



RESEARCH ARTICLE

Stability conditions for polarised varieties

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Received: 2 May 2022; Revised: 10 September 2023; Accepted: 30 September 2023

2020 Mathematics Subject Classification: Primary – 32Q26; Secondary – 14L24, 32Q15

Abstract

We introduce an analogue of Bridgeland’s stability conditions for polarised varieties. Much as Bridgeland stability is modelled on slope stability of coherent sheaves, our notion of Z -stability is modelled on the notion of K -stability of polarised varieties. We then introduce an analytic counterpart to stability, through the notion of a Z -critical Kähler metric, modelled on the constant scalar curvature Kähler condition. Our main result shows that a polarised variety which is analytically K -semistable and asymptotically Z -stable admits Z -critical Kähler metrics in the large volume regime. We also prove a local converse and explain how these results can be viewed in terms of local wall crossing. A special case of our framework gives a manifold analogue of the deformed Hermitian Yang–Mills equation.

Contents

1	Introduction	2
2	Z-stability and Z-critical Kähler metrics	6
2.1	Stability conditions	6
2.1.1	Z -stability	6
2.1.2	K -stability	10
2.2	Z -critical Kähler metrics	11
3	Z-critical metrics on asymptotically Z-stable manifolds	15
3.1	Preliminaries on analytic Deligne pairings	16
3.2	The Z -energy	18
3.3	Moment maps	22
3.3.1	Moment maps in finite dimensions	22
3.3.2	Moment maps in infinite dimensions	25
3.4	The core analytic argument	28
3.4.1	Understanding the model operator	30
3.4.2	The approximate solution	34
3.4.3	Understanding the expansion of the operator	36
3.4.4	Properties of the linearisation	38
3.4.5	Applying the quantitative inverse function theorem	42
3.4.6	Analysis over the Kuranishi space	44
3.5	Solving the finite-dimensional problem	47
3.6	Existence implies stability	49

4 The higher rank case **50**

4.1 Stability 51

4.2 Z-energy 51

4.3 Final steps 55

1. Introduction

Two notions of stability have dominated much of algebraic geometry over the last 20 years: These are the notions of *K-stability* of a polarised variety [63, 24] and *Bridgeland stability* of an object in a triangulated category [5]. Bridgeland stability is modelled on the more classical notion of *slope stability* of a coherent sheaf over a polarised variety, and slope stability can be viewed as the ‘large volume limit’ of Bridgeland stability. One then expects to obtain moduli spaces of Bridgeland stable objects (and one frequently does [65, 53, 1]), with the usefulness of Bridgeland stability arising from the fact that one can vary the stability condition, which often leads to a good geometric understanding of the birational geometry of these moduli spaces. This, in turn, frequently leads to interesting geometric consequences [4].

In the simplest case that the object of the triangulated category in question is a holomorphic vector bundle, there is a differential-geometric counterpart to Bridgeland stability, though the dictionary is not exact and theory is in its infancy. This counterpart is the notion of a *Z-critical connection* [15], recently introduced by the author, McCarthy and Sektnan, which concretely is a solution to a partial differential equation on the space of Hermitian metrics on the holomorphic vector bundle. Z-critical connections should play an analogous role to Hermite–Einstein metrics in the study of slope stability of vector bundles, and indeed the ‘large volume limit’ of the Z-critical condition is the Hermite–Einstein condition.

K-stability of a polarised variety originated directly through from Kähler geometry, through the search for *constant scalar curvature Kähler (cscK) metrics* on smooth polarised varieties, whose existence is conjectured by Yau, Tian and Donaldson is to be equivalent to K-stability [69, 63, 24]. Already through the early work of Fujiki and Schumacher it was apparent that the cscK condition (hence, *a posteriori*, the K-stability condition) should be the appropriate condition to form moduli of polarised varieties, and there is now much compelling evidence for this [30, 31, 16, 39], especially in the Fano setting [51, 47, 68]. With these moduli spaces being increasingly well understood, it is natural to ask what the geometry of these spaces is and whether their birational geometry can be understood through other notions of stability; this is a heavily studied problem for moduli spaces of curves [37]. Thus, one is led to the question: Is there an analogue of Bridgeland stability for polarised varieties?

Here, we begin a programme to answer this question. The definitions and techniques in the present work are most relevant in the ‘large volume’ regime, where categorical input is less necessary, and the links with differential geometry are currently strongest.

The main input into a Bridgeland stability condition is a *central charge*; our analogue for varieties is essentially a complex polynomial in cohomology classes of the polarised variety (X, L) , including Chern classes of X . Fixing such a central charge Z , one obtains a complex number $Z(X, L)$ with *phase* $\varphi(X, L) = \arg Z(X, L)$, which we always assume to be nonzero. On the differential-geometric side, we introduce the notion of a *Z-critical Kähler metric*, which is a solution to a partial differential equation of the form

$$\text{Im}(e^{-i\varphi(X,L)} \tilde{Z}(\omega)) = 0,$$

where $\tilde{Z}(\omega)$ is a complex-valued function defined using representatives of the cohomology classes associated to the central charge $Z(X, L)$, with appropriate Chern–Weil representatives chosen to represent the Chern classes. We also require the positivity condition $\text{Re}(e^{-i\varphi(X,L)} \tilde{Z}(\omega)) > 0$. The Z-critical condition is then equivalent to asking that the function

$$\tilde{Z}(\omega) : X \rightarrow \mathbb{C}$$

has constant argument. The equation has formal similarities to the notion of a Z -critical connection on a holomorphic vector bundle, leading us to mirror the terminology.

