ON TIDY SUBGROUPS OF
LOCALLY COMPACT TOTALLY DISCONNECTED GROUPS

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Willis has established a structure theory of locally compact totally disconnected groups. An important feature of this theory is the notion of a tidy subgroup. In this paper we provide several results regarding these subgroups.

In recent years, Willis (see [3, 4]) has established several structure theorems for locally compact totally disconnected groups. An important feature of this structure theory is the notion of a tidy subgroup, whose definition is recalled:

DEFINITION: (Willis) Let $G$ be a totally disconnected locally compact group and let $f$ be a bicontinuous automorphism of $G$. Let $U$ be a compact open subgroup of $G$ and set

$$U_+ = \bigcap_{n=0}^{\infty} f^n(U) \quad \text{and} \quad U_- = \bigcap_{n=0}^{\infty} f^{-n}(U).$$

Then $U$ is said to be tidy for $f$ if it satisfies:

1. \quad $U = U_+ U_- = U_- U_+$, and
2. \quad $\bigcup_{n=0}^{\infty} f^n(U_+)$ and $\bigcup_{n=0}^{\infty} f^{-n}(U_-)$ are closed in $G$.

Willis has proved the existence of tidy subgroups and has described properties that are possessed by these subgroups. Using Willis' results, we provide some additional observations regarding tidy subgroups.

Throughout this paper, $G$ will denote a locally compact totally disconnected group. Let $f$ be a bicontinuous automorphism of $G$. We begin by recalling the following result of Willis [3, Lemma 1].

LEMMA 1. Let $U$ be a compact open subgroup of $G$. Then there is an integer $k$ such that for all $l \geq k$, $U(l) = \bigcap_{n=0}^{l} f^n(U)$ satisfies T1.
REMARK. Note that $U(l)_+ = \bigcap_{n=0}^{\infty} f^n(U(l)) = U_+$, while $U(l)_- = \bigcap_{n=0}^{\infty} f^{-n}(U(l)) = U(l) \cap U_-$. Therefore, for $l \geq k$, $U(l)_+ = U(k)_+$ and $U(l)_- \subseteq U(k)_-$. Let $U$ be a compact open subgroup of $G$, and let $U_{++} = \bigcup_{n=0}^{\infty} f^n(U_+)$. The following proposition gives necessary and sufficient conditions for $U_{++}$ to be closed.

**Proposition 2.** Let $U$ be a compact open subgroup of $G$. Then $U_{++}$ is closed if and only if $U_{++} \cap U \subseteq f^l(U_+)$ for some $l \geq 0$.

**Proof:** Suppose $U_{++}$ is closed. Since $U_{++}$ is a countable union of compact subgroups, by the Baire Category Theorem, $f^k(U_+)$ must be open for some $k$. Thus $U_+$ is open. Since $\{ f^n(U_+) \cap U \}_{n=0}^{\infty}$ is an open cover of $U_{++} \cap U$ and is monotonically increasing, $U_{++} \cap U \subseteq \bigcup_{n=0}^{l} f^n(U_+) \cap U \subseteq f^l(U_+) \cap U \subseteq f^l(U_+)$. Conversely, if $U_{++} \cap U \subseteq f^l(U_+)$, then $U_{++} \cap U = f^l(U_+) \cap U$, which is closed. Therefore $U_{++}$ is closed by [2, Chapter III, Proposition 2.4].

**Remark.** Willis has shown ([3, Lemma 3]) that if $U$ satisfies T1, then $U_{++}$ is closed if and only if $U_{++} \cap U = U_+$ (that is, $l = 0$ in Proposition 2). Moreover, [3, Lemma 3] also shows that $U$ is tidy if and only if $U_{++}$ is closed.

Let $V$ be a compact open subgroup of $G$ satisfying T1. Define $L_V = \{ g \in G : f^n(g) \in V \mbox{ for all but finitely many integers } n \}$ and let $L_V$ be the closure of $L_V$. Then $L_V \subseteq V_{++}$ and $L_V$ is a compact $f$-invariant subgroup of $G$ ([3, Lemma 6]).

**Proposition 3.** Let $U$ be a compact open subgroup of $G$. If $U$ satisfies T1, then $U$ is tidy if and only if $L_U = L_U \subset U_{++}$.

**Proof:** If $U_{++}$ is closed, then $L_U = L_U \subset U_{++}$. Conversely, if $U_{++}$ is not closed, then the proof of [3, Lemma 3] shows that there exists a sequence $z_1, z_2, \ldots \in U_{++} \cap U$ such that $\lim_{n \to \infty} z_n = w \notin U_{++}$. In fact, the $z_n$ are contained in $L_U$. Thus $L_U = L_U \subset U_{++}$. 

