\varepsilon}u^{1/3}$, and particularly gives (1.1) when $u = (q,r) = 1$ and $q \ll x^{2/3-\varepsilon}$. In fact we have (using (2.3))

$$u^{-1}(d(u))^c \gg P_1 \geq \frac{d(u)}{u},$$
$$P_2 = 0 \text{ for } u = 1, \quad d(u) \gg P_2 \geq (d(u)\log u)u^{-1} \text{ for } u > 1,$$

and

$$\frac{q}{\varphi(q)} \sum_{t|q} \frac{\mu(t)\log t}{t} = -\sum_{p|q} \frac{\log p}{p-1}, \quad \sum_{p|q} \frac{\log p}{p-1} \ll \log\log 6q.$$

(To show the equality it suffices to assume that q is squarefree, and then use mathematical induction on the number of distinct prime factors of q; the $\ll$ estimate follows from a familiar technique). Thus using (2.3) we get

$$P_1(\log x + 2\gamma - 1) - P_2 - 2P_1 \frac{q}{\varphi(q)} \sum_{t|q} \frac{\mu(t)\log t}{t}$$
$$> \sum_{\substack{(a,q)=u \\ p|a \Rightarrow p|u \\ a \leq x^{1-\varepsilon}}} \frac{d(a)}{a} (\log x - \log a - c(\log\log x)^2) + o(x^{\varepsilon-1}) \gg \frac{d(u)}{u} \log x.$$

For $u = (q,r) = 1$, Theorem 1.1 implies (1.1) for $q \ll x^{2/3-\varepsilon}$. In fact this follows from using $L(s,\chi_0) = \zeta(s)\prod_{p|q}(1-p^{-s})$, and (for s near 1)

$$(s-1)\zeta(s) = 1 + \gamma(s-1) + \cdots,$$
$$\prod_{p|q}\Big(1 - \frac{1}{p^s}\Big) = \frac{\varphi(q)}{q}\Big\{1 - (s-1)\Big(\sum_{p|q} \frac{\log p}{p-1}\Big) + \cdots\Big\},$$
$$x^s/s = x + (x\log x - x)(s-1) + \cdots.$$

Note that if $u = (q,r) > x^{6\varepsilon}$ and q satisfies

$$\max(x^{2/3+8\varepsilon}u^{-1}, x^{5/12+2\varepsilon}u^{1/4}) < q < x^{2/3-4\varepsilon}u,$$

then Theorem 1.1 gives $\Delta(x;q,r) \ll x^{1/3-\varepsilon}$, which is better than that given by [5, Lemma C] , [4, Theorem 3 and its Corollary], Selberg unpublished result (see [4]).

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2 Proof of Theorem 1.1

We need two easy lemmas; the first is well known [6, p. 100] and the second is a special case of Lemma 2.5 (ii) of [9].

Lemma 2.1 *For $\xi \geq 1$, integers $t, q \geq 1$, $(t,q)=1$, there holds*

$$\sum_{\substack{u \leq \xi \\ u \equiv t \pmod q}} 1 = \frac{\xi}{q} + \psi\Big(\frac{-t}{q}\Big) - \psi\Big(\frac{\xi - t}{q}\Big), \psi(u) = u - [u] - 1/2.$$

Lemma 2.2 *Let $k, \xi \geq 1$, k be an integer, γ the Euler constant. Then*

$$\sum_{\substack{u \leq \xi \\ (u,k)=1}} \frac{1}{u} = \frac{\varphi(k)}{k}(\log \xi + \gamma) - \sum_{r \mid k} \frac{\mu(r)\log r}{r} + O\Big(\frac{1}{\xi}d(k)\Big).$$

Now let $u = (q,r)$, $\rho = q/u$. A positive integer n for which $n \equiv r \pmod q$ can be uniquely written as $n = ab$, here $u|a$ and $p|a \Rightarrow p|u$, $(a/u,\rho)=1$ and $(b,u)=1$. Thus we get

$$D(x;q,r) = \sum_{\substack{u|a \\ p|a \Rightarrow p|u \\ (a/u,\rho)=1 \\ a \leq x}} d(a)\Big(\sum_{\substack{vst \equiv \lambda (\bmod \rho) \\ \lambda = r/u \\ v = a/u \\ (st,u)=1 \\ st \leq x/a}} 1\Big). \tag{2.1}$$