On the algebro-geometric side, the notion of stability involves *test configurations*, which are the \mathbb{C}^* -degenerations $(\mathcal{X}, \mathcal{L})$ of (X, L) crucial to the definition of K -stability. We associate a numerical invariant $Z(\mathcal{X}, \mathcal{L})$ to each test configuration, which is again a complex number whose phase we denote $\varphi(\mathcal{X}, \mathcal{L})$. The notion of Z -stability we introduce, which is roughly analogous to Bridgeland stability, means that for each test configuration the phase inequality

$$\operatorname{Im}\left(\frac{Z(\mathcal{X}, \mathcal{L})}{Z(X, L)}\right) > 0$$

holds. These definitions allow us to state the following analogue of the Yau–Tian–Donaldson conjecture:

Conjecture 1.1. *Let (X, L) be a smooth polarised variety with discrete automorphism group. Then the existence of a Z -critical Kähler metric in $c_1(L)$ is equivalent to Z -stability of (X, L) .*

We should say immediately that this conjecture is only plausible in sufficiently ‘large volume’ regions of the space of central charges; this is a condition which we expect to be explicit in concrete situations. Away from this region, categorical phenomena should enter. Thus, Conjecture 1.1 should be seen as a first approximation of a larger conjecture involving a more categorical framework. When the values $Z(X, L)$ and $Z(\mathcal{X}, \mathcal{L})$ lie in the upper half plane, the inequality is equivalent to asking for the phase inequality $\varphi(\mathcal{X}, \mathcal{L}) > \varphi(X, L)$ to hold, and the ‘large volume’ hypothesis should imply that for the relevant test configuration, $Z(\mathcal{X}, \mathcal{L})$ does lie in the upper half plane. We also note that, much as with the Yau–Tian–Donaldson conjecture, it seems reasonable that one may need to impose a uniform notion of stability [13, Conjecture 1.1]; see [46] for recent progress.

Here, we prove the ‘large volume limit’ of this conjecture, for what seems to be the most interesting class of central charge. For this admissible class of central charge defined in Section 3, when one scales the polarisation L to kL for $k \gg 0$, the central charge takes values in the upper half plane and the leading order term in k of the phase inequalities $\varphi_k(X, L) < \varphi_k(\mathcal{X}, \mathcal{L})$ is simply the usual inequality on the Donaldson–Futaki invariant involved in the definition of K -stability. It follows that the natural notion of *asymptotic Z -stability* implies *K -semistability*. A K -semistable polarised variety conjecturally admits a test configuration with central fibre K -polystable, and we say that (X, L) is *analytically K -semistable* if there is a test configuration whose central fibre is a smooth polarised variety admitting a cscK metric. We in addition assume that the deformation theory of the central fibre is unobstructed, to aid the analytic argument in the following, which is our main result:

Theorem 1.1. *Let (X, L) be an analytically K -semistable variety which has discrete automorphism group. Then (X, kL) admits Z -critical Kähler metrics for all $k \gg 0$ provided it is asymptotically Z -stable.*

In particular, when (X, L) itself admits a cscK metric and has discrete automorphism group, we prove the existence of Z -critical Kähler metrics for all $k \gg 0$. The converse, namely that existence of Z -critical Kähler metrics implies asymptotic Z -stability, also holds in a weak, local sense. To discuss the sense in which this is true, we must discuss some of the elements of the proof of Theorem 1.1. We denote the cscK degeneration of (X, L) by (X_0, L_0) and consider the Kuranishi space B of (X_0, L_0) ; that the deformation theory of (X_0, L_0) is unobstructed implies that B is smooth. This space admits a universal family $(\mathcal{X}, \mathcal{L}) \rightarrow B$, and from its construction \mathcal{L} admits a relatively Kähler metric which induces the cscK metric on (X_0, L_0) . There are then three steps:

- (i) We reduce to the above finite-dimensional moment map problem on B by perturbing the relatively Kähler metric on \mathcal{L} in such a way that the only obstruction to solving the Z -critical equation arise from the automorphisms of the central fibre (X_0, L_0) . This uses a quantitative version of the implicit function theorem and occupies much of the paper.

- (ii) We show that the Z -critical equation can, locally, be viewed as a moment map on a given orbit. More precisely, the automorphism group of (X_0, L_0) acts on B , and on each orbit in B we show that with respect to a natural Kähler metric we produce on B , the condition that the Kähler metric on the fibre is Z -critical is essentially the moment map for the action of the associated maximal compact subgroup action. This can be viewed as an orbit-wise analogue of the Fujiki–Donaldson moment map picture for the cscK equation [30, 23], but we take a new approach that gives weaker results but much greater flexibility. It is then important that the phase inequalities involved in the definition of Z -stability correspond exactly to the weight inequalities arising from the finite-dimensional moment map problem.
- (iii) We show that, in our local finite-dimensional moment map problem, stability implies the existence of a zero of the moment map, which thus produces Z -critical Kähler metrics by the first step. This relies on a local version of the Kempf–Ness theorem proven in [15, Section 4.2].

This basic strategy is analogous to work of Brönnle and Székelyhidi [6, 61], with the difference arising from the fact we consider a *sequence* of moment maps and a strictly K -semistable manifold.

As part of step (ii), we obtain analogues of several important tools in the study of cscK metrics, such as the Futaki invariant associated to holomorphic vector fields, and an energy functional analogous to Mabuchi’s K -energy. The local moment map picture also quite formally produces the local converse to Theorem 1.1. Let us say that (X, L) is *locally asymptotically Z -stable* if the phase inequality holds for all test configurations produced from the Kuranishi space B of its cscK degeneration.

Theorem 1.2. *With the above setup, (X, kL) admits Z -critical Kähler metrics for all $k \gg 0$ if and only if it is locally asymptotically Z -stable.*

Thus, we have proven a version of the large volume limit of Conjecture 1.1. There is an interesting interpretation of this result in terms of local wall-crossing. Wall-crossing phenomena arise when one can vary the stability condition, and one then expects the resulting moduli spaces to undergo birational transformations. The strictly stable locus is unchanged by suitably small changes of the stability condition, and the interesting question concerns the semistable locus. The above then demonstrates that the algebro-geometric walls, governed by Z -stability, agree with the differential-geometric walls, governed by the existence of Z -critical Kähler metrics.

Our results can be seen as manifold analogues of results established in [15] for holomorphic vector bundles. There it is proven that the existence of Z -critical connections on a holomorphic vector bundle is equivalent to asymptotic Z -stability of the bundle; the latter notion is a variant of Bridgeland stability. The strategy employed in [15] is different: There, a local version of the Kempf–Ness theorem is used to provide a good choice of initial connection [15, Section 4.2.1], after which analytic aspects of Z -critical connections enters. Here, the analysis is considerably more involved, leading us to perform the key analytic step first. To ensure that we stay in the realm of Kähler geometry, we perturb the fibrewise Kähler metric rather than perturbing the almost complex structure (the latter approach has its origins in the fundamental work of Székelyhidi [61]); this new approach is crucial to allowing us to employ the local version of the Kempf–Ness theorem.

