Can. J. Math. Vol. 44 (5), 1992 pp. 1050–1060

THE RAMSEY PROPERTY FOR FAMILIES OF GRAPHS WHICH EXCLUDE A GIVEN GRAPH

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ABSTRACT. For graphs A, B and a positive integer r, the relation $A \longrightarrow (B)^1_r$ means that whenever Δ is an r-colouring of the vertices of A, then there is an embedding ϕ of B into A such that $\Delta \circ \phi$ is constant. A class of graphs $\mathcal{F}$ has the *Ramsey property* if, for every $B \in \mathcal{F}$, there is an $A \in \mathcal{F}$ such that $A \longrightarrow (B)^1_2$. For a given finite graph G, let Forb(G) denote the class of all finite graphs which do not embed G. It is known that, if G is 2-connected, then Forb(G) has the Ramsey property, and Forb(G) has the Ramsey property if and only if Forb($\bar{G}$) also has the Ramsey property. In this paper we show that if neither G nor its complement $\bar{G}$ is 2-connected, then either (i) G has a cut point adjacent to every other vertex, or (ii) G has a cut point adjacent to every other vertex except one. We show that Forb(G) has the Ramsey property if G is a path of length 2 or 3, but that Forb(G) does not have the Ramsey property if (i) holds and G is not the path of length 2.

1. **Introduction.** We only consider finite, undirected, simple graphs, $\mathbf{K}_n$ denotes the complete graph on n vertices. If A is a graph and X is a subset of the set of vertices $V(A)$, we denote by $A|X$ the induced subgraph on X, also we write $A - X$ instead of $A|\big(V(A) - X\big)$. For any vertex x of A we denote by $\Gamma_A(x)$ the subgraph $A|\{y : \{x, y\}$ an edge of $A\}$. As usual $\bar{A}$ denotes the complement of the graph A. A graph is *connected* if any two vertices may be joined by a path. The graph A is k-connected if $A - X$ is connected for any set $X \subseteq V(A)$ with $|X| < k$. If A is not a complete graph, the *connectivity* of A is the largest integer k such that A is k-connected. If A is connected, a *cutpoint* of A is a vertex u such that $A - u$ is not connected. For graphs A, B, an *embedding* of A in B is a map $\phi: V(A) \longrightarrow V(B)$ such that $\forall a, a' \in V(A)$, $\{a, a'\}$ is an edge of A if and only if $\{\phi(a), \phi(a')\}$ is an edge of B; in other words if A is isomorphic to some induced subgraph of B.

For graphs A, B and a positive integer r, the relation $A \longrightarrow (B)^1_r$ means that whenever Δ is an r-colouring of the vertices of A, then there is an embedding ϕ of B into A such that $\Delta \circ \phi$ is constant. A class of graphs $\mathcal{F}$ has the *Ramsey property* if, for every $B \in \mathcal{F}$, there is an $A \in \mathcal{F}$ such that $A \longrightarrow (B)^1_2$. It is easily seen that if $\mathcal{F}$ is Ramsey, then it has the seemingly stronger property that, for any positive integer r, for every $B \in \mathcal{F}$, there is an $A \in \mathcal{F}$ such that $A \longrightarrow (B)^1_r$. It also follows immediately from the definition that $\mathcal{F}$ is Ramsey if and only if the class $\bar{\mathcal{F}} = \{\bar{A} : A \in \mathcal{F}\}$ of complementary graphs is also Ramsey. For a set of graphs $\mathcal{L}$ we denote by Forb($\mathcal{L}$) the family of all graphs A which do not embed any member $L \in \mathcal{L}$. In particular, if $\mathcal{L} = \{G\}$ we write Forb(G) instead

The second author has been supported in part by NSERC grant 63-1325.
Received by the editors October 30, 1990.

of Forb($\mathcal{L}$). It is known [2] (see Theorem 1.2 below) that, if G is a 2-connected graph, then the class of graphs Forb(G) is Ramsey.

A *hypergraph* $\mathcal{H}$ is a pair (V, E), where $V = V(\mathcal{H})$ is the set of vertices, and $E = E(\mathcal{H}) \subseteq \wp(V)$ is the set of edges of $\mathcal{H}$. $\mathcal{H}$ is *r-uniform* if $|e| = r$ for every $e \in E$. A *circuit* of length n in $\mathcal{H}$ is a finite sequence of distinct vertices $x_1, \ldots, x_n$ such that there are distinct hyperedges $e_1, \ldots, e_n$ such that $x_i, x_{i+1} \subseteq e_i$, where $x_{n+1} = x_1$. In particular, if two hyperedges intersect in two or more points, they form a circuit of length 2. The *girth* of $\mathcal{H}$ is the length of the smallest circuit in $\mathcal{H}$. A subset $X \subseteq V(\mathcal{H})$ is *independent* if it contains no hyperedge of $\mathcal{H}$. The *chromatic number* of $\mathcal{H}$ is the least cardinal k such that $V(\mathcal{H})$ is a union of k independent subsets.

We shall make frequent use of the following theorem of Erdös & Hajnal [1].

THEOREM 1.1 ([1]). *For any positive integers r, k, l there is an r-uniform hypergraph $\mathcal{H}$ of girth l with no independent set of size $\frac{1}{k}|V(\mathcal{H})|$ (and so has chromatic number $> k$).*

To illustrate how Theorem 1.1 is used in the present context, we begin by reproving the fact mentioned above.

THEOREM 1.2 [2]. *If $\mathcal{L}$ is a finite set of 2-connected graphs, then* Forb(L) *is Ramsey.*

PROOF. Let $B \in$ Forb $\mathcal{L}$, and let $\mathcal{H}$ be a $|B|$-uniform hypergraph of chromatic number 3 and girth g, where $g > 3$ and g exceeds the number of vertices of every $L \in \mathcal{L}$. Consider the graph A on $V(\mathcal{H})$ in which an isomorphic copy of B is placed in each hyperedge of $\mathcal{H}$; note that two distinct hyperedges meet in only one point, so that A can be constructed in this way. Obviously $A \longrightarrow (B)^1_2$ since $\mathcal{H}$ is 3-chromatic. We need only check that $A \in$ Forb $\mathcal{L}$. Suppose for a contradiction that A embeds some $L \in \mathcal{L}$. Since B does not embed L and L is 2-connected, A must contain vertices which form a circuit in $\mathcal{H}$. But this contradicts the fact that g exceeds the number of vertices of L. ∎

The question arises whether there is an graph G such that Forb(G) is not Ramsey?