**Corollary 4.** Let $U$ be a compact open subgroup of $G$ satisfying T1 but not T2. Let $V = U \cap L_U$. Then $V_{++}$ is not closed and $V_{++}$ is dense in $L_U$.

**Proof:** Now $V_{++} = \bigcup_{n=0}^{\infty} f^n(V_+) = \bigcup_{n=0}^{\infty} f^n((U \cap L_U)_+) = \bigcup_{n=0}^{\infty} f^n(U_+ \cap L_U) = \bigcup_{n=0}^{\infty} (f^n(U_+) \cap L_U) = U_{++} \cap L_U \supseteq L_U$. Thus $V_{++}$ is dense in $L_U$. Since $L_U \notin U_{++}$ (Proposition 3), $V_{++} \neq L_U$. Therefore, $V_{++}$ is not closed.

**Remark.** $V = U \cap L_U$ is a compact open subgroup of $L_U$. By Lemma 1, $V(l) = \bigcap_{n=0}^{l} f^n(V)$ satisfies T1 for some integer $l$. However, since $V(l)_+ = V_+$, $V(l)$ is not tidy in $L_U$. 

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In [4] Willis shows how to construct tidy subgroups in three steps. Given an arbitrary compact open subgroup \( U \) of \( G \), the first step is to find a compact open subgroup \( V \subseteq U \) such that \( V \) satisfies T1. This step is precisely Lemma 1. The second step is to identify a particular compact \( f \)-invariant subgroup \( L \) of \( G \). Here we take \( L = L_V \). The third step uses \( L_V \) and \( V \) to produce a tidy subgroup \( W \). Define \( V' = \{ v \in V : l v l^{-1} \in Vl \text{ for all } l \in L \} \). Then \( V' \) is an open subgroup of \( V \) and \( W = V'L \) is a tidy subgroup of \( G \) ([4, Theorem 3.1]). Due to this construction we have the next proposition.

**Proposition 5.** Let \( V \) be a compact open subgroup of \( G \) satisfying T1. Then there exists a tidy subgroup \( V'L \) such that \( V_+ = (V'L)_+ \).

**Proof:** Let \( L = L_V \) and let \( V' \) be as above. Then \( V'L \) is a tidy subgroup of \( G \). By [4, Lemma 3.4], there exists an integer \( p \) such that \( f^{-p}(V_+) \subseteq V'_+ \). Then \( V_+ = \bigcup_{n=0}^{\infty} f^n(V_+) \subseteq \bigcup_{n=0}^{\infty} f^n(f^{-p}(V_+)) \subseteq \bigcup_{n=0}^{\infty} f^n(V'_+) \subseteq V'_+ \subseteq (V'L)_+ \). Thus \( V_+ \subseteq (V'L)_+ \) by [4, Lemma 3.7]. Since \( (V'L)_+ \) is closed, \( V_+ \subseteq (V'L)_+ \). On the other hand, since \( V'_+ \subseteq V_+ \) and \( L \subseteq V_+ \), \( V'_+ L = (V'L)_+ \subseteq V_+. \) Therefore, \( V_+ = (V'L)_+ \).

**Remark.** Given any compact open subgroup \( U \) of \( G \), there exists an integer \( l \) such that \( U(l) = \bigcap_{n=0}^{l} f^n(U) \) satisfies T1. Then \( U(l)_+ = U_+ \) so \( U(l)_+ = U_+ \). Thus, by Proposition 5, there exists a tidy subgroup \( U' \) such that \( U_+ = U'_+ \).

**Lemma 6.** Let \( U \) be a compact open subgroup of \( G \) and let \( z \in G \).

\( a \) \hspace{1cm} \( z \notin U_+ \) if and only if there is a sequence \( 0 < n_1 < n_2 < \cdots < n_k < \cdots \) such that \( z \notin \bigcup_{k=1}^{\infty} f^{n_k}(U) \).

\( b \) \hspace{1cm} If \( U \) is tidy, then \( z \notin U_+ \) if and only if there exists \( l \geq 0 \) such that \( z \notin \bigcup_{n=l}^{\infty} f^n(U) \).

**Proof:** (a) If \( z \notin f^l(U_+) \), then there exists \( l' > l \) such that \( z \notin f^{l'}(U) \). Thus if \( z \notin U_+ \), then there exists \( 0 < n_1 < n_2 < \cdots < n_k < \cdots \) such that \( z \notin \bigcup_{k=1}^{\infty} f^{n_k}(U) \).

Conversely, suppose \( z \notin \bigcup_{k=1}^{\infty} f^{n_k}(U) \). Since \( z \notin f^{n_k}(U) \), \( f^{-n_k}(z) \notin U \), which implies that \( f^{-n_k}(z) \notin U_+ \). Thus \( z \notin f^{n_k}(U_+) \) for all \( n_k \), where \( n_k \to \infty \). Since \( \{ f^n(U_+) \}_{n=0}^{\infty} \) is monotonically increasing, \( z \notin \bigcup_{n=0}^{\infty} f^n(U_+) = U_+ \).