Continuing with the comparison with the bundle story, we must mention that the general notion of a Z -critical connection is modelled on the specific notion of a *deformed Hermitian Yang–Mills connection* associated with a special central charge of particular relevance to mirror symmetry. Indeed, the deformed Hermitian Yang–Mills equation was introduced through Strominger–Yau–Zaslow (SYZ) mirror symmetry to be the mirror of the special Lagrangian equation [45]. The quite beautiful theory of this equation on holomorphic line bundles has developed with speed over the past few years [40, 8, 11, 12], and these developments have emphasised that the special form of the central charge in this case has significant geometric implications. We thus emphasise that there is a direct analogue of the deformed Hermitian Yang–Mills equation for manifolds, which one might call the *deformed cscK equation* and which seems to be the natural avenue for further research. Fixing normal coordinates for the Kähler metric ω in which $\text{Ric } \omega$ is diagonal, let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $\text{Ric } \omega$ and let $\sigma_j(\omega)$ denote

the j^{th} elementary symmetric polynomial in these eigenvalues. Then this equation takes the form

$$\operatorname{Im} \left(e^{-i\varphi(X,L)} \left(\sum_{j=0}^n (-i)^j (\sigma_j(\omega) - \Delta\sigma_{j-1}(\omega)) \right) \right) = 0.$$

We remark that the name is misleading, as it is only truly a ‘deformation’ of the cscK equation in the large volume limit. We also remark that the phase range in which existence of solutions to the deformed Hermitian Yang–Mills equation is equivalent to stability is the full supercritical phase range [8], which emphasises that in explicit situations one should expect the large volume hypothesis of Conjecture 1.1 to be similarly explicit.

The simplest new partial differential equation (PDE) we consider in the present work, to which Theorem 1.1 applies, takes the form

$$S(\omega) + \frac{1}{k} \left(\frac{2}{n(n-2)} \Delta S(\omega) - \frac{\operatorname{Ric} \omega^2 \wedge \omega^{n-2}}{\omega^n} \right) = \text{const.},$$

which is an elliptic, sixth-order fully nonlinear PDE in the Kähler potential and which is exactly the Z-critical equation for a special (k -dependent) central charge. An important feature of the equation is that in the large volume regime $k \rightarrow \infty$, the constant of ellipticity degenerates to zero. Much of our analytic work is devoted to this PDE, and in the large volume regime $k \gg 0$ we view the general Z-critical equation as a perturbation of this model equation.

Categorification

The approach we take in the present work is to consider explicitly defined central charges, as opposed to an axiomatic approach more closely analogous to the theory of Bridgeland stability conditions. In a sequel to this paper [14], an axiomatic approach to stability conditions on general stacks is developed (with the relevant stack here being the stack of polarised schemes), motivated by the more explicit approach taken here. To explain this, it is clearer to view the central charge as a function on schemes endowed with a \mathbb{C}^* -action. The key properties are then *additivity* of the central charge, which essentially asks that the central charge is additive under composition of commuting one-parameter subgroups and *equivariant constancy* of the central charge, which asks that the value of the central charge is constant in equivariant flat families. We refer to [14] for further details.

Stability of maps

While we have thus far emphasised the case of polarised varieties and while our main result only holds in that setting, the basic framework is more general and links with interesting questions in enumerative geometry. While for a broad and interesting class of central charge, the ‘large volume condition’ is K-stability, in general one obtains the notion of twisted K-stability [13], which is linked to the existence of twisted cscK metrics. The appropriate geometric context in which to study twisted K-stability is when one has a map $p : (X, L) \rightarrow (Y, H)$ of polarised varieties, where it is essentially equivalent to K-stability of the map p [18, 19].

From the moduli theoretic point of view, one expects to be able to form moduli of K-stable maps to a fixed (Y, H) . The definition of K-stability of maps generalises Kontsevich’s notion when (X, L) is a curve, and the resulting (entirely conjectural) higher-dimensional moduli spaces would thus be higher-dimensional analogues of the moduli space of stable maps; there is also a version of theory involving divisors, as a higher-dimensional analogue of the maps of marked curves used in Gromov–Witten theory [2][18, Section 5.3]. What seems most interesting is that our work suggests that there should be variants of stability of maps even in the curves case, which may even lead to an understanding of wall-crossing phenomena for Gromov–Witten invariants; this seems likely to require developing a more categorical approach to the problem as discussed above.

2. Z-stability and Z-critical Kähler metrics

Here, we define the key algebro-geometric and differential-geometric criteria of interest to us: Z-stability and Z-critical Kähler metrics. The definitions involve a central charge, which involves various Chern classes of X . The differential geometry is substantially more complicated when higher Chern classes (rather than merely the first Chern class) appear in the central charge, and so we postpone the definitions and results in that case to Section 4. The difference is roughly analogous to the difference between the theory of Z-critical connections on holomorphic line bundles and bundles of higher rank, and so we call the situation in which higher Chern classes appear the ‘higher rank case’. The analogy is far from exact, and the case in which only the first Chern class and its powers appear in the central charge already exhibits many of the main difficulties in the study of Z-critical connections on arbitrary rank vector bundles.

2.1. Stability conditions

2.1.1. Z-stability

We work throughout over the complex numbers, in order to preserve links with the complex differential geometry. We also fix a normal polarised variety (X, L) of dimension n , with L an ample \mathbb{Q} -line bundle. Normality implies that the canonical class K_X of X exists as a Weil divisor, we always assume that K_X exists as a \mathbb{Q} -line bundle.

In addition to our ample line bundle, we will fix a *stability vector*, a *unipotent cohomology class* and a *polynomial Chern form*; we define these in turn.

Definition 2.1. A *stability vector* is a sequence of complex numbers

$$\rho = (\rho_0, \dots, \rho_n) \in \mathbb{C}^{n+1}$$

such that $\rho_n = i = \sqrt{-1}$.

The condition $\rho_n = i$ is a harmless normalisation condition which, when it is not satisfied, can be achieved by multiplying the stability vector by a fixed complex number. In Bridgeland stability, one normally assumes $\rho \in (\mathbb{C}^*)^{n+1}$; this will be unnecessary for us.

Definition 2.2. A *unipotent cohomology class* is a complex cohomology class $\Theta \in \bigoplus_j H^{j,j}(X, \mathbb{C})$ which is of the form $\Theta = 1 + \Theta'$, where $\Theta' \in H^{>0}(X, \mathbb{C})$.