2. **Graphs such that G and $\bar{G}$ are not 2-connected.**

To answer the question stated at the end of the last section, we need only consider those graphs G such that neither G nor its complement $\bar{G}$ is 2-connected. In this section we give a description of such graphs.

Denote by $\mathcal{M}$ the class of those graphs G with the property that there is a cut point $u \in V(G)$ which is joined by an edge to every other vertex. Also, denote by $\mathcal{K}$ the class of graphs G such that there is a cut point $u \in V(G)$ which is joined by an edge to every other vertex except one. For example, $P_2 \in M$ and $P_3 \in K$, where P_n denotes the path of length of n.

We say that the graph G is *n-partite* if there is a partition of $V(G)$ into n disjoint non-empty sets A_i $(1 \leq i \leq n)$ such that $\{x, y\}$ is an edge of G whenever x, y belong to different A_i's.

LEMMA 2.1. *If $\bar{G}$ is disconnected, then either $G \in \mathcal{M}$ or G has connectivity $k > 1$.*

PROOF. Since $\bar{G}$ is disconnected, G is n-partite for some $n \geq 2$. Therefore G is connected and has connectivity $k \geq 1$. If $k = 1$, then there is a cut point u. Therefore, $G - u$ is disconnected and its complement $\overline{G - u} = \bar{G} - u$ is connected. It follows that $\{u\}$ is a component of $\bar{G}$, and hence $G \in \mathcal{M}$. ∎

THEOREM 2.2. *If neither G nor $\bar{G}$ is 2-connected, then $G \in \mathcal{M} \cup \bar{\mathcal{M}} \cup \mathcal{K} \cup \bar{\mathcal{K}}$.*

PROOF. By Lemma 2.1 we can assume that G and $\bar{G}$ are both connected and have connectivity 1. Let u be a cutpoint of G and v a cutpoint of $\bar{G}$. Then $u \neq v$ since $G - v$ is connected and $G - u$ is not, and by Lemma 2.1 either $\bar{G} - u \in \mathcal{M}$ or $\bar{G} - u$ has connectivity $k \geq 2$.

Suppose that $\bar{G} - u \in \mathcal{M}$. Then there is a vertex w joined in $\bar{G}$ to all other points of $\bar{G} - \{u, w\}$, and $\bar{G} - \{u, w\}$ is disconnected. Since G is connected, it follows that $\{u, w\}$ is an edge of G. If u is joined to every other vertex by an edge of G, then $G \in \mathcal{M}$. Suppose that u is not joined to all other points in G. If $w = v$, then $\bar{G} \in \mathcal{K}$, and so $G \in \bar{\mathcal{K}}$. On the other hand, if $w \neq v$, then $\bar{G} - v$ has the two components $\{u\}$ and $\bar{G} - \{u, v\}$. Therefore, u is joined to every vertex in $G - v$, and since $\{u, v\}$ is not an edge of G, it follows that $G \in \mathcal{K}$.

Suppose then that $\bar{G} - u$ is 2-connected. Then $\bar{G} - \{u, v\}$ is connected, and so the components of $\bar{G} - v$ are $\{u\}$ and $\bar{G} - \{u, v\}$. Therefore, u is joined to all points of $G - \{u, v\}$ by edges of G. But $\{u, v\}$ is not an edge of G since $\bar{G}$ is connected. Since u is a cut point of G it follows that $G \in \mathcal{K}$. ∎

3. **Amalgamation properties.**

The family of graphs $\mathcal{F}$ has the *join-embedding* property if

$$\forall A, B \in \mathcal{F}\ \exists C \in \mathcal{F} \quad (\exists \text{ embeddings } \phi: A \longrightarrow C, \psi: B \longrightarrow C). \tag{1}$$

$\mathcal{F}$ has the *amalgamation property* if

$$\forall A, B \in \mathcal{F}, a \in V(A), b \in V(B)\ \exists C \in \mathcal{F}\big(\exists \text{ embeddings } \phi: A \longrightarrow C, \psi: B \longrightarrow C \text{ such that } \phi(a) = \psi(b)\big). \tag{2}$$

If the condition in (2) holds, we say that C *amalgamates* A and B on $a \simeq b$. Finally, we say that $\mathcal{F}$ has the *disjoint amalgamation property* if ϕ, ψ in (2) can be chosen so that, in addition,

$$\phi\big(V(A - a)\big) \cap \psi\big(V(B - b)\big) = \emptyset.$$

and, in this case we say that C *disjointly amalgamates* A and B on $a \simeq b$.

LEMMA 3.1. *For any graph G,* Forb(G) *has the join-embedding property.*

PROOF. Let $A, B \in$ Forb(G). We can assume that $V(A)$ and $V(B)$ are disjoint. If G is connected, then the disjoint sum $A \oplus B \in$ Forb(G). If G is disconnected $\overline{\bar{A} \oplus \bar{B}} \in$ Forb(G). ∎

For the next theorem we need the following known fact which follows easily by induction on k: *If the outdegrees in a directed graph $\mathcal{D}$ are at most k, then the chromatic number of $\mathcal{D}$ is at most 3^k.*

THEOREM 3.2. *If $\mathcal{F}$ is Ramsey and has the join-embedding property, then $\mathcal{F}$ has the disjoint amalgamation property.*