Writing $Y = x/a$, and letting $\overline{sv}$ be the unique integer such that $\overline{sv} \cdot sv \equiv 1 \pmod \rho$, $0 \leq \overline{sv} < \rho$, we get

$$\sum_{\substack{vst \equiv \lambda (\bmod \rho) \\ \lambda = r/u \\ v = a/u \\ (st,u)=1 \\ st \leq x/a}} 1 = \sum_{\substack{s \leq \sqrt{Y} \\ (s,q)=1}} \Big(2 \sum_{\substack{t \equiv \lambda \overline{sv} (\bmod \rho) \\ (t,u)=1 \\ t \leq Y/s}} 1 - \sum_{\substack{t \equiv \lambda \overline{sv} (\bmod \rho) \\ (t,u)=1 \\ t \leq \sqrt{Y}}} 1\Big).$$

We have

$$\sum_{\substack{t \equiv \lambda \overline{sv} (\bmod \rho) \\ (t,u)=1 \\ t \leq Y/s}} 1 = \sum_{\substack{\delta | u \\ (\delta,\rho)=1}} \mu(\delta)\Big(\sum_{\substack{\omega \equiv k \pmod \rho \\ \omega \leq Y/(s\delta)}} 1\Big),$$

where k is the unique integer such that $k\delta \equiv \lambda \overline{sv} \pmod \rho$, and $0 \leq k < \rho$. Thus by Lemma 2.1 we have

$$\sum_{\substack{t \equiv \lambda \overline{sv} (\bmod \rho) \\ (t,u)=1 \\ t \leq Y/s}} 1 = \frac{Y}{s\rho}P(u,\rho) + \sum_{\substack{\delta | u \\ (\delta,\rho)=1}} \mu(\delta)\Big\{\psi\Big(\frac{Y/(s\delta) - k}{\rho}\Big) - \psi\Big(\frac{-k}{\rho}\Big)\Big\}. \tag{2.2}$$

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A similar result holds for $\sum_{t\equiv\lambda\overline{sv}(\bmod\rho),(t,u)=1,t\le\sqrt{Y}}1$ with s being replaced by $\sqrt{Y}$. Then

$$\sum_{\substack{(a,q)=u\\p|a\Rightarrow p|u\\a>x}}\frac{d(a)}{a}\le x^{0.5\varepsilon-1}\sum_{\substack{u|a\\p|a\Rightarrow p|u\\a>x}}d(a)a^{-0.5\varepsilon}\tag{2.3}$$

$$\le x^{0.5\varepsilon-1}\prod_{p|u}(1-2^{-\varepsilon})^{-2}=O(x^{0.5\varepsilon-1}d(u)^{c(\varepsilon)})=O(x^{\varepsilon-1}),$$

and similarly

$$\sum_{\substack{(a,q)=u\\p|a\Rightarrow p|u\\a>x}}\frac{d(a)\log a}{a}=O(x^{\varepsilon-1}),$$

$$\sum_{\substack{u|a\\p|a\Rightarrow p|u}}d(a)a^{-\varphi}\ll d(u)u^{-\varphi}\sum_{p|m\Rightarrow p|u}d(m)m^{-\varphi}\tag{2.4}$$

$$\ll d(u)u^{-\varphi}\prod_{p|u}(1-2^{-\varepsilon})^{-2}\ll u^{-\varphi}(d(u))^{c(\varphi)}$$

(for any constant $\varphi>0$). From (2.2), using Lemma 2.2, we have
(2.5)

$$\sum_{\substack{vst\equiv\lambda(\bmod\rho)\\\lambda=r/u,\,v=a/u\\(st,u)=1,\,st\le x/a}}1=P(u,\rho)\Big(\frac{2Y}{\rho}\Big(\sum_{\substack{s\le\sqrt{Y}\\(s,q)=1}}\frac{1}{s}\Big)-\frac{\sqrt{Y}}{\rho}\Big(\sum_{\substack{s\le\sqrt{Y}\\(s,q)=1}}1\Big)\Big)$$

$$=P(u,\rho)\Big(\frac{2Y}{\rho}\Big(\frac{\varphi(q)}{q}\Big(\frac{1}{2}\log Y+\gamma\Big)-\sum_{r|q}\frac{\mu(r)\log r}{r}+O\Big(\frac{d(q)}{\sqrt{Y}}\Big)\Big)$$

$$-\frac{\sqrt{Y}}{\rho}\Big(\frac{\varphi(q)}{q}\sqrt{Y}+O(d(q))\Big)\Big)+E$$

$$=P(u,\rho)\Big(\frac{Y\varphi(q)}{\rho q}(\log Y+2\gamma-1)-\frac{2Y}{\rho}\sum_{r|q}\frac{\mu(r)\log r}{r}+O\Big(\frac{\sqrt{Y}}{\rho}d(q)\Big)\Big)+E,$$

and where E consists of the following sums

$$S_1=\sum_{\substack{s\le\sqrt{Y}\\(s,q)=1}}\psi\Big(\frac{Y/(s\delta)-k}{\rho}\Big),\quad S_2=\sum_{\substack{s\le\sqrt{Y}\\(s,q)=1}}\psi\Big(\frac{-k}{\rho}\Big),$$

$$S_3=\sum_{\substack{s\le\sqrt{Y}\\(s,q)=1}}\psi\Big(\frac{\sqrt{Y}/\delta-k}{\rho}\Big).$$