Note that Θ' must satisfy

$$\overbrace{\Theta' \cdot \dots \cdot \Theta'}^{j \text{ times}} = 0$$

for $j \geq n + 1$. A typical example of a choice of Θ is to fix a class $\beta \in H^{1,1}(X, \mathbb{R})$ and set $\Theta = e^{-\beta}$, which is analogous to a ‘B-field’ in Bridgeland stability.

Definition 2.3. A *polynomial Chern form* is a sum of the form

$$f(K_X) = \sum_{j=0}^n a_j K_X^j,$$

where $a_j \in \mathbb{C}$ and K_X^j denotes the j^{th} -intersection product $K_X \cdot \dots \cdot K_X$, viewed as a cycle. We always assume the normalisation condition $a_0 = a_1 = 1$, and interpret $K_X^0 = 1$ as a cycle.

As mentioned above, in the current section we restrict ourselves to central charges only involving $c_1(X) = c_1(-K_X)$, with the case of higher Chern classes postponed to Section 4.

Definition 2.4. A polynomial central charge is a function $Z : \mathbb{N} \rightarrow \mathbb{C}$ taking the form

$$Z_k(X, L) = \sum_{l=0}^n \rho_l k^l \int_X L^l \cdot f(K_X) \cdot \Theta,$$

for some ρ and Θ . A central charge is a polynomial central charge with k fixed, such that $Z(X, L) \neq 0$. We often set $\varepsilon = k^{-1}$ and denote the induced quantity by $Z_\varepsilon(X, L)$.

We will sometimes simply call a polynomial central charge a central charge when the dependence on k is clear from context. The definition is motivated by an analogous definition of Bayer in the bundle setting [3, Theorem 3.2.2]. For a polynomial central charge it is automatic that $Z_k(X, L)$ lies in the upper half plane in \mathbb{C} for $k \gg 0$ since $\text{Im}(\rho_n) > 0$. Thus, we can make the following definition.

Definition 2.5. We define the phase of X to be

$$\varphi_k(X, L) = \arg Z_k(X, L),$$

the argument of the nonzero complex number. We denote this by $\varphi(X, L)$ when k is fixed, and for fixed (X, L) often simply denote this by φ .

Here, we consider \arg as a function $\arg : \mathbb{C} \rightarrow \mathbb{R}$ by setting $\arg(1) = 0$. We now turn to our definition of stability, which depends on a choice of central charge Z . As in the definition of K-stability of polarised varieties, we require the notion of a test configuration, which is essentially a \mathbb{C}^* -degeneration of (X, L) to another polarised scheme.

Definition 2.6 [63][24, Definition 2.1.1]. A test configuration for (X, L) consists of a pair $\pi : (\mathcal{X}, \mathcal{L}) \rightarrow \mathbb{C}$, where:

- (i) \mathcal{X} is a normal polarised variety such that $K_{\mathcal{X}}$ is a \mathbb{Q} -line bundle;
- (ii) \mathcal{L} is a relatively ample \mathbb{Q} -line bundle;
- (iii) there is a \mathbb{C}^* -action on $(\mathcal{X}, \mathcal{L})$ making π an equivariant flat map with respect to the standard \mathbb{C}^* -action on \mathbb{C} ;
- (iv) the fibres $(\mathcal{X}_t, \mathcal{L}_t)$ are each isomorphic to (X, L) for each $t \neq 0 \in \mathbb{C}$.

A test configuration is a product if $(\mathcal{X}_0, \mathcal{L}_0) \cong (X, L)$, hence inducing a \mathbb{C}^* -action on (X, L) ; it is further trivial if this \mathbb{C}^* -action is the trivial one.

Remark 2.1. One typically does not require $K_{\mathcal{X}}$ to be a \mathbb{Q} -line bundle in the usual definition of a test configuration, but one should not expect this discrepancy to play a significant role in either K-stability or the theory of Z-stability we are describing.

A test configuration admits a canonical compactification to a family over \mathbb{P}^1 by equivariantly compactifying trivially over infinity [66, Section 3]. This compactification produces a flat family endowed with a \mathbb{C}^* -action, which we abusively denote $(\mathcal{X}, \mathcal{L}) \rightarrow \mathbb{P}^1$, such that each fibre over $t \neq \infty \in \mathbb{P}^1$ is isomorphic to (X, L) . The reason to compactify is that it allows us to perform intersection theory on the resulting projective variety \mathcal{X} .

It will also be convenient to be able to consider classes on X as inducing classes on \mathcal{X} , so we pass to a variety with a surjective map to X as follows. There is a natural equivariant birational map

$$f : (X \times \mathbb{P}^1, p_1^*L) \dashrightarrow (\mathcal{X}, \mathcal{L}),$$

with $p_1 : X \times \mathbb{P}^1 \rightarrow X$ the projection, so we take an equivariant resolution of indeterminacy of the form:

$$\begin{array}{ccc} \mathcal{Y} & & \\ q \downarrow & \searrow r & \\ X \times \mathbb{P}^1 & \dashrightarrow & \mathcal{X}, \end{array}$$

where we may assume \mathcal{Y} is smooth. In particular, the unipotent cohomology class Θ on X involved in the definition of a central charge induces a class $(q \circ p_1)^*\Theta$ on \mathcal{Y} , which we still denote Θ . The classes \mathcal{L} and $K_{\mathcal{X}}$ on \mathcal{X} induce also classes $r^*\mathcal{L}$ and $r^*K_{\mathcal{X}/\mathbb{P}^1}$ on \mathcal{Y} , we in addition set $K_{\mathcal{X}/\mathbb{P}^1} = K_{\mathcal{X}} - \pi^*K_{\mathbb{P}^1}$ to be the relative canonical class. Thus, to a given intersection number $L^d \cdot K_{\mathcal{X}}^j \cdot U$ we can associate the intersection number on \mathcal{Y} which we (slightly abusively) denote

$$\int_{\mathcal{X}} \mathcal{L}^{l+1} \cdot K_{\mathcal{X}/\mathbb{P}^1}^j \cdot \Theta = \int_{\mathcal{Y}} (r^*\mathcal{L})^{l+1} \cdot r^*(K_{\mathcal{X}/\mathbb{P}^1}^j) \cdot \Theta,$$

which is computed in \mathcal{Y} . In computing this intersection number, note that $\dim \mathcal{X} = \dim \mathcal{Y} = n + 1$. The following elementary result justifies the notation omitting \mathcal{Y} .