PROOF. We first show that $\mathcal{F}$ has the ordinary amalgamation property. Suppose for a contradiction that this is false. Then there are $A, B \in \mathcal{F}$, $a \in V(A)$, $b \in V(B)$ which witness this failure. Since $\mathcal{F}$ has the join-embedding property and is Ramsey, there are $C, D \in \mathcal{F}$ such that $C \longrightarrow (D)^1_2$ and D embeds both A and B. Colour a vertex x of C blue if there is an embedding $\phi: B \longrightarrow C$ such that $\phi(b) = x$; otherwise, colour x red. Now consider any embedding $\psi: D \longrightarrow C$. By our choice of D, there are embeddings $\alpha: A \longrightarrow D$, $\beta: B \longrightarrow D$. Clearly, $\psi(\beta(b))$ is blue. If $x = \psi(\alpha(a))$ is coloured blue, then there is some embedding $\phi: B \longrightarrow C$ such that $x = \phi(b)$. Since $\psi \circ \alpha$ is also an embedding of A into C with $\psi(\alpha(a)) = x$, this contradicts our assumption that A, B cannot be amalgamated on $a \simeq b$ in any graph $C \in \mathcal{F}$. It follows therefore, that $x = \psi(\alpha(a))$ is red. This shows that every copy of D in C contains both blue and red vertices, and this contradicts the fact that $C \longrightarrow (D)^1_2$.

We now show that $\mathcal{F}$ has the stronger disjoint amalgamation property. As above, we assume that this is false and that $A, B \in \mathcal{F}$, $a \in V(A)$, $b \in V(B)$ witness this, so that no $C \in \mathcal{F}$ disjointly amalgamates A and B on $a \simeq b$. Since $\mathcal{F}$ has the amalgamation property and is Ramsey, there are $C, D \in \mathcal{F}$ such that $C \longrightarrow (D)^1_r$, where $r = 3^{|B|-1}$, and D amalgamates A and B on $a \simeq b$. Let $\alpha: A \longrightarrow D$, $\beta: B \longrightarrow D$ be embeddings such that $\alpha(a) = \beta(b)$. For $x \in V(C)$, if there is an embedding $\psi: D \longrightarrow C$ such that $\psi(\alpha(a)) = \psi(\beta(b)) = x$, then we choose one such embedding, say ψ_x, and define $T_x = \psi_x(\beta(B - b))$; if there is no such ψ, we put $T_x = \emptyset$. Now consider the directed graph $\mathcal{D}$ on $V(C)$ in which there is a directed edge from x to y if and only if $y \in T_x$. The outdegree of each vertex of $\mathcal{D}$ is at most $|B| - 1$, and so the chromatic number is at most $3^{|B|-1}$. Let $\Delta: V(C) \longrightarrow 3^{|B|-1}$ be any vertex colouring of $\mathcal{D}$ such that no two vertices having the same colour are joined in $\mathcal{D}$. Now let $\chi: D \longrightarrow C$ be any embedding and let $x = \chi(\beta(b)) = \chi(\alpha(a))$. Since C does not disjointly amalgamate A and B on $a \simeq b$, it follows that there is some $y \in \chi(\alpha(A - a)) \cap \psi_x(\beta(B - b))$. Now $y \in T_x$ and so $\Delta(x) \neq \Delta(y)$. Thus $\chi(D)$ contains two vertices x, y with different colours for the colouring Δ. But this contradicts the fact that $C \longrightarrow (D)^1_r$. ∎

4. $\mathrm{Forb}(P_2)$ **and** $\mathrm{Forb}(P_3)$ **are both Ramsey.** The fact that $\mathrm{Forb}(P_2)$ is Ramsey follows immediately from the fact that $G \in \mathrm{Forb}(P_2)$ if and only if G is a disjoint union of complete graphs. For, if $B \in \mathrm{Forb}(P_2)$ and B has k components each of size at most l, then $A \longrightarrow (B)^1_2$, where A is the graph consisting of $2k - 1$ disjoint copies of the complete graph $\mathbf{K}_{2l-1}$. The fact that $\mathrm{Forb}(P_3)$ is Ramsey is not quite so obvious.

For disjoint subsets U, V of $V(G)$ let $[U, V] = \{\{u, v\} : u \in U, v \in V\}$. A *series-parallel* partition of G is a partition $V(G) = U \cup V$ into two disjoint, non-empty sets U, V such that either $[U, V] \subseteq E(G)$ or $[U, V] \subseteq E(\bar{G})$. The next theorem gives a useful characterization of P_3-free graphs.

THEOREM 4.1. *If* $G \in \mathrm{Forb}(P_3)$ *and* $|V(G)| > 1$, *then there is a series-parallel partition of* G.

PROOF. The proof is by induction on $|V(G)|$. Since $P_3 \cong \overline{P_3}$, we may assume that G is connected and that $|V(G)| > 2$. Let $a \in V(G)$. By the induction hypothesis, $V(G-a) = U \cup V$, where U, V are non-empty disjoint sets and either $[U, V] \subseteq E(G)$ or $[U, V,] \subseteq E(\bar{G})$. If a is joined to every other vertex of G, then $\{a\} \cup \big(V(G) - \{a\}\big)$ is a series-parallel partition of $V(G)$. Thus we may assume that there are are $u \in U$, $v \in V$ such that $\{a, u\} \notin E(G)$, $\{a, v\} \in E(G)$. Suppose that $[U, V] \subseteq E(\bar{G})$. Then, since G is connected, there is a path $u = x_0, \ldots, x_r = a$, v which is an induced subgraph of G, and so G embeds P_3. Therefore, $[U, V] \subseteq E(G)$. Let $W = \{z \in V : \{a, z\} \in E(G)\}$. If $W = V$ then $[U \cup \{a\}, V]$ is a series-parallel partition, so we can assume that W, $V - W$ are both non-empty. Suppose there are $x \in W$ and $y \in V - W$ such that $\{x, y\} \in E(\bar{G})$. Then a, x, u, y is an induced P_3. Therefore, $[W, V - W] \subseteq E(G)$, and so $[U \cup (V - W \cup \{a\}, W]$ is a series-parallel partition of G. ∎

THEOREM 4.2. $\mathrm{Forb}(P_3)$ *is Ramsey.*

PROOF. As before we shall denote by $A \oplus B$ the disjoint sum of the graphs A, B. Also, we shall denote by $A \odot B$ the graph on $A \times B$ in which two vertices (a, b), (a', b') are joined by an edge if and only if either $(i) b = b'$ and $\{a, a'\} \in E(A)$, or $(ii) \{b, b'\} \in E(B)$.

