A characterization of boolean spaces

C.E. Dickerson and M.E. Moore

A boolean space is a compact Hausdorff space which is zero-dimensional. In this paper, a boolean space $X$ is characterized in terms of its ring of real-valued functions $C(X)$. The result is sharpened for the case when $X$ is an $F$-space (every finitely generated ideal of $C(X)$ is principal).

1. Introduction

A boolean space is a compact Hausdorff space which is zero-dimensional. The purpose of this paper is to characterize a boolean space $X$ in terms of its ring of real-valued continuous functions $C(X)$. The result will be sharpened for the case when $X$ is an $F$-space (every finitely generated ideal of $C(X)$ is principal).

2. $B$-rings

Let $S$ be a commutative ring with identity 1, and let $\{M_\alpha \mid \alpha \in A\}$ be the set of all maximal ideals of $S$. The Jacobson radical of $S$ is the set $J(S) = \cap\{M_\alpha \mid \alpha \in A\}$. $S$ is called a $B$-ring if for each integer $n \geq 3$ and each $s_1, \ldots, s_n \in S$ such that $(s_1, \ldots, s_{n-2}) \notin J(S)$ and $1 \in (s_1, \ldots, s_n)$, there exists $t \in S$ such that $1 \in (s_1, \ldots, s_{n-2}, s_{n-1} + ts_n)$; see [4] for details. Here, the notation $(s_1, \ldots, s_n)$ means the ideal of $S$ generated by $s_1, \ldots, s_n$.

Since every set of the form $M_x = \{f \in C(X) \mid f(x) = 0\}$ is a maximal ideal of $C(X)$, it follows that if $g \in J(C(X))$ then $g \notin M_x$ for each

Received 29 October 1974.
$x \in X$ so that $g(x) = 0$ for each $x \in X$, or equivalently, $g = 0$.

Hence, $J(C(X)) = (0)$. We can now simplify the definition of $B$-rings in the special case of $C(X)$.

**PROPOSITION 2.1.** $C(X)$ is a $B$-ring if and only if $f, g, h \in C(X)$ with $f \neq 0$ and $1 \in (f, g, h)$ implies there exists $t \in C(X)$ such that $1 \in (f, g + th)$.

**Proof.** The direct implication is obvious. To see the converse let $n \geq 3$ with $(f_1, \ldots, f_{n-2}) \notin (0)$ and $1 \in (f_1, \ldots, f_n)$; then $f_1^2 + \cdots + f_{n-2}^2 \neq 0$ and $2 \left( f_1^2 + \cdots + f_n^2 \right) = \emptyset$, where $Z \left( f_1^2 + \cdots + f_n^2 \right)$ denotes the zero set of the function $f_1^2 + \cdots + f_n^2$. Note that $Z \left( f_1^2 + \cdots + f_{n-2}^2 + f_{n-1}^2 + f_n^2 \right) = \emptyset$ must also hold. Consequently, $1 \in \left( f_1^2 + \cdots + f_{n-2}^2 + f_n \right)$, $1 \in \left( f_1^2 + \cdots + f_{n-2}^2 + f_n - tf_n \right)$. By hypothesis, there exists $t \in C(X)$ such that $1 \in \left( f_1^2 + \cdots + f_{n-2}^2 + f_n - tf_n \right)$. From this we see that $Z \left( f_1^2 + \cdots + f_{n-2}^2 + [f_n - tf_n]^2 \right) = \emptyset$. Therefore, $1 \in \left( f_1, \ldots, f_{n-2}, f_n - tf_n \right)$.

3. $B$-rings and boolean spaces

In this section we shall assume that $X$ is a compact Hausdorff space. We begin by proving a lemma similar to Lemma 4.3 of [1].

**LEMMA 3.1.** Let $f, g, h \in C(X)$ and denote $g^{-1}(0, \infty)$ as $P(g)$ and $g^{-1}(-\infty, 0)$ as $N(g)$. If there is a connected subset $Z$ of $Z(f)$ such that $Z \cap P(g) \neq \emptyset$ and $(Z \cap N(g)) \neq \emptyset$, then for each $t \in C(X)$, $1 \notin (f, g + th)$.

**Proof.** Note that there must be $x, y \in Z$ such that $(g + th)(x) > 0$ and $(g + th)(y) < 0$. Since $Z$ is connected, the continuity of $g + th$ implies the existence of some $z \in Z$ such that $(g + th)(z) = 0$. This shows that $Z(f) \cap Z(g + th) \neq \emptyset$, or equivalently, $1 \notin (f, g + th)$.

**LEMMA 3.2.** If $C(X)$ is a $B$-ring, then for each closed connected set $Z$ and each closed set $S$, $Z \cap S$ must be connected.
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Proof. The proof follows Lemma 4.5 of [1]. Suppose that $Z$ is a closed connected set and that $S$ is a closed set such that $Z \cap S$ is not connected. Write $Z \cap S = F_1 \cup F_2$ where $F_1, F_2$ are disjoint non-empty closed subsets of $Z \cap S$, hence closed subsets of $X$. Since $X$ is assumed to be a compact Hausdorff space, and therefore normal, there are open sets $U_1 \supseteq F_1$ and $U_2 \supseteq F_2$ whose closures are disjoint. Put $U = U_1 \cup U_2$. The closed sets $Z - U$ and $S - U$ are disjoint, hence contained in disjoint open sets $V_1, V_2$ respectively. By Urysohn’s Lemma, choose $f, g, h \in C(X)$ such that $f(Z) = 0$ and $f(S - V_1 - U) = 1$, $g(U_1) = 1$ and $g(U_2) = -1$, $h(S) = 0$ and $h(U_2 - S - U) = 1$. Then $f, g, h$ satisfy the hypothesis of the previous lemma and $1 \notin (f, g, h)$. It follows that $C(X)$ is not a $B$-ring.