Lemma 2.2. *This intersection number is independent of resolution of indeterminacy \mathcal{Y} chosen.*

Proof. Given two such resolutions of indeterminacy \mathcal{Y} and \mathcal{Y}' , there is a third resolution of indeterminacy \mathcal{Y}'' with commuting maps to both \mathcal{Y} and \mathcal{Y}' . The result then follows from an application of the push-pull formula in intersection theory. □

Definition 2.7. Let $(\mathcal{X}, \mathcal{L})$ be a test configuration and Z be a polynomial central charge. We define the *central charge* of $(\mathcal{X}, \mathcal{L})$ to be

$$Z_k(\mathcal{X}, \mathcal{L}) = \sum_{l=0}^n \frac{\rho_l k^l}{l+1} \int_{\mathcal{X}} \mathcal{L}^{l+1} \cdot f(K_{\mathcal{X}/\mathbb{P}^1}) \cdot \Theta,$$

and set $\varphi_k(\mathcal{X}, \mathcal{L}) = \arg Z_k(\mathcal{X}, \mathcal{L})$ when $Z_k(\mathcal{X}, \mathcal{L}) \neq 0$. Note that $f(K_{\mathcal{X}/\mathbb{P}^1}) = \sum_{j=0}^n a_j K_{\mathcal{X}/\mathbb{P}^1}^j$ arises from the polynomial Chern form. With k fixed we denote these by $Z(\mathcal{X}, \mathcal{L})$ and $\varphi(\mathcal{X}, \mathcal{L})$, respectively.

The stability condition, for fixed k , is then the following.

Definition 2.8. We say that (X, L) is

- (i) *Z-stable* if for all nontrivial test configurations $(\mathcal{X}, \mathcal{L})$ we have

$$\operatorname{Im} \left(\frac{Z(\mathcal{X}, \mathcal{L})}{Z(X, L)} \right) > 0.$$

- (ii) *Z-polystable* if for all test configurations $(\mathcal{X}, \mathcal{L})$ we have

$$\operatorname{Im} \left(\frac{Z(\mathcal{X}, \mathcal{L})}{Z(X, L)} \right) \geq 0,$$

with equality holding only for product test configurations.

- (iii) *Z-semistable* if for all test configurations $(\mathcal{X}, \mathcal{L})$ we have

$$\operatorname{Im} \left(\frac{Z(\mathcal{X}, \mathcal{L})}{Z(X, L)} \right) \geq 0.$$

- (iv) *Z-unstable* otherwise.

The natural asymptotic notion is the following.

Definition 2.9. We say that (X, L) is *asymptotically Z-stable* if for all nontrivial test configurations $(\mathcal{X}, \mathcal{L})$ and for all $k \gg 0$ we have

$$\operatorname{Im} \left(\frac{Z_k(\mathcal{X}, \mathcal{L})}{Z_k(X, L)} \right) > 0.$$

Asymptotic Z-polystability, semistability and instability are defined similarly.

Note that, as $\text{Im}(\rho_n) > 0$ by assumption, both $Z_k(X, L)$ and $Z_k(\mathcal{X}, \mathcal{L})$ are nonvanishing and lie in the upper half plane for $k \gg 0$. Here, strictly speaking to ensure that $Z_k(\mathcal{X}, \mathcal{L})$ lies in the upper half plane we may need to modify \mathcal{L} to $\mathcal{L} + \mathcal{O}(m)$ for some $\mathcal{O}(m)$ pulled back from \mathbb{P}^1 ; this leaves the various stability inequalities unchanged by Lemma 2.4 below. Thus, asymptotic Z-stability can be rephrased as asking for all test configurations $(\mathcal{X}, \mathcal{L})$ to have for $k \gg 0$

$$\varphi_k(\mathcal{X}, \mathcal{L}) > \varphi_k(X, L).$$

Remark 2.3. In Bridgeland stability, much work goes into ensuring that the central charge has image in the upper half plane, and this is one of the most challenging aspects of constructing Bridgeland stability conditions. We have essentially ignored this, at the expense of having a notion that should only be the correct one near the ‘large volume regime’ when k is taken to be large; this should be thought of as producing a ‘large volume’ region in the space of central charges.

We note that in the better understood story of deformed Hermitian Yang–Mills connections, the link between analysis and a simpler (noncategorical) stability conditions holds in the ‘supercritical phase’ [11, 8], which can be thought of as an explicit description of the ‘large volume regime’. Away from the large volume situation, it seems likely that categorical techniques must be used and, for example, more structure should be required of the stability vector by analogy with Bayer’s hypotheses [3, Theorem 3.2.2]. Thus, our algebro-geometric definitions should be seen as the first approximation of a larger story, which is appropriate only in an explicit large volume region.

The factor $l + 1$ in the definition of $Z_k(\mathcal{X}, \mathcal{L})$ ensures that the key inequality defining stability is invariant under certain changes of \mathcal{L} . For this, note that one can modify the polarisation of a test configuration $(\mathcal{X}, \mathcal{L})$ by adding the pullback $\mathcal{O}(m)$ of the (m^{th} tensor power of the) hyperplane line bundle from \mathbb{P}^1 for any j .

Lemma 2.4. *The phase inequality remains unchanged under the addition of $\mathcal{O}(m)$. That is,*

$$\text{Im}\left(\frac{Z(\mathcal{X}, \mathcal{L} + \mathcal{O}(m))}{Z(X, L)}\right) = \text{Im}\left(\frac{Z(\mathcal{X}, \mathcal{L})}{Z(X, L)}\right).$$