We first show that $\mathrm{Forb}(P_3)$ is closed under the operation $\odot$. Suppose for a contradiction that A, B are P_3-free and that (a_0, b_0), (a_1, b_1), (a_2, b_2), (a_3, b_3) is an induced path in $A \odot B$. If the b_i are all equal, then $a_0, \ldots, a_3$ is an induced P_3 in A. Similarly, if all the b_i are distinct, then $b_0, \ldots, b_3$ is an induced P_3 in B. Hence there are $\{i, j, k\} \subseteq \{0, 1, 2, 3\}$ such that $b_i = b_j \neq b_k$ and $|k - i| = 1$, $|k - j| > 1$. Therefore, $\{b_i, b_k\} \in E(B)$, and since $b_j = b_i$, it follows that $\{a_j, b_j\}$ is joined to $\{a_k, b_k\}$ in $A \odot B$; but this is a contradiction since $|k - j| > 1$.

Let $B \in \mathrm{Forb}(P_3)$. We want to show that there is some $A \in \mathrm{Forb}(P_3)$ such that $A \longrightarrow (B)^1_2$. If there is such an A we denote one such graph by $R(B)$. Note that if $B_1, B_2 \in \mathrm{Forb}(P_3)$ and if $R(B_1)$, $R(B_2)$ both exist, then $R(B_1) \odot R(B_2) \longrightarrow (B_1 \odot B_2)^1_2$. For consider any two-colouring Δ: $V\big(R(B_1)\big) \times V\big(R(B_2)\big) \longrightarrow 2$. For each vertex y of $R(B_2)$ let $V(y) = \big\{(x, y) : x \in V\big(R(B_1)\big)\big\}$. Then $R(B_1) \odot R(B_2) | V(y)$ is isomorphic to $R(B_1)$ and so there are $\epsilon_y \in \{0, 1\}$ and an embedding ϕ_y of B_1 into $R(B_1)$ such that $\Delta\big(\phi_y(x), y\big) = \epsilon_y\big(\forall x \in V(B_1)\big)$. Also, there are $\epsilon \in \{0, 1\}$ and an embedding ψ of B_2 into $R(B_2)$ such that $\epsilon_{\psi(y)} = \epsilon\big(\forall y \in V(B_2)\big)$. Now consider the embedding χ of $B_1 \odot B_2$ into $R(B_1) \odot R(B_2)$ given by $\chi(x, y) = \big(\phi_{\psi(y)}(x), \psi(y)\big)$. Clearly, $\Delta\big(\chi(x, y)\big) = \epsilon_{\psi(y)} = \epsilon$.

We now show that $R(B)$ exists for all $B \in \mathrm{Forb}(P_3)$ by induction on $|B| = |V(B)|$. By Theorem 4.1, since $\bar{P}_3 \cong P_3$, we can assume that $B = C \oplus D$ is the disjoint union of two non-empty sugraphs. By the induction hypothesis $R(C)$ and $R(D)$ both exist. Clearly, $F \longrightarrow (D \oplus D)^1_2$, where $F = R(D) \oplus R(D) \oplus R(D)$, and by the above, $A = R(C) \odot F \longrightarrow \big(C \odot (D \oplus D)\big)^1_2$. But $C \odot (D \oplus D) \cong (C \odot D) \oplus (C \odot D)$, and since $C \odot D$ embeds both C and D, it follows that $A \longrightarrow (C \oplus D)^1_2$, *i.e.* $A \longrightarrow (B)^1_2$. ∎

5. **Graphs G such that** Forb(G) **is not Ramsey.** In the last section we proved that Forb(G) is Ramsey for $G = P_2$ or $G = P_3$. The main result, which will be proved in this and the next section, is that Forb(G) is not Ramsey if $G \in \mathcal{M} - \{P_2\}$. It is not known if the same is true for $G \in \mathcal{K} - \{P_3\}$, although Zhu and Sauer [4] have proved this for a certain subset of these G's.

THEOREM 5.1. Forb(G) *is not Ramsey if* $G \in \mathcal{M} - \{P_2\}$.

PROOF. Let $G \in \mathcal{M} - \{P_2\}$, $|V(G)| = n$. By Lemma 3.1 and Theorem 3.2, in order to show that Forb(G) is not Ramsey, it will be enough to construct two graphs $A(G)$, $B(G) \in$ Forb(G) and two vertices a, b in these graphs such that $A(G)$ and $B(G)$ cannot be disjointly amalgamated on $a \simeq b$.

Since $G \in \mathcal{M}$, there is a cutpoint u of G which is adjacent to every other vertex of G. Let K be a component of $G - u$ of minimum cardinality and let $C = V(G) - (K \cup \{u\})$. For an integer $r \geq 2$, let $\mathcal{H}_r$ be a $|C|$-uniform hypergraph having chromatic number $r + 1$ and girth ≥ 4, and let $W = V(\mathcal{H}_r)$. For each hyperedge E of $\mathcal{H}_r$, let ψ_E be a fixed $1 - 1$ map from E onto C. We now define a graph $A_r(G) \in$ Forb(G) as follows. The vertex set of $A_r(G)$ is $W \cup \{x\}$, where $x \notin W$. Two distinct vertices y, y' of $A_r(G)$ are joined by an edge if and only if either (i) $x \in \{y, y'\}$, or (ii) $\{y, y'\} \not\subseteq E$ for any $E \in E(\mathcal{H}_r)$, or (iii) $y, y' \in E \in \mathcal{H}_r$ and $\{\psi_E(y), \psi_E(y')\} \in E(G)$. Thus $A_r(G)|E \cong G|C$ for any hyperedge E.

