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COINCIDENCE SETS OF COINCIDENCE PRODUCING MAPS

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ABSTRACT. A theorem by H. Robbins shows that every closed and non-empty subset of the unit ball B^n in Euclidean n-space is the fixed point set of a selfmap of B^n. This result is extended to coincidence producing maps of B^n, where a map $f:X\to Y$ is coincidence producing (or universal) if it has a coincidence with every map $g:X\to Y$. The main result implies that if $f:B^n, S^{n-1}\to B^n, S^{n-1}$ is coincidence producing and $A\subset B^n$ closed and non-empty, then there exist a map $f':B^n, S^{n-1}\to B^n, S^{n-1}$ and a map $g:B^n\to B^n$ such that $f'\mid S^{n-1}$ is homotopic to $f\mid S^{n-1}$ and A is the coincidence set of f' and g.

1. **The problem.** In 1967 H. Robbins [5] gave a simple proof of the fact that every closed and non-empty subset of the unit ball B^n in Euclidean n-space R^n is the fixed point set of a map $f:B^n\to B^n$. This result has since been extended by several authors, and it is known that the class of spaces which have the property that every closed and non-empty subset can be realized as the fixed point set of a continuous selfmap includes e.g. all compact topological manifolds, locally finite simplicial complexes with the weak topology, 1-dimensional Peano continua and locally compact metrizable topological groups. See [9] for a survey and for references.

Here we consider a related problem for the coincidence set

$$\text{coin}(f, g)=\{x\in B^n \mid f(x)=g(x)\}$$

of two maps $f, g:B^n\to B^n$. Clearly every closed subset A of B^n can occur as such a coincidence set, and Robbins' idea of proof suffices to show that if A is also non-empty, then f can be prescribed not only as the identity (as in the fixed point case), but also in a more general manner (see the Lemma in §2). But a less trivial extension of Robbins' problem arises if f is prescribed as a *coincidence producing* map, where a map $f:X\to Y$ is called coincidence producing [7] (or, equivalently, ***universal*** [2]) if $\text{coin}(f, g)\neq\varnothing$ for every $g:X\to Y$.

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It was shown in [6] that $f: B^n \to B^n$ is coincidence producing if it maps the boundary S^{n-1} of B^n essentially onto itself, and hence we ask whether any given closed and non-empty subset A of B^n can be realized as the coincidence set coin(f, g) if f is prescribed as such a coincidence producing map. In general this is not true (see §3b), and we can only prove that for any given map $f: B^n, S^{n-1} \to B^n, S^{n-1}$ (whether coincidence producing or not) there exists a map $f': B^n, S^{n-1} \to B^n, S^{n-1}$ with $f' \mid S^{n-1}$ homotopic to $f \mid S^{n-1}$ so that coin$(f', g) = A$ for some $g: B^n \to B^n$ (see the Theorem in §2).

After [6] was published, Holsztyński [3], [4] showed that a map $f: B^n \to B^n$ is coincidence producing if and only if the restriction of f to $f^{-1}(S^{n-1})$ is essential. It is an open problem to find conditions on $A \subset B^n$ which ensure that A is the coincidence set of a prescribed coincidence producing map $f: B^n \to B^n$ and any map $g: B^n \to B^n$, and to find an analogue of the Theorem in §2 for arbitrary coincidence producing maps.

2. **The result.** The following lemma is a straightforward extension of Theorem 1 in [5] and its proof, but it will be useful.

LEMMA. *Let $f: B^n \to B^n$ be a map and A a closed non-empty subset of B^n. If there exists a point $c \in B^n$ so that $f^{-1}(c) \subset A$, then $A = \text{coin}(f, g)$ for some map $g: B^n \to B^n$.*

Proof. Define g by

$$\overrightarrow{Og(x)} = \overrightarrow{Of(x)} + \tfrac{1}{2} d(x, A) \overrightarrow{f(x)c},$$

where O is the origin and d the Euclidean metric.

Now we state and prove our result. The symbol $\simeq$ denotes a homotopy.

THEOREM. *Let $f: B^n, S^{n-1} \to B^n, S^{n-1}$ be a map and A a closed and non-empty subset of B^n. Then there exists a map $f': B^n, S^{n-1} \to B^n, S^{n-1}$ so that*

(i) $f' \mid S^{n-1} \simeq f \mid S^{n-1}$,

(ii) coin$(f', g) = A$ for some $g: B^n \to B^n$.