Proof. A single intersection number changes as

$$\int_{\mathcal{X}} (\mathcal{L} + \mathcal{O}(m))^{l+1} \cdot K_{\mathcal{X}/\mathbb{P}^1}^j \cdot \Theta = \int_{\mathcal{X}} \mathcal{L}^{l+1} \cdot K_{\mathcal{X}/\mathbb{P}^1}^j \cdot \Theta + m(l + 1) \int_X L^l \cdot K_X^j \cdot \Theta,$$

since by flatness intersecting with $\mathcal{O}(1)$ can be viewed as intersecting with a fibre $\mathcal{X}_t \cong X$ for $t \neq 0$, and $\mathcal{L}, K_{\mathcal{X}/\mathbb{P}^1}$ and Θ restrict to L, K_X and Θ respectively on X . It follows that

$$Z(\mathcal{X}, \mathcal{L} + \mathcal{O}(m)) = Z(\mathcal{X}, \mathcal{L}) + mZ(X, L),$$

which means since $m \in \mathbb{Q}$ is real

$$\begin{aligned} \text{Im}\left(\frac{Z(\mathcal{X}, \mathcal{L} + \mathcal{O}(m))}{Z(X, L)}\right) &= \text{Im}\left(\frac{Z(\mathcal{X}, \mathcal{L}) + mZ(X, L)}{Z(X, L)}\right), \\ &= \text{Im}\left(\frac{Z(\mathcal{X}, \mathcal{L})}{Z(X, L)}\right). \end{aligned}$$

□

Example 2.5. A central charge of special interest is

$$\begin{aligned} Z_k(X, L) &= - \int_X e^{-ikL} \cdot e^{-K_X}, \\ &= - \sum_{j=0}^n \frac{(-i)^j}{j!(n-j)!} \int_X (kL)^j \cdot (-K_X)^{n-j}. \end{aligned}$$

This can be viewed as an analogue of the central charge on the Grothendieck group $K(X)$ (in the sense of Bridgeland stability) associated to the deformed Hermitian Yang–Mills equation on a holomorphic line bundle [12, Section 9].

We will not consider a completely arbitrary central charge in the present work, as we require that the large volume limit of our conditions is ‘nondegenerate’ in a suitable sense. Let Θ_1 denote the $(1, 1)$ -part of the unipotent cohomology class $\Theta \in \oplus_j H^{j,j}(X, \mathbb{C})$.

Definition 2.10. We say that Z is

- (i) *nondegenerate* if $\text{Re}(\rho_{n-1}) < 0$ and Θ_1 vanishes;
- (ii) *of map type* if $\text{Re}(\rho_{n-1}) < 0$ and there is a map $p : X \rightarrow Y$ such that Θ is the pullback of a cohomology class from Y and with $-\Theta_1$ is the class of the pullback of an ample line bundle from Y .

The motivation for these definition is through the link with K -stability and its variants.

2.1.2. K-stability

The definition of asymptotic Z -stability given is motivated not only by the vector bundle theory, but also by the notion of K -stability of polarised varieties due to Tian and Donaldson [63, 24]. As before, we take (X, L) to be a normal polarised variety such that K_X is a \mathbb{Q} -line bundle.

Definition 2.11. We define the *slope* of (X, L) to be the topological invariant, computed as an integral over X

$$\mu(X, L) = \frac{-K_X \cdot L^{n-1}}{L^n}.$$

We further define the *Donaldson–Futaki invariant* of a test configuration $(\mathcal{X}, \mathcal{L})$ to be

$$\text{DF}(\mathcal{X}, \mathcal{L}) = \int_{\mathcal{X}} \left(\frac{n\mu(X, L)}{n+1} \mathcal{L}^{n+1} + \mathcal{L}^n \cdot K_{\mathcal{X}/\mathbb{P}^1} \right).$$

We remark that this is not Donaldson’s original definition but rather is proven by Odaka and Wang to be an equivalent one [50, Theorem 3.2] [66, Section 3] (see also [24, Proposition 4.2.1]).

Definition 2.12. We say that (X, L) is

- (i) *K-stable* if for all nontrivial test configurations $(\mathcal{X}, \mathcal{L})$ for (X, L) we have $\text{DF}(\mathcal{X}, \mathcal{L}) > 0$;
- (ii) *K-polystable* if for all test configurations we have $\text{DF}(\mathcal{X}, \mathcal{L}) \geq 0$, with equality exactly when $(\mathcal{X}, \mathcal{L})$ is a product;
- (iii) *K-semistable* if for all test configurations we have $\text{DF}(\mathcal{X}, \mathcal{L}) \geq 0$;
- (iv) *K-unstable* otherwise.

The following is immediate from the definitions.

Lemma 2.6. *K-semistability is equivalent to asymptotic Z-semistability where*

$$Z_k(X, L) = \int_X (ik^n L^n - k^{n-1} K_X \cdot L^{n-1}).$$

That is, with $\rho = (0, 0, \dots, -1, i)$, $\Theta = 0$.

Of course, the same is true for K -stability and K -polystability, modulo our slightly nonstandard requirement that $K_{\mathcal{X}}$ is a \mathbb{Q} -line bundle, which is irrelevant for K -semistability as in that situation one can assume \mathcal{X} is smooth.

Example 2.7. K -semistability of maps can be recovered as a special case of Z -stability. Indeed, supposing $p : (X, L) \rightarrow (Y, H)$ is a map of polarised varieties, then setting

$$Z_k(X, L) = \int_X (ik^n L^n - k^{n-1}(K_X + p^*H) \cdot L^{n-1})$$

recovers the notion of K -semistability of the map p [18, Definition 2.9]. That is, we take Θ to be (the class of) p^*H .

Slightly more generally, twisted K -stability fits into this picture [13, Definition 2.7], though this notion is less geometric than K -stability of maps and we hence do not discuss it. Similarly, the ‘fully degenerate’ case $a_j = 0$ for $j \leq n - 1$ produces variants of J -stability [42, Section 2] and has links with Z -stability of holomorphic line bundles [15, Conjecture 1.6]. In general, asymptotic Z -stability is related to K -stability as follows:

Proposition 2.13. *For an arbitrary central charge Z , asymptotic Z -semistability implies*

- (i) K -semistability if Z is nondegenerate;
- (ii) K -semistability of the map p if Z is of map type.