Thus, using (2.3) and (2.4) we find that other terms apart from E of (2.5) contribute to (2.1) the "main terms" of Theorem 1.1, together with the following error term

$$O\big(x^{0.5\varepsilon}+x^{1/2}u^{1/2}q^{-1}(d(q))^c\big)=O(x^{1/2}(uq)^{-1/4}),$$

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because $x^{1/2}(uq)^{-1/4} \gg x^{0.5\varepsilon}$. It remains to deal with the contribution of S_1, for the treatments for S_2 and S_3 would be similar and easier. Let $\sqrt{Y} > 1000$. We have

$$S_1 = \sum_N S(N) + O(1), S(N) = \sum_{\substack{s \sim N \\ (s,q)=1}} \psi\Big(\frac{Y/(s\delta) - k}{\rho} \Big),$$

where N takes $O(\log 2Y)$ positive integer values, $10 < N < 2\sqrt{Y}$. Let c be a large constant (which will be clear later). For any $1 < H < x^2$ using the well-known Fourier expansion treatment of the function $\psi(\cdot)$ ([10]), we have

$$S(N) \ll \log x \Big(\frac{N}{H} + \sum_{1 \le h \le H^2} \min\Big(\frac{1}{h}, Hh^{-2} \Big) \Big| \sum_{\substack{s \sim N \\ (s,q)=1}} e\Big(h \frac{Y/(s\delta) - \lambda \overline{\delta} \overline{s v}}{\rho} \Big) \Big| \Big). \tag{2.6}$$

Let

$$S_h(N) = \sum_{\substack{s \sim N \\ (s,q)=1}} e\Big(h \frac{Y/(s\delta) - \lambda \overline{s \delta v}}{\rho} \Big),$$

$$T_h(M) = \sum_{\substack{N \le s \le M \\ (s,q)=1}} e\Big(-h \frac{\lambda \overline{s \delta v}}{\rho} \Big), \quad N \le M \le 2N.$$

By Abel's partial summation we get

$$\begin{aligned} S_h(N) &= \sum_{M \sim N} e\Big(h \frac{Y/(M\delta)}{\rho} \Big) (T_h(M) - T_h(M-1)) \\ &= \sum_{M \sim N} \Big(e\Big(h \frac{Y/(M\delta)}{\rho} \Big) - e\Big(h \frac{Y/((M+1)\delta)}{\rho} \Big) \Big) T_h(M) \\ &\quad + e\Big(h \frac{Y/(([2N]+1)\delta)}{\rho} \Big) T_h([2N]). \end{aligned}$$

Since (using the inequality $e(\theta) - 1 \ll |\theta|$ for any real number θ)

$$e\Big(h \frac{Y/(M\delta)}{\rho} \Big) - e\Big(h \frac{Y/((M+1)\delta)}{\rho} \Big) \ll hY(\delta N^2 \rho)^{-1},$$

we have (this treatment is similar to that of Hooley [6, p. 108])

$$S_h(N) \ll (1 + hY(\delta N \rho)^{-1}) \max_{M \sim N} |T_h(M)|. \tag{2.7}$$

We have

$$T_h(M) \ll d(u) \max_{t|u, (t,\rho)=1} \Big| \sum_{\substack{E \le w \le F \\ (w,\rho)=1}} e\Big(-h \frac{\lambda \overline{t w \delta v}}{\rho} \Big) \Big|, \quad E, F \approx N/t.$$

We split the inner sum into subsums in each of which the variable runs through exactly one complete residue class (mod q) (each of them is indeed a Ramanujan

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sum), together with one sum which has the form (after suitably reducing the variable mod q)

$$S' := \Big| \sum_{\substack{1\le w\le Q\\ (w,\rho)=1}} e\Big(-h\frac{\lambda \overline{tw\delta v}}{\rho}\Big)\Big|, Q < \rho.$$

For this sum we appeal to the technique in proving Theorem 3 of [7, §7.7], which then results in a complete Kloosterman sum. Thus, by using Estermann's estimate for Kloosterman's sum [2], we always have (the technique of [7, §7.7] will add an additional factor $\log 2\rho$.)