We need to show that $A_r(G)$ does not embed G. Suppose for a contradiction that α is an embedding of G into $A_r(G)$. Assume first that K contains at least two different vertices. If a, b belong to different components of $G - u$, then $\alpha(a)$ and $\alpha(b)$ must belong to the same hyperedge E of $\mathcal{H}_r$. It follows that $\alpha\big(V(G - u)\big) \subseteq E$. But this is impossible since $|E| = |C| < |V(G - u)|$. Let us now assume that $V(K) = \{v\}$. Let T be a largest induced subgraph of C such that $\bar{T}$ is a connected component of $\bar{C}$. Observe that to every vertex $a \in V(T)$ there is an edge E_a of H which contains both vertices $\alpha(v)$ and $\alpha(a)$. Because the girth of H is at least four there is only one such edge E_a for every vertex $a \in V(T)$. If $a, b \in V(T)$ are two vertices for which $E_a \neq E_b$, then $\alpha(a)$ and $\alpha(b)$ are adjacent in $A_r(G)$ because H does not contain a circle of length three. Then a and b are adjacent vertices of T. But this means that $V\big(T|\alpha(E_a)\big)$ is disconnected from $V\big(T|\alpha(E_B)\big)$ in $\bar{T}$ in contradiction to $\bar{T}$ being connected. Hence there is some edge E of H such that $V\big(\alpha(T)\big) \cup \{\alpha(v)\} \subseteq E$. There is an embedding ϕ_E from $A_r(G)|E$ to C. Observe that the complement of the graph $A_r(G) \,\Big|\, \Big(V\big(\alpha(T)\big) \cup \{\alpha(v)\}\Big)$ is connected. Hence the complement of the graph $\phi_E\Big(A_r(G) \,\Big|\, \Big(V\big(\alpha(T)\big) \cup \{\alpha(v)\}\Big)\Big)$ is connected. This is in contradiction to the choice of T as a largest connected component of $\bar{C}$.

The remainder of the proof splits into several different cases.

CASE 1: $|K| = 1$. In this case we put $A(G) = A_m(G)$, where $m = 3(n - 1)$. Also, we let $B(G)$ be the graph on $m + 1$ points $\{x_0, \ldots, x_m\}$ in which $\{x_i, x_j\}$ is an edge if and only if either $|i - j| = 1$ or $i = 3r, j = 3s$ and $\{f(r), f(s)\} \in E(G)$, where $f\colon n - 1 \longrightarrow V(G - u)$ is a fixed surjection.

https://doi.org/10.4153/CJM-1992-064-7 Published online by Cambridge University Press

We have already shown that $A(G) \in \mathrm{Forb}(G)$. We now verify that $B(G) \in \mathrm{Forb}(G)$ also. Suppose β is an embedding of G in $B(G)$. Then $\beta(u) = x_{3p}$ for some p since u has degree greater than two. But the size of the largest component of $B(G)|\{y : \{x_{3p}, y\} \in E\big(B(G)\big)\}$ is $\max\{1, t-1\}$, where t is the size of the largest component in $G - u$. Thus there cannot be an embedding unless $t = 1$. But in this case $G - u$ has no edges, $B(G)$ is a path and x_{3p} has degree at most two.

We now show that if D is any graph in which $A(G)$ and $B(G)$ can be disjointly amalgamated on $x \simeq x_0$, where x is the special vertex of $A(G)$ joined to every other vertex, then $D \notin \mathrm{Forb}(G)$. Without loss of generality we may assume that $V\big(A(G)\big), V\big(B(G)\big) \subseteq V(D)$, $x = x_0$ and $V\big(A(G)\big) \cap V\big(B(G)\big) = \{x\}$ and that the identity maps on $A(G)$ and $B(G)$ are embeddings in D. If $v \in V(D) - V\big(B(G)\big)$ is such that $\{v, x_i\} \in E(D)$ for all $i \leq m$, then $D|\{v\} \cup \{x_{3i} : i < n\}$ is an isomorphic copy of G. Therefore, for each $v \in V(D) - V\big(B(G)\big)$, there is a least index $i(v) \leq m$ such that $\{v, x_{i(v)}\} \notin E(D)$. Note that $i(a) \neq 0$ if $a \in V\big(A(G) - x\big)$ since $x = x_0$ is joined to every other vertex of $A(G)$. Consider the vertex colouring of $A(G) - x$ in which a is coloured $i(a)$. Since $V\big(A(G) - x\big) = W = V(\mathcal{H}_m)$ and $\mathcal{H}_m$ has chromatic number $m + 1$, there are $1 \leq i \leq m$ and some hyperedge E of $\mathcal{H}_m$ such that $\{a, x_i\} \notin E(D)$ and $\{a, x_{i-1}\} \in E(D)$ for all $a \in E$. But $D|E$ is isomorphic to $G|C$. Therefore, $D|E \cup \{x_{i-1}, x_i\}$ is isomorphic to G. ∎

Before considering the other cases in detail we give a construction which will be useful for these cases.

For graphs D, Z we say that Z is *t-dense* in D if, for any subset $Y \subseteq V(D)$ of cardinality $|Y| \geq \frac{1}{t}|V(D)|$, there is an embedding of Z into $D|Y$; this is stronger than the assertion that $D \longrightarrow (Z)^1_t$.

For an integer $t \geq 1$ let $\mathcal{M}_t$ be an $(n-1)$-uniform hypergraph with girth ≥ 4 and having no independent set of size $\frac{1}{t}|V(\mathcal{M})|$. For each hyperedge E of $\mathcal{M}$, let ϕ_E be a surjective map from E onto $V(G - u)$. Let D_t be a graph such that $V(D_t) = V(\mathcal{M}_t)$ and $\{a, b\}$ is an edge if and only if $\{a, b\} \subseteq E$ for some hyperedge E and $\{\phi_E(a), \phi_E(b)\} \in E(G)$. Since $\mathcal{M}_t$ contains no 'large' independent set, it follows that $G - u$ is t-dense in D_t. We also have the following fact.