Proof. We only give the proof for K -semistability, as the proof is the same for the map type situation. By nondegeneracy, there is an expansion

$$Z_k(X, L) = k^n i \int_X L^n + k^{n-1} \rho_{n-1} \int_X K_X \cdot L^{n-1} + O(k^{n-2}),$$

where we have used that $\Theta_1 = 0$ and that our normalisation for the polynomial Chern form assumes $a_0 = a_1 = 1$. Thus,

$$Z_k(\mathcal{X}, \mathcal{L}) = \frac{i}{n+1} k^n \int_X \mathcal{L}^{n+1} + \frac{\rho_{n-1}}{n} k^{n-1} \int_X K_{\mathcal{X}/\mathbb{P}^1} \cdot \mathcal{L}^{n-1} + O(k^{n-2}),$$

meaning that

$$\operatorname{Im} \left(\frac{Z_k(\mathcal{X}, \mathcal{L})}{Z_k(X, L)} \right) = \frac{-\operatorname{Re}(\rho_{n-1})}{nL^n} \operatorname{DF}(\mathcal{X}, \mathcal{L}) k^{-1} + O(k^{-2}).$$

Thus, since $\operatorname{Re}(\rho_{n-1}) < 0$ by nondegeneracy, the asymptotic Z -stability hypothesis demands that this be negative for $k \gg 0$, forcing $\operatorname{DF}(\mathcal{X}, \mathcal{L}) \geq 0$. □

2.2. Z -critical Kähler metrics

We now turn to the differential-geometric counterpart of stability and thus assume that (X, L) is a smooth polarised variety. We wish to define a notion of a ‘canonical metric’ in $c_1(L)$, adapted to the central charge Z . We recall our notation that the central charge takes the form

$$Z_k(X, L) = \sum_{l=0}^n \rho_l k^l \int_X L^l \cdot \left(\sum_{j=0}^n a_j K_X^j \right) \cdot \Theta,$$

with the induced phase being denoted $\varphi_k(X, L) = \arg Z_k(X, L)$; we take k to be fixed and omit it from our notation.

Associated to any Kähler metric $\omega \in c_1(L)$ is its Ricci form

$$\text{Ric } \omega = -\frac{i}{2\pi} \partial \bar{\partial} \log \omega^n \in c_1(X) = c_1(-K_X)$$

and a Laplacian operator Δ . We also fix a representative of the unipotent class Θ , which we denote $\theta \in \Theta$. When Z is nondegenerate in the sense of Definition 2.10 so that $\Theta_1 = 0$, we always take the $(1, 1)$ -component $\theta_1 \in \Theta_1$ to vanish, and similarly when Z is of map type we take θ_1 to be the pullback of a Kähler metric from Y . To the intersection number $L^l \cdot (-K_X)^j \cdot \Theta$ we associate the function

$$\frac{\omega^l \wedge \text{Ric } \omega^j \wedge \theta}{\omega^n} - \frac{j}{l+1} \Delta \left(\frac{\omega^{l+1} \wedge \text{Ric } \omega^{j-1} \wedge \theta}{\omega^n} \right) \in C^\infty(X, \mathbb{C}), \tag{2.1}$$

with the second term taken to be zero when $j = 0$. The presence of the Laplacian terms will be crucial to link with the algebraic geometry. By linearity, this produces a function $\tilde{Z}(\omega)$ defined in such a way that

$$\int_X \tilde{Z}(\omega) \omega^n = Z(X, L);$$

as with our algebro-geometric discussion, we always assume that $Z(X, L) \neq 0$.

Definition 2.14. We say that ω is a *Z-critical Kähler metric* if

$$\text{Im}(e^{-i\varphi(X,L)} \tilde{Z}(\omega)) = 0$$

and the positivity condition $\text{Re}(e^{-i\varphi(X,L)} \tilde{Z}(\omega)) > 0$ holds.

When we consider a k -dependent central charge Z_k , we define $\tilde{Z}_k(\omega)$ by replacing ω with $k\omega$. We view this as a partial differential equation on the space of Kähler metrics in $c_1(L)$ or equivalently on the space of Kähler potentials with respect to a fixed Kähler metric. Viewed on the space of Kähler potentials, for a generic choice of central charge ensuring the presence of a nonzero term involving the Laplacian, the equation is a sixth-order fully nonlinear partial differential equation. The condition is equivalent to asking that the function

$$\tilde{Z}(\omega) : X \rightarrow \mathbb{C}$$

has constant argument, which must then equal that of $Z(X, L) \in \mathbb{C}$, as we have assumed the positivity condition $\text{Re}(e^{-i\varphi(X,L)} \tilde{Z}(\omega)) > 0$ (in fact, one only needs that this function is never zero, and the sign is irrelevant).

Remark 2.8. The presence of the Laplacian term is crucial to obtain a link with algebraic geometry and in practice arises when deriving the Z -critical equation as the Euler–Lagrange equation of an associated energy functional in Proposition 3.5.

Remark 2.9. In the vector bundle theory, rather than working with arbitrary connections one works with ‘almost-calibrated connections’ [12, Section 8.1]. This is a positivity condition which depends on the choice of $\theta \in \Theta$ and which is trivial in the large volume limit [15, Lemma 2.8] and is analogous to the positivity condition $\text{Re}(e^{-i\varphi(X,L)} \tilde{Z}(\omega)) > 0$ that we have imposed. The notion of a ‘subsolution’ also plays a prominent role in the bundle theory [11], which for example forces the equation to be elliptic in that situation [15, Lemma 2.32]. We note that, also in the manifold case, ellipticity of the Z -critical equation cannot hold in general, and hence for this reason and others it is natural to ask if there is a manifold analogue of the notion of a subsolution.

The appearance of the phase is justified by the following.

Lemma 2.10. For any Kähler metric $\omega \in c_1(L)$, the integral

$$\int_X \operatorname{Im}(e^{-i\varphi(X,L)} \tilde{Z}(\omega)) \omega^n = 0$$

vanishes.

Proof. Since $\int_X \tilde{Z}(\omega) = Z(X, L)$ and $\varphi(X, L) = \arg(Z(X, L))$, we see

$$e^{-i\varphi(X,L)} = \frac{r(X, L)}{Z(X, L)}$$

with $r(X, L)$ real. Thus,

$$\int_X \operatorname{Im}(e^{-i\varphi(X,L)} \tilde{Z}(\omega)) \omega^n = \operatorname{Im}\left(\frac{r(X, L)}{Z(X, L)} Z(X, L)\right) = 0.$$

□

The Z -critical condition can be reformulated as follows. The analogous reformulation, in the special case of the deformed Hermitian Yang–Mills equation [40], has been crucial to all progress in understanding the equation geometrically, and an analogous reformulation holds for Z -critical connections on holomorphic line bundles [15, Example 2.24].