(b) If no such \( l \) exists, then there exists \( n_1 < n_2 < \cdots < n_k < \cdots \) such that \( z \in f^{n_k}(U) \) for all \( n_k \). By [3, Lemma 6], \( f^{-n_1}(z) \in L_U U_+ \). Since \( U \) is tidy,
\[ L_U = L_U = U_+ \cap U_- \subseteq U_+ \text{ ([3, Corollary to Lemma 3])}. \] Therefore, \( L_U U_+ = U_+ \), and \( z \in f^{n_1}(U_+) \subseteq U_{++} \).

The converse follows from part (a).

**Proposition 7.** Suppose \( U \) is a compact open subgroup of \( G \) satisfying T1. Then \( U \) is tidy if and only if the following two conditions hold.

1. Let \( z \in G \). Then \( z \notin U_{++} \) if and only if there exists \( l \geq 0 \) such that \( z \notin \bigcup_{n=l}^{\infty} f^n(U) \).
2. \( \bigcup_{n=l}^{\infty} f^n(U) \) is closed for each \( l \geq 0 \).

**Proof:** If \( U \) is tidy, then condition (1) is satisfied by Lemma 6(b). Since \( U \) is tidy, \( f^l(U) \) is tidy for each \( l \geq 0 \), so \( \bigcup_{n=l}^{\infty} f^n(U) \) is closed by [3, Proposition 1].

Conversely, conditions (1) and (2) together imply that \( U \) is tidy because from (1) we have that \( G \setminus U_{++} = \bigcup_{l=0}^{\infty} \left( G \setminus \bigcup_{n=l}^{\infty} f^n(U) \right) \) and from (2) we have \( G \setminus \bigcup_{n=l}^{\infty} f^n(U) \) is open for each \( l \). Therefore, \( G \setminus U_{++} \) is an open subset, so \( U_{++} \) is closed.

**Proposition 8.** Let \( U \) be a compact open subgroup of \( G \) satisfying T1. Then \( U \) is tidy if and only if \( \bigcup_{n \in F} f^n(U) \) is closed, where \( F \) is any subset of non-negative integers.

**Proof:** By Lemma 6(a), we have that \( G \setminus U_{++} = \bigcup_{F \in \mathcal{F}} \left( G \setminus \bigcup_{n \in F} f^n(U) \right) \), where \( \mathcal{F} \) is the family of all infinite subsets of non-negative integers. Thus, if we assume that \( \bigcup_{n \in F} f^n(U) \) is closed for each \( F \), then \( U_{++} \) is also closed.

Conversely, suppose \( U \) is tidy. To show that \( \bigcup_{n \in F} f^n(U) \) is closed for any \( F \in \mathcal{F} \), we can use the same proof that Willis used to show that \( \bigcup_{n=0}^{\infty} f^n(U) \) is closed ([3, Proposition 1]).

Recall that for every locally compact group, there is a topology on the space of closed subgroups (see [1, Chapter VIII, Section 5]). In this topology, a sequence of closed subgroups \( \{S_n\} \) converges to \( S \) if for every compact set \( K \) and for every neighbourhood \( V \) of \( e \), there exists an integer \( N \) such that for all \( n > N \), \( S \cap K \subseteq VS_n \) and \( S_n \cap K \subseteq VS \).

**Proposition 9.** Let \( U \) be a tidy subgroup of \( G \). Then \( \{f^n(U)\} \) converges to \( U_{++} \).

**Proof:** Let \( K \) be a compact set and let \( V \) be a neighbourhood of \( e \).

Since \( U_{++} \) is closed, \( U_+ \) is open and \( U_{++} \cap K \) is compact. Thus \( U_{++} \cap K \subseteq f^{N_1}(U_+) \) for some \( N_1 \). Therefore, if \( z \in U_{++} \cap K \), then \( z \in f^{N_1}(U_+) \), which implies...
that \( z \in f^n(U) \) for all \( n > N_1 \). Thus \( U_{++} \cap K \subseteq Vf^n(U) \) for all \( n > N_1 \).