LEMMA 5.2. *$D = D_t$ does not embed $G - K$.*

PROOF. Suppose α is an embedding of $G - K$ in D. Let $\mathcal{E} = \{E : \alpha(u) \in E \in E(\mathcal{M})\}$. Since $\mathcal{M}$ has girth ≥ 4, it follows that $E \cap E' = \{\alpha(u)\}$ for $E \neq E'$ in $\mathcal{E}$, and whenever $\{a, b\} \in E(G - K)$ there is some $E \in \mathcal{E}$ such that $\{\alpha(a), \alpha(b)\} \subseteq E$. Thus α maps each connected component of $G - K$ into a unique $E \in \mathcal{E}$. If $B \neq K$ is a component of $G - u$ of largest size, then there is some $E \in \mathcal{E}$ such that $\alpha(B) \subseteq E$. Thus $\alpha(B) \cup \{\alpha(u)\}$ is a subset of some connected component, say A, in $D|E$. But this is impossible since $|A| > |B|$ and there is an embedding ϕ_E of $D|E$ into $G - u$. ∎

CASE 2: $G - u$ HAS JUST TWO COMPONENTS EACH ISOMORPHIC TO $\mathbf{K}_k$. Let $t = k+1$, $D = D_t$, $d = |D|$, $m = d(k+1)$, and let $V(D) = \{a_i : i \in d\}$. In this case we define the graph $B(G)$ on the set $\{x_i : i \in m\}$ in which $\{x_i, x_j\}$ is an edge if and only if either

$1 \leq |i-j| \leq k$ or if $i \equiv j \bmod k+1$ and $\{a_p, a_q\} \in E(D_t)$, where $p = [i/k+1]$ and $q = [j/k+1]$ (and $[x]$ is the integer part of x). Thus, $B(G)$ embeds $k+1$ disjoint copies of D_t.

Note that, since the hyperedges of M_t interesect in at most one point, for any vertex a of D_t, the graph $\Gamma_{D_t}(a)$ consists of a number of disjoint copies of $\mathbf{K}_{k-1}$. Therefore, for any vertex x_i of $B(G)$, the graph $\Gamma_{B(G)}(x_i)$ does not contain two vertex-disjoint $\mathbf{K}_k$'s, and so $B(G)$ does not embed G.

For this case we let $A = A(G)$ be the complete graph $\mathbf{K}_{2^m.k}$, and x any vertex of $A(G)$. We claim that A and $B = B(G)$ cannot be disjointly amalgamated at $x \simeq x_0$ in any graph $J \in \text{Forb}(G)$. Assume to the contrary that there is such a graph J. We may assume that A, B are induced subgraphs of J with the single common vertex $x = x_0$. Consider the colouring Δ of $A - x$ which associates to ever vertex a of $A - x$ the set of all $x_i \in V(B)$ adjacent to a in J. Let $S \subseteq V(B)$ be any subset with the property that there is some $x_i \in S$ such that $i + k < m$ and $S \cap \{x_j : i < j \leq i + k\} = \emptyset$. Then $|\Delta^{-1}(S)| < k$. For, if $T \subseteq \Delta^{-1}(S)$ and $|T| = k$, then $J|(T \cup \{x_j : i \leq j \leq i+k\})$ is isomorphic to G. It follows that there is some vertex $y \in V(A)$ such that $\Delta(y)$ is not such a set S. Since $x_0 \in \Delta(y)$, it follows that, for every set of indices $I \subseteq m$ consisting of k consecutive integers, there is some $i \in I$ such that x_i is joined to y in J. Thus $|\Delta(y)| \geq \frac{m}{k}$ and so $\Delta(y)$ contains at least $\frac{m}{k(k+1)} = \frac{d}{k} > \frac{d}{k+1}$ vertices from one of the $k+1$ disjoint copies of D_t in B. Since $G - u$ is t-dense in D_t, it follows that $\Delta(y)$ embeds $G - u$. This contradicts our assumption that $J \in \text{Forb}(G)$. ∎

6. **The remaining cases.** In order to complete the proof in the remaining cases we will define three graphs B_0, B_1, B_2 (which depend upon G). These three graphs will have a common vertex set V and a special vertex $x_0 \in V$, and will be increasing in the sense that $E(G_0) \subseteq E(G_1) \subseteq E(G_2)$. We do not claim that these three graphs all belong to Forb(G), but, in each case, at least one of them is a member of Forb(G). We will also define a graph $A = A(G) \in \text{Forb}(G)$ and $x \in V(A)$, and show that, for each $i \in 3$, A and B_i cannot be disjointly amalgamated on $x \simeq x_0$ in any graph $J \in \text{Forb}(G)$. The theorem, of course, follows from this.

For the remainder of the proof we let $t = k^2$, $D = D_t$, $d = |V(D)|$, where D_t is the graph defined in the preceding section after Lemma 5.2. We put $A = A_r(G)$, where $r = (k+2)^d$, and, as before, $\mathbf{x}$ is the special vertex of A joined to every other vertex.

We now proceed to describe the three graphs B_0, B_1, B_2. The common vertex set is $V = \{x_0\} \cup Y \cup Z$, where $Y = \{y_{ij} : i \in d, j \in k\}$ and $Z = \{z_{ijl} : i \in d, j \in k, l \in k\}$. Let $Y_i = \{y_{ij} : j \in k\}$, $Z_{ij} = \{z_{ijl} : l \in k\}$ and $P_{jl} = \{z_{ijl} : i \in d\}$. For each $i \in d, j \in k$, $l \in k$ let $\phi_i \colon Y_i \longrightarrow K$, $\sigma_{ij} \colon Z_{ij} \longrightarrow K$, $\psi_{jl} \colon P_{jl} \longrightarrow V(D)$ be surjective maps; assume also that $\phi_i(y_{i0})$ and $\sigma_{ij}(z_{ij0})$ are vertices of K having minimal degree, and that $\phi_i(y_{i1})$ is a vertex of K having maximal degree.