THEOREM 3.3. Let $X$ be a compact Hausdorff space. If $C(X)$ is a $B$-ring, then $X$ is a boolean space.

Proof. Let $x \in X$. If $C$ is the connected component of $X$ containing $x$, then $C$ is a closed connected set. If $C \neq \{x\}$ then it would follow that the discrete set $C \cap \{x, y\} = \{x, y\}$ must be connected, where $y \in C - \{x\}$. We conclude that $C = \{x\}$ and, hence, $X$ is totally disconnected. By compactness, $X$ is zero-dimensional.

Next we prove the converse of Theorem 3.3. We begin by defining $A(X)$ to be all those functions $f \in C(X)$ whose range is a finite set. In particular, $A(X)$ contains the constant functions. It is well known that for compact spaces $X$, we may apply the Stone-Weierstrass Theorem to conclude that $A(X)$ is dense in $C(X)$, under the topology of uniform convergence, if $X$ is zero-dimensional.

THEOREM 3.4. If $X$ is a boolean space, then $C(X)$ is a $B$-ring.

Proof. If $f, g, h \in C(X)$ with $1 \notin (f, g, h)$, then a straightforward computation shows that there exist $\delta, \varepsilon > 0$ such that if $f', g', h' \in C(X)$ with $|f - f'| < \varepsilon$, $|g - g'| < \varepsilon$, and $|h - h'| < \varepsilon$ then $|f'| + |g'| + |h'| > \delta$. Let $\xi = \min(\varepsilon, \delta/3)$ and choose $f', g', h' \in A(X)$ within $\xi$ of $f, g, h$ respectively. Note then that $|f'| + |g'| + |h'| > \delta$.

Since functions in $A(X)$ have finite range, it follows that there
exist functions \( u, v, w \in A(X) \) satisfying \( uf' = |f'| \), \( vg' = |g'| \), \( wh' = |h'| \), and \( |u| = |v| = |w| = 1 \). Define \( \sigma \in A(X) \) by
\[
\sigma = 1/(|f'| + |g'| + |h'|) < 1/\delta .
\]
Choosing \( p = wa \), \( g = va \), and \( t = w/v \) gives \( 1 = pf' + q(g'+th') \). Thus, we have appropriately written the identity in the subring \( A(X) \).

Now set \( d_1 = f' - f \), \( d_2 = g' - g \), and \( d_3 = h' - h \); then \( |d_i| < \xi \) for each \( i \) and
\[
1 = p(f+d_1) + q((g+d_2)+t(h+d_3)) = |pf+q(g+th)| + |pd_1+qd_2+qtd_3| .
\]
Letting \( s = |pd_1+qd_2+qtd_3| \) it follows that \( 1 - s \leq |pf+q(g+th)| \). By direct calculation,
\[
s \leq |p|\cdot|d_1| + |q|\cdot|d_2| + |q|\cdot|d_3| 
\leq (1/\delta)\cdot\xi + (1/\delta)\cdot\xi + (1/\delta)\cdot\xi \leq 1 .
\]
This gives that \( 0 < 1 - s \leq |pf+q(g+th)| \) so that \( pf + q(g+th) \) is a unit in \( C(X) \). Since \( pf + q(g+th) \in (f, g+th) \), it follows that \( 1 \in (f, g+th) \).

Thus, we have shown that a compact space \( X \) is a boolean space if and only if \( C(X) \) is a \( B \)-ring. It is interesting to note that we did not need \( f \neq 0 \).

By assuming \( X \) is Lindelöf and using the Stone-Čech compactification of \( X \), one can easily show that \( X \) is zero-dimensional if and only if \( C^*(X) \) is a \( B \)-ring.

4. \( SB \)-rings and boolean \( F \)-spaces

Let \( S \) be a commutative ring with identity. \( S \) is called an \( SB \)-ring if for each \( s, c, d, e \in S \) with \( s \in (c, d, e) \) and \( c \not\in J(S) \), it follows that \( s \in (c, d+te) \) for some \( t \in S \); see [4] for details.

A topological space \( X \) is called an \( F \)-space if every finitely generated ideal of \( C(X) \) is principal. \( X \) is called a \( T \)-space if \( C(X) \) is an Hermite ring; and \( X \) is called a \( U \)-space if for each \( f \in C(X) \) there exists a unit \( u \in C(X) \) such that \( f = u|f| \). In [1] it is shown that every \( U \)-space is a \( T \)-space.
THEOREM 4.1. Suppose $X$ is a compact $F$-space. Then $X$ is a boolean space if and only if $C(X)$ is an SB-ring.

Proof. Since every SB-ring is a $B$-ring [4, p. 457], it suffices to show that if $X$ is a boolean $F$-space then $C(X)$ is an SB-ring. Now, every boolean $F$-space is a $U$-space [1, Theorem 5.5]. Hence, $X$ is a $T$-space and $C(X)$ is a Hermite ring. Since Hermite $B$-rings are SB-rings [4, Theorem 3.3], it follows that $C(X)$ is an SB-ring.

References