Proof. We assume $n \geq 2$, as the Theorem can easily be proved directly if $n = 1$.

Case 1). $A \not\subset S^{n-1}$. Select a point $a \in A$ with $a \notin S^{n-1}$, consider B^n as the cone aS^{n-1} resp. OS^{n-1}, and define $f': B^n, S^{n-1}, a \to B^n, S^{n-1}, O$ with $f' \mid S^{n-1} = f \mid S^{n-1}$ as the cone on $f \mid S^{n-1}$. Then apply the Lemma.

Case 2). $A \subset S^{n-1}$ and A not finite. Homotope $f \mid S^{n-1}$ to $f'' \mid S^{n-1}: S^{n-1} \to S^{n-1}$ so that $(f'' \mid S^{n-1})^{-1}(c)$ is a finite set $\{b_1, b_2, \ldots, b_k\}$ for some $c \in S^{n-1}$, and define $f'': B^n, S^{n-1}, O \to B^n, S^{n-1}, O$ as the cone on $f'' \mid S^{n-1}$. It is now possible to select points $a_1, a_2, \ldots, a_k$ in A and an orientation-preserving homeomorphism $h: B^n \to B^n$ (i.e. a homeomorphism where $h \mid S^{n-1}$ is homotopic to the identity map of S^{n-1}) so that $h(a_j) = b_j$ for $j = 1, 2, \ldots, k$. Finally apply the

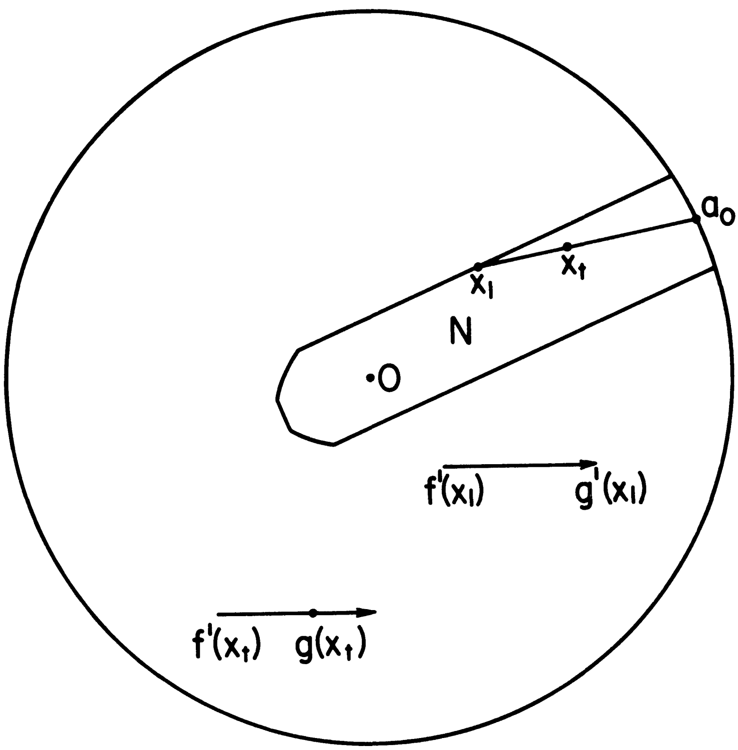
Lemma to obtain $g'': B^n \to B^n$ with coin$(f'', g'') = h(A)$, and let $f' = f'' \circ h$ and $g = g'' \circ h$.

Case 3). $A \subset S^{n-1}$ and A finite. First select a point $a_0 \in A$, let $A' = A \cup \{O\} - \{a_0\}$, and use Case 1) to obtain maps $f': B^n, S^{n-1} \to B^n, S^{n-1}$ and $g': B^n \to B^n$ with $f'(O) = 0$, $f' \mid S^{n-1} = f \mid S^{n-1}$ and coin$(f', g') = A'$. Then take a closed convex neighbourhood $N \subset B^n$ of the segment $[a_0, O]$ with $N \cap A = \{a_0\}$. Let Bd N denote the boundary of N in R^n, label the points of Bd $N - \{a_0\}$ as x_1 and the points of $N - \{a_0\}$ as x_t $(0 < t \leq 1)$, where $\overrightarrow{Ox_t} = \overrightarrow{Oa_0} + t\overrightarrow{a_0x_1}$, and let $\overrightarrow{v(x_1)}$ be the (free) vector $\overrightarrow{f'(x_1)g'(x_1)}$. Define $g(x_t)$, for $x_t \in N - \{a_0\}$, by