The edges of B_0 are as follows. Two distinct vertices $a, b \in V$ are joined by an edge of B_0 if and only if one of the following conditions is satisfied:

- $\{a, b\} \subseteq Y_i$ for some $i \in d$ and $\{\phi_i(a), \phi_i(b)\} \in E(G)$.

- $\{a,b\} \subseteq Z_{ij}$ for some $i \in d, j \in k$ and $\{\sigma_{ij}(a), \sigma_{ij}(b)\} \in E(G)$.
- $\{a,b\} \subseteq P_{jl}$ for some $j,l \in k$ and $\{\psi_{jl}(a), \psi_{jl}(b)\} \in E(D)$.
- $\{a,b\} = \{x_0, y\}$ for some $y \in Y$.
- $\{a,b\} = \{y_{ij}, z_{ijl}\}$ for some $i \in d, j \in k, l \in k$.

$\{a,b\}$ is an edge of B_1 if and only if it is an edge of B_0, or

- $\{a,b\} = \{y_{e0}, y_{f0}\}$ for some $e,f \in d$ $(e \neq f)$.

Finally, $\{a,b\}$ is an edge of B_2 if and only if it is an edge of B_1, or

- $\{a,b\} = \{y_{i,j+1}, z_{ij0}\}$ for some $i \in d, j \in k$ (and $j+1$ is taken modulo k).

We now show that, if A is as described at the beginning of this section, and if $B = B_i$ for some $i \in 3$, then A and B cannot be disjointly amalgamated on $a \simeq b$ in any graph $J \in \mathrm{Forb}(G)$.

Assume for a contradiction that A, B are induced subgraphs of $J \in \mathrm{Forb}(G)$ and that $x = x_0$. For each vertex $a \in W = V(\mathcal{H}_r)$, we shall define a function $f_a \colon d \longrightarrow \{x\} \cup Y \cup \{q\}$, where $q \notin V = V(J)$, as follows. Let $i \in d$. If a is not joined to any vertex of Y_i in J, put $f_a(i) = x$. Suppose now that a is joined to some vertex $y \in Y_i$. If there is some $j \in k$ such that $\{a, y_{ij}\} \in E(J)$ and a is not joined (in J) to some $z \in Z_{ij}$, then put $f_a(i) = y_{ij}$, where j is the least index with this property. If, on the other hand, a is joined to some $z \in Z_{ij}$ whenever a is joined to y_{ij}, then put $f_a(i) = q$. This defines the function f_a for each $a \in W$. Suppose for some hyperedge $E \in E(\mathcal{H}_r)$, we have $f_a(i) = x$ for some $i \in d$ and all $a \in E$. Then $J|E \cup \{x\} \cup Y_i$ is isomorphic to G, a contradiction. Similarly, if there are a hyperedge $E \in E(\mathcal{H}_r)$ and $i \in d, j \in k$ such that $f_a(i) = y_{ij}$, then $J|E \cup \{y_{ij}\} \cup Z_{ij}$ is an isomorphic copy of G, again a contradiction. Because of this, and because $\mathcal{H}_r$ has chromatic number greater than $r = (k+2)^d$, it follows that, for some $a \in W, f_a$ is the function which assumes the constant value q. Therefore, for some $j \in k$ and $l \in k$, a is adjacent to at least $\frac{1}{k^2}$ of the vertices in P_{jl}. Since $J|P_{jl} \cong D$ and $G - u$ is k^2-dense in D, it follows that J contains an isomorphic copy of G.

All that remains is to prove our earlier claim that, if G is not one of the graphs covered in Cases 1 & 2, then one of the graphs B_i $(i \in 3)$ belongs to $\mathrm{Forb}(G)$.

CASE 3: THE CONNECTED COMPONENTS OF $G - u$ ARE NOT ALL ISOMORPHIC. In this case we show $B = B_0 \in \mathrm{Forb}(G)$. Suppose not and that α defines an embedding of G into B. Let J be a connected component of C which is not isomorphic to K. Since the connected components of $\Gamma_B(x_0)$ are all isomorphic to K, it follows that $\alpha(u) \neq x_0$. The connected components of $\Gamma_B(y_{ij})$ are $Q = \{x_0\} \cup \big(Y_i \cap \Gamma_B(y_{ij})\big)$, and Z_{ij}. Thus, if $\alpha(u) = y_{ij}$, then $G - u$ has just the two connected components J and K. Moreover, K is isomorphic to $B|Z_{ij}$, and so J is isomorphic to $B|Q$. It follows that Q has exactly k elements, so that y_{ij} must be adjacent to every other vertex of Y_i in B. Therefore, since x_0 is also adjacent to every other vertex in Q, it follows that $J \cong B|Q \cong B|Y_i \cong K$, and this is a contradiction. The only remaining possibility is that $\alpha(u) = z_{ijl}$ for some $i \in d$, $j \in k$, $l \in k$. However, $\Gamma_B(z_{ijl}) \subseteq P_{jl} \cup Z_{ij} \cup \{y_{ij}\}$. Since $P_{jl} \cong D$ it does not embed $G - K$ by Lemma 5.2, and it follows that there is some component $L \ncong K$ of G such that $\alpha(L) \nsubseteq P_{jl}$. Consequently, $\alpha(L) \subseteq Z_{ij} \cup \{y_{ij}\}$. But, since $|L| \geq |K| = k$ this implies that

z_{ijl} is joined to every other vertex of Z_{ij} so that $L \cong J|(Z_{ij} - \{z_{ijk}\}) \cup \{y_{ij}\} \cong J|Z_{ij} \cong K$, and this is a contradiction. ■

CASE 4: THE CONNECTED COMPONENTS OF $G - u$ ARE PAIRWISE ISOMORPHIC TO K, $K > 1$, AND EITHER $G - u$ HAS AT LEAST THREE COMPONENTS OR K HAS NO VERTEX OF DEGREE $k - 1$. We will prove in this case that $B = B_1 \in \text{Forb}(G)$. Suppose for a contradiction that α is an embedding of G into B. Suppose $\alpha(u) = x_0$. Since $\Gamma_B(x_0) = Y$, it follows that $\alpha(K) \cap S \neq \emptyset$, where $S = \{y_{i0} : i \in d\}$. Since $B|S$ is a complete graph, it follows that $\alpha(L) \cap S = \emptyset$ for every other component L of $G - u$. But this is a contradiction since the connected components of $B|Y - S$ have cardinality at most $k - 1$.