$$S' \ll d(\rho)\rho^{1/2}(h,\rho)^{1/2}(\log 2\rho).$$

As for those Ramanujan sums, for each of them we have an estimate $\ll (h,\rho)$ (see [1, p. 149]). Therefore,

$$T_h(M) \ll d(\rho)(N/\rho)(h,\rho) + d^2\rho^{1/2}(h,\rho)^{1/2}(\log 2\rho).$$

Let $L = \log x$. From (2.5), (2.6), and (2.7), for any $1 < H < x^2$ we have

$$L^{-2}d^{-3}(q)S(N) \ll \frac{N}{H} + \Big(\frac{YH}{\delta N\rho} + 1\Big)\rho^{1/2} + \frac{N}{\rho} + \frac{HY}{\rho^2}.$$

Taking into account the trivial estimate we have for all $H \in [0,\infty)$

$$L^{-2}d^{-3}(q)S(N) \ll \frac{N}{H} + \Big(\frac{YH}{\delta N\rho} + 1\Big)\rho^{1/2} + \frac{NH}{x^2} + \frac{N}{\rho} + \frac{HY}{\rho^2}.$$

We can use the following lemma to choose an optimal H .

Lemma 2.3 *For $M,N \ge 1, A_m, B_n, u_m, v_n > 0$, we can find a number $\omega \in (0,\infty)$ such that*

$$\sum_{1\le m\le M} A_m\omega^{u_m} + \sum_{1\le n\le N} B_n\omega^{-v_n} \ll \sum_{\substack{1\le m\le M\\ 1\le n\le N}} (A_m^{v_n}B_n^{u_m})^{1/(u_m+v_n)}.$$

Proof It can be deduced from [8, Lemma 6] by taking $Q_1 \to 0$ and $Q_2 \to \infty$. In fact, we can choose (as is clear from [8, p. 209]) $\omega = \min_{1\le i\le M, 1\le j\le N}(B_jA_i^{-1})^{1/(u_i+v_j)}$. ∎

Now by Lemma 2.3 we have

$$L^{-2}d^{-3}(q)S(N) \ll N\rho^{-1} + \sqrt{Y\delta^{-1}\rho^{-1/2}} + \sqrt{NY\rho^{-2}} + \frac{N}{x} + \rho^{1/2}.$$

Thus

$$L^{-3}d^{-3}(q)S_1 \ll \sqrt{Y\rho^{-1/2}} + \sqrt{Y^{1.5}\rho^{-2}} + \rho^{1/2} + Y^{0.5}/x.$$

The same bound holds also for S_2 and S_3. Using (2.3) to treat the summation for a in (2.1), we find that $\sqrt{Y\rho^{-1/2}}$ and $\sqrt{Y^{1.5}\rho^{-2}}$ contribute respectively the terms $(d(u))^c x^{1/2}(uq)^{-1/4}$ and $(d(u))^c u^{1/4}q^{-1}x^{3/4}$. Thus to obtain Theorem 1.1 we only need to estimate (see (2.1)) $\rho^{1/2}S''$, here

$$S'' := \sum_{\substack{u|a\\ p|a\Rightarrow p|u\\ (a/u,\rho)=1\\ a\le x}} d(a).$$

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Using $d(mn) \le d(m)d(n)$, we have

$$S'' \ll d(u) \sum_{p|b \Rightarrow p|u} d(b) \Big(\frac{x}{ub}\Big)^{0.5\varepsilon} \ll x^{0.5\varepsilon} \frac{d(u)}{u^{0.5\varepsilon}} \prod_{p|u} (1 - p^{-0.5\varepsilon})^{-2}$$

$$\ll x^{0.5\varepsilon} \frac{d(u)}{u^{0.5\varepsilon}} \prod_{p|u} (1 - 2^{-0.5\varepsilon})^{-2} \ll x^{0.5\varepsilon} \frac{1}{u^{0.5\varepsilon}} (d(u))^{c(\varepsilon)} \ll x^{0.5\varepsilon}.$$

Thus $\rho^{1/2} S''$ contributes to $D(x; q, r)$ a term $\ll (\frac{q}{u})^{1/2} x^{\varepsilon}$.

In case $(q, r) = u = 1$, there is only one term $a = 1$ in the summation

$$\sum_{\substack{u|a \\ p|a \Rightarrow p|u \\ (a/u, \rho) = 1 \\ a \le x}} .$$

Thus certain treatments can be omitted. The proof of Theorem 1.1 is finished.

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