$$\overrightarrow{Og(x_t)} = \overrightarrow{Of'(x_t)} + t\overrightarrow{v(x_1)}$$

if this yields a point in B^n (see the Figure). Otherwise let $g(x_t)$ be the point of intersection with S^{n-1} of the ray from $f'(x_t)$ in the direction of $\overrightarrow{v(x_1)}$. Finally extend $g \mid N - \{a_0\}$ to $g: B^n \to B^n$ by $g(a_0) = f'(a_0)$ and $g(x) = g'(x)$ for $x \in B^n - N$. It is easy to check that g is continuous and that coin$(f', g) = A$.

REMARK. If $n \geq 3$, then a simple proof of the Cases 2) and 3) can be obtained with the help of Lemma 7.2 on p. 351 of [1], which shows that there exist



Figure

$f'' \mid S^{n-1} \simeq f \mid S^{n-1}$ and $c \in S^{n-1}$ so that $(f'')^{-1}(c) = b$ is a singleton. From this fact f' and g can be obtained as in the proof of Case 2).

3. Discussion of the result

a) It is possible to strengthen the Theorem by requiring that $g(S^{n-1}) \subset S^{n-1}$. A proof can be obtained by using ideas from the proof of the Theorem as well as from those of Theorems 1 and 2 in [8], and is left as an exercise to the reader.

b) The Theorem is false, however, if $f = f'$ is required. To see this, let $f: B^n \to B^n$ be given by

$$f(x) = \begin{cases} 2x & \text{for } 0 \leq \|x\| \leq \frac{1}{2}, \\ x/\|x\| & \text{for } \frac{1}{2} \leq \|x\| \leq 1. \end{cases}$$

If $B^n_{1/2} = \{x \in B^n \mid \|x\| \leq \frac{1}{2}\}$, then $f \mid B^n_{1/2} : B^n_{1/2} : B^n_{1/2} \to B^n$ is coincidence producing by Theorem A of [4], so there exists no map $g: B^n \to B^n$ with $\operatorname{coin}(f, g) \subset B^n - B^n_{1/2}$. In fact it follows from this theorem that if $f, g: B^n \to B^n$, then $\operatorname{coin}(f, g) \cap C \neq \varnothing$ for every closed $C \subset B^n$ on which $f \mid C: C \to B$ is coincidence producing. Hence we ask the

QUESTION. Given a map $g: B^n \to B^n$ and a closed subset A of B^n, is $A = \operatorname{coin}(f, g)$ for some map $g: B^n \to B^n$ if and only if $A \cap C \neq \varnothing$ for every closed $C \subset B^n$ on which $f \mid C: C \to B^n$ is coincidence producing?

REFERENCES

1. J. Dugundji, *Topology*, Allyn and Bacon, Inc., Boston, 1966.

2. W. Holsztyński, *Une généralisation du théorèm de Brouwer sur les points invariants*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 603–606.

3. ——, *Universality of the product mappings onto products of I^n and snakelike spaces*, Fund. Math. **64** (1969), 147–155.

4. ——, *On the product and composition of universal mappings of manifolds into cubes*, Proc. Amer. Math. Soc. **58** (1976), 311–314.

5. H. Robbins, *Some complements to Brouwer's fixed point theorem*, Israel J. Math. **5** (1967), 225–226.

6. H. Schirmer, *A Brouwer type coincidence theorem*, Canad. Math. Bull. **9** (1966), 443–446.

7. ——, *Coincidence producing maps onto trees*, Canad. Math. Bull. **10** (1967), 417–423.

8. ——, *On fixed sets of homeomorphisms of the n-ball*, Israel J. Math. **7** (1969), 46–50.

9. ——, *Fixed point sets of continuous selfmaps*, Fixed Point Theory, Proceedings, Sherbrooke, Quebec 1980, Lecture Notes in Mathematics **886,** Springer-Verlag, 1981, 417–428.

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