Suppose that $\alpha(u) = y_{i0}$ for some $i \in d$. The connected components of $\Gamma_B(y_{i0})$ are Z_{i0} and $T = (S - \{y_{i0}\}) \cup \{x_0\} \cup U$, where U is the set of vertices in Y_i adjacent to y_{i0}. If $x_0 \notin \alpha(G)$, then the only possible connected components of $\alpha(G - u)$ are P, Q, R, where $P \subseteq S - \{y_{i0}\}$, $Q \subseteq U$ and $R \subseteq Z_{i0}$. We must have $Q = \emptyset$ since $|U| < k$, and so $G - u$ has two components each isomorphic to $\mathbf{K}_k$, and this was dealt with in Case 2. Similarly, if $x_0 \in \alpha(G)$, then $G - u$ must have two connected components each isomorphic to K and, moreover, K must contain a vertex joined to every other vertex.

Suppose that $\alpha(u) = y_{ij}$ for some $i \in d$ and $j \in k - \{0\}$. The connected components of $\Gamma_B(y_{ij}\})$ are $(Y_i - \{y_{ij}) \cup \{x_0\}$ and Z_{ij}. Again we see that x_0 is in the image of G and so K contains a vertex adjacent to every other vertex.

Finally, if $\alpha(u) = z_{ijl}$ for some $i \in d, j \in k, l \in l$, we use exactly the same argument as for the preceding case.

CASE 5: $G - u$ HAS TWO CONNECTED COMPONENTS EACH ISOMORPHIC TO K, K IS NOT A COMPLETE GRAPH AND HAS A VERTEX OF DEGREE $k - 1$. In this case we show that $B = B_2 \in \text{Forb}(G)$. Assume that the two components of $G - u$ are K and K', and that $\alpha : G \longrightarrow B$ is an embedding. The same argument used in Case 4 shows that $\alpha(u) \neq x_0$. Suppose $\alpha(u) = y_{i0}$. Since y_{i0} has degree $< k - 1$ in $B|Y_i$, we can assume it is not adjacent to $y_{i,k-1}$ and so $\alpha(K)$ and $\alpha(K')$ are subsets either of $Z_{i0} \cup \{z_{i,k-1,0}\}$ or of $(\{x_0\} \cup Y_i \cup S) - \{y_{i0}\}$ where, as before, $S = \{y_{r0} : r \in d\}$. If $x_0 \notin \alpha(G - u)$, then $\alpha(G - u)$ fails to have two components of size k. So we can assume that $x_0 \in \alpha(K)$ and $\alpha(K) \cap S \neq \emptyset$, and also that $z_{i00} \in \alpha(K')$. Therefore, $y_{il} \notin \alpha(G)$ since it is adjacent ot z_{i00}. Since y_{il} is adjacent to y_{i0}, it follows that $|\alpha(K) \cap Y_i| < p$, where $p < k - 1$ is the minimum degree of a vertex in the graph K. Since K is not a complete graph, $\alpha(K) \cap Y_i \neq \emptyset$ and so $B|\alpha(K)$ contains a vertex of degree $< p$, and therefore is not isomorphic to K.

Suppose $\alpha(u) = y_{ij}$ for some $i \in d, j \in k - \{0\}$. In this case $\alpha(K \cup K') \subseteq \{x_0\} \cup (Y_i - \{y_{ij}\}) \cup Z_{ij} \cup \{z_{i,j-1,0}\}$. Suppose $Z_{ij} = \alpha(K')$. Then $y_{i,j+1} \notin \alpha(K \cup K')$ since $y_{i,j+1}$ is adjacent to z_{ij0}. Therefore, we must have $\alpha(K) = \{x_0\} \cup (Y_{ij} - \{y_{ij}, y_{i,j+1}\}) \cup \{z_{i,j-1,0}\}$. It follows that K has a vertex of degree one, and hence exactly one vertex of degree $k - 1$. Therefore, we must have $j = 1$. But then $B|\{x_0\} \cup (Y_i - \{y_{ij}, y_{i,j+1}\}) \cup \{z_{i,j-1,0}\}$ contains no vertex of degree $k - 1$, and this is a contradiction. Similarly, if $Z_{ij} \not\subseteq \alpha(K \cup K')$, then $y_{i,j+1}$ together with points of Z_{ij} must form one component of $\alpha(G - u)$, say $\alpha(K')$. But then we are led to conclude, just as before, that $\alpha(K)$ contains a vertex of degree one and so K has just one vertex of degree $k - 1$, whereas $\alpha(K)$ contains no vertex of degree $k - 1$.

The only remaining possibility is that $\alpha(u) = z_{ijl}$ for some $i \in d, j \in k, l \in k$. In this case, since $\Gamma_B(z_{ijl}) \subseteq \{y_{ij}, y_{i,j+1}\} \cup Z_{ij} \cup P_{jl}$, for some connected component of $G - u$, say K', it must be the case that $\alpha(K') \subseteq P_{jl}$. But this is impossible since $P_{ijl} \cong D$ and, by Lemma 5.2, D does not embed $G - K$. ∎

REFERENCES

1. P. Erdős and A. Hajnal, *On chromatic number of graphs and set systems*, Acta. Math. Acad. Sci. Hung. **17**(1966), 61–99.
2. J. Nešetřil and V. Rödl, *Partitions of vertices*, Comment. Math. Univ. Carolina **17**(1976), 85–95.
3. ———, *Partitions of finite relational and set systems*, J. Combin. Theory (A) **22**, 289–312.
4. N. Sauer and X. Zhu, *Graphs which do not embed a given graph and the Ramsey property*, manuscript.

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