ON TOPOLOGICAL SPACES WITH A UNIQUE COMPATIBLE QUASI-UNIFORMITY

by LAWRENCE M. BROWN

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It is shown in [**2**] that a *uqu* space satisfies the following conditions.

(DC) There is no infinite, strictly decreasing sequence of open sets with open intersection.

(IC) There is no infinite, strictly increasing sequence of open sets.

In this note we show that for a transitive space these conditions are sufficient for the space to be *uqu*. This will follow as a consequence of the following result.

THEOREM 1. *Let $\mathscr{S}$ be a complete lattice of sets under the operations of intersection and union, in which all chains are finite. Then $\mathscr{S}$ is finite.*

Proof. This result is a consequence of standard theorems on lattice theory. See for example Corollary 2 of [**1**, p. 59] or Theorem 31 of [**3**, p. 116]. However, for the sake of completeness we give a direct proof. Since $\mathscr{S}$ is a complete lattice we have

$$A_x = A_x^{\mathscr{S}} = \cap \{S \in \mathscr{S} \mid x \in S\} \in \mathscr{S}$$

for all $x \in X = \cup \{S \mid S \in \mathscr{S}\}$. Also, for $S \in \mathscr{S}$, $S = \cup \{A_x \mid x \in S\}$ and so it will suffice that the number of distinct sets A_x $(x \in X)$ is finite. We will call A_a *minimal* if $x \in A_a$ implies $A_x = A_a$. Clearly distinct minimal sets are disjoint. Since all chains in $\mathscr{S}$ are finite we see there are just finitely many distinct minimal sets, say $A_{a_1}, \ldots, A_{a_n}$, that there exist a finite number of sets $A_{b_1}, \ldots, A_{b_m}$, which cover X, and that for any $x \in X$ there exist i, j with $A_{a_i} \subseteq A_x \subseteq A_{b_j}$. Hence we may complete the proof by showing that if $A_a \subset A_b$ then there are just finitely many distinct A_x with $A_a \subseteq A_x \subseteq A_b$. We will write $A_u < A_v$ if A_v is an immediate successor of A_u for the ordering $\subseteq$. Given $A_x \subset A_y$ there exists $z \in X$ with $A_x < A_z \subseteq A_y$, and hence there is a finite sequence $A_a < A_{z_1} < \ldots < A_{z_t} < A_b$, since $\mathscr{S}$ contains no infinite chains. Moreover, for the same reason, the number of terms in such a sequence is bounded and each A_x can only have a finite number of distinct immediate successors A_z, so there are only finitely many distinct sequences $A_a < A_{z_1} < \ldots < A_{z_t} < A_b$. Finally since each A_x with $A_a \subseteq A_x \subseteq A_b$ belongs to one of these sequences the proof is complete.

An examination of the above proof shows that we have actually established the slightly more general result that if $\mathscr{S}$ is a set of sets, and if $\mathscr{S}'$, the set of finite unions of the sets A_x $(x \in X)$ is a subset of $\mathscr{S}$ and contains no infinite chains, then $\mathscr{S}$ is finite.

We return now to the question of *uqu* spaces. We recall that an open cover $\mathscr{C}$ of the topological space $(X, \mathscr{T})$ is called a Q-cover if

$$A_x^{\mathscr{C}} = \cap \{C \in \mathscr{C} \mid x \in C\} \in \mathscr{T}$$

for all $x \in X$. The quasi-uniformity $\mathscr{U}_Q$ with subbase

$$\{U_{\mathscr{C}} = \cup \{\{x\} \times A_x^{\mathscr{C}} \mid x \in X\} \mid \mathscr{C} \quad \text{is a } Q\text{-cover for} \quad (X, \mathscr{T})\}$$

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is the largest compatible transitive quasi-uniformity, and $(X, \mathcal{T})$ is called transitive if $\mathcal{U}_Q$ is actually the largest compatible quasi-uniformity.

Clearly a *uqu* space is transitive.

THEOREM 2. *A transitive space which satisfies the conditions* (DC) *and* (IC) *is uqu.*

Proof. By Corollary 3.5 of **[2]** it is sufficient to show that $\mathcal{U}_Q$ is totally bounded. Hence the result will follow if we can show that for any Q-cover $\mathcal{C}$ the number of distinct sets $A_x^{\mathcal{C}}$ $(x \in X)$ is finite. For if these sets are given by $A_{x_1}, \ldots, A_{x_n}$ and we set $B_i = \{x \mid A_{x_i} = A_x\}$, then $B_i \times B_i \subseteq U_{\mathcal{C}}$ and $B_1, B_2, \ldots, B_n$ cover X. But if we let $\mathcal{S}$ be the set of arbitrary unions of the sets $A_x^{\mathcal{C}}$ $(x \in X)$ together with $\varnothing$, we see $\mathcal{S}$ is a complete lattice of open sets and it follows from (DC) and (IC) that all chains in $\mathcal{S}$ are finite. Hence the result follows at once from Theorem 1.

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REFERENCES

1. G. Birkhoff, *Lattice theory*, 3rd ed., Amer. Math. Soc. Colloquium Publications 25 (1967).

2. W. F. Lindgren, Topological spaces with a unique compatible quasi-uniformity, *Canad. Math. Bull.* **14** (1971), 369–372.

3. G. Szasz, *Einführung in die Verbandtheorie* (Teubner, Leipzig, 1962).

HACETTEPE UNIVERSITY
DEPARTMENT OF MATHEMATICS
BEYTEPE CAMPUS
ANKARA
TURKEY

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