Now we want to show that there exists an integer \( N_2 \) such that for all \( n > N_2, f^n(U) \cap K \subseteq VU_{++} \). Suppose not. Then there exists a sequence \( 0 < n_1 < n_2 < \cdots < n_i < \cdots \) such that \( f^{n_i}(U) \cap K \not\subseteq VU_{++} \). That is, there exists \( z_i \in f^{n_i}(U) \cap K \) such that \( z_i \not\in VU_{++} \). Since \( \bigcup_{i=1}^{\infty} f^{n_i}(U) \) is closed (Proposition 8), \( \bigcup_{i=1}^{\infty} f^{n_i}(U) \cap K \) is compact. Thus, by passing to a subsequence if necessary, we may assume that the sequence \( \{z_i\} \) converges to \( z \). Since \( z_i \not\in U_{++} \) and \( U_{++} \) is open, \( z \not\in U_{++} \). By Proposition 7 there exists an \( l \) such that \( z \not\in \bigcup_{n=l}^{\infty} f^n(U) \). Now \( \bigcup_{n=l}^{\infty} f^n(U) \) is closed (Proposition 7), so there exists a neighbourhood \( W \) of \( e \) such that \( zW \cap \bigcup_{n=l}^{\infty} f^n(U) = \emptyset \). Given \( W \), there exists \( N_3 \) such that for all \( i > N_3 \), \( z_i \in zW \). Choose \( m > N_3 \) such that \( n_m > l \). Then \( z_m \in zW \cap \bigcup_{n=l}^{\infty} f^n(U) \), which is a contradiction. Thus, there exists \( N_2 \) such that for all \( n > N_2 \), \( f^n(U) \cap K \subseteq VU_{++} \).

If \( U \) is a compact open subgroup of \( G \), then \( U_+ \cap U_- \) is a compact \( f \)-invariant subgroup of \( G \). In particular, if \( U \) is tidy, the next proposition shows that \( U_+ \cap U_- \) is a maximal compact \( f \)-invariant subgroup of \( U_{++} \).

**Proposition 10.** Let \( U \) be a tidy subgroup of \( G \). Then \( U_+ \cap U_- \) is a maximal compact \( f \)-invariant subgroup of \( U_{++} \).

**Proof:** Suppose \( M \) is a compact \( f \)-invariant subgroup of \( U_{++} \) which contains \( U_+ \cap U_- \). Since \( \{f^n(U_+)\}_{n=0}^{\infty} \) is an open cover of \( U_{++} \) and is monotonically increasing, \( M \subseteq f^l(U_+) \) for some \( l \geq 0 \). If \( l = 0 \), then \( M \subseteq U_+ \subseteq U \). Thus \( M = U_+ \cap U_- \). If \( l > 0 \), then \( f^{-l}(M) \subseteq U_+ \subseteq U \). Since \( f^{-l}(M) \) is \( f \)-invariant and \( U_+ \cap U_- \subseteq f^{-l}(M) \subseteq U_+ \), \( f^{-l}(M) = U_+ \cap U_- \). Thus \( M = f^l(U_+ \cap U_-) = U_+ \cap U_- \).

**Corollary 11.** Let \( U \) be a compact open subgroup of \( G \). If \( U \) satisfies T1, then \( U \) is tidy if and only if \( U_+ \cap U_- \) is a maximal compact \( f \)-invariant subgroup of \( U_{++} \).

**Proof:** If \( U \) is tidy, then \( U_{++} \) is closed. By Proposition 10, \( U_+ \cap U_- \) is a maximal compact \( f \)-invariant subgroup of \( U_{++} \).

Conversely, suppose \( U_+ \cap U_- \) is a maximal compact \( f \)-invariant subgroup of \( U_{++} \). Since \( U_+ \cap U_- \subseteq L_U \subseteq L_U \subseteq U_{++} \) and \( L_U \) is a compact \( f \)-invariant subgroup, \( L_U = L_U = U_+ \cap U_- \). Thus \( U \) is tidy by [3, Corollary to Lemma 3].

We conclude this paper by considering compact totally disconnected groups.

**Proposition 12.** Suppose \( G \) is compact. Let \( U \) be a compact open subgroup of \( G \).

\((a)\) \( U_{++} \) is closed if and only if \( f(U_+) = U_+ \).
U is tidy if and only if $f(U) = U = U_+ = U_-$. 

Proof:

(a) Suppose $U_+ = \bigcup_{n=0}^{\infty} f^n(U_+)$ is closed. Then $U_+ = \bigcup_{n=0}^{\infty} f^n(U_+)$ is compact. Thus there exists an $l$ such that $U_+ = f^l(U_+)$ for all $n \geq l$. Hence $f(U_+)$ is closed. Conversely, if $f(U_+)$ is closed, then $U_+ = f(U_+)$ is also compact. 

(b) If $U$ is tidy, then $U$ satisfies T1, so $[f(U), f(U) \cap U] = [f(U)_+, f(U)_+]$ (see [3, p. 354]). Thus $f(U) \subseteq U$. But $U$ is also tidy for $f^{-1}$, so $f^{-1}(U) \subseteq U$. Hence $f(U) = U$. 

References