Lemma 2.11. Write

$$\tilde{Z}(\omega) = \operatorname{Re} \tilde{Z}(\omega) + i \operatorname{Im} \tilde{Z}(\omega).$$

Then ω is a Z -critical Kähler metric if and only if

$$\arctan\left(\frac{\operatorname{Im} \tilde{Z}(\omega)}{\operatorname{Re} \tilde{Z}(\omega)}\right) = \varphi(\omega) \pmod{2\pi\mathbb{Z}}.$$

Proof. We calculate

$$\operatorname{Im}(e^{-i\varphi(X,L)} \tilde{Z}(\omega)) = \operatorname{Im}\left(e^{-i\varphi(X,L)} \exp\left(i \arctan\left(\frac{\operatorname{Im} \tilde{Z}(\omega)}{\operatorname{Re} \tilde{Z}(\omega)}\right)\right)\right),$$

which vanishes if and only if

$$\arctan\left(\frac{\operatorname{Im} \tilde{Z}(\omega)}{\operatorname{Re} \tilde{Z}(\omega)}\right) = \varphi(X, L) \pmod{2\pi\mathbb{Z}}.$$

□

Example 2.12. Consider the central charge

$$Z(X, L) = - \int_X e^{-iL} \cdot e^{-K_X} = - \sum_{j=0}^n \frac{(-i)^j}{j!(n-j)!} \int_X L^j \cdot (-K_X)^{n-j}$$

described in Example 2.5. The induced representative $\tilde{Z}(\omega)$ is given by

$$\tilde{Z}(\omega) = - \sum_{j=0}^n \frac{(-i)^j}{j!(n-j)!} \left(\frac{\omega^{n-j} \wedge \operatorname{Ric} \omega^j}{\omega^n} - \frac{j}{n-j+1} \Delta \left(\frac{\operatorname{Ric} \omega^{j-1} \wedge \omega^{n-j+1}}{\omega^n} \right) \right),$$

which produces what one might call the *deformed cscK equation*

$$\text{Im}(e^{-i\varphi(X,L)} \tilde{Z}(\omega)) = 0, \tag{2.2}$$

which is the manifold analogue of the deformed Hermitian Yang–Mills equation on a holomorphic line bundle. Strictly speaking this equation does not conform to our normalisation of the central charge, but the central charge $-n!(-i)^{3n+1} \overline{Z(X,L)}$ (with $Z(X,L)$ denoting the complex conjugate of $Z(X,L)$), which produces an equivalent partial differential equation, does.

Each component of this equation, of the form

$$\frac{\text{Ric } \omega^j \wedge \omega^{n-j}}{\omega^n} - \frac{j}{n-j+1} \Delta \left(\frac{\text{Ric } \omega^{j-1} \wedge \omega^{n-j+1}}{\omega^n} \right),$$

has appeared previously in the work of Chen–Tian [10, Definition 4.1] and Song–Weinkove [58, Section 2] in relation to the Kähler–Ricci flow. To understand the equation more fully, choose a point p and normal coordinates at p so that $\text{Ric } \omega$ is diagonal with diagonal entries $\lambda_1, \dots, \lambda_n$. Letting $\sigma_j(\omega)$ be the j^{th} elementary symmetric polynomial in these eigenvalues so that

$$(\omega + t \text{Ric } \omega)^n = \sum_{j=0}^n t^j \sigma_j(\omega) \omega^n,$$

the deformed cscK equation takes the much simpler form

$$\text{Im} \left(e^{-i\varphi(X,L)} \left(\sum_{j=0}^n (-i)^j (\sigma_j(\omega) - \Delta \sigma_{j-1}(\omega)) \right) \right) = 0.$$

This is a close analogue of the deformed Hermitian Yang–Mills equation on a holomorphic line bundle, but the presence of the terms involving the Laplacian seems to present significant new challenges.

We also remark that Schlitzer–Stoppa have studied a coupling of the deformed Hermitian Yang–Mills equation to the constant scalar curvature equation [55], which should be related to a combination of Bridgeland stability of the bundle and K-stability of the polarised variety, and which is of quite a different flavour to Equation (2.2).

We now focus on the large volume regime of the Z -critical equation.

Lemma 2.13. *Suppose the central charge Z_k is of map type, with $\theta_1 \in \Theta_1$ a real $(1, 1)$ -form. Then there is an expansion as $k \rightarrow \infty$ of the form*

$$\text{Im}(e^{-i\varphi_k} \tilde{Z}_k(\omega)) = k^{-1} (\text{Re}(\rho_{n-1}) L^n) (S(\omega) - \Lambda_\omega \theta_1 - n \mu_{\Theta_1}(X, L)) + O(k^{-2}),$$

where $\mu_{\Theta_1}(X, L) = \frac{-L^{n-1} \cdot (K_X + \Theta_1)}{L^n}$.

Proof. We first calculate

$$\text{Im} \left(\frac{\tilde{Z}_k(\omega)}{Z_k(X, L)} \right) = \frac{\text{Im } \tilde{Z}_k(\omega) \text{Re } Z_k(X, L) - \text{Re } \tilde{Z}_k(\omega) \text{Im } Z_k(X, L)}{\text{Re } Z_k(X, L)^2 + \text{Im } Z_k(X, L)^2}.$$

Since

$$\begin{aligned} Z_k(X, L) &= i L^n k^n + \rho_{n-1} L^{n-1} \cdot (K_X + \Theta_1) k^{n-1} + O(k^{n-2}), \\ \tilde{Z}_k(\omega) &= i - \frac{\rho_{n-1}}{n} (S(\omega) - \Lambda_\omega \theta_1) k^{-1} + O(k^{-2}), \end{aligned}$$

this is given by

$$\operatorname{Im}\left(\frac{\tilde{Z}_k(\omega)}{Z_k(X, L)}\right) = k^{-n-1}(\operatorname{Re}(\rho_{n-1})(S(\omega) - \Lambda_\omega\theta_1 - n\mu_{\Theta_1}(X, L)) + O(k^{-n-2})).$$

Writing $Z_k(X, L) = r_k e^{i\varphi_k}$, we have

$$\operatorname{Im}(e^{-i\varphi_k(X, L)}\tilde{Z}_k(\omega)) = r_k(X, L) \operatorname{Im}\left(\frac{\tilde{Z}_k(\omega)}{Z_k(X, L)}\right),$$

which since $r_k = L^n k^n + O(k^{n-1})$ implies the result. □

Thus, up to multiplication by the nonzero (in fact strictly negative) constant $\operatorname{Re}(\rho_{n-1})$, the ‘large volume limit’ of the Z -critical equation is the *twisted cscK equation*

$$S(\omega) - \Lambda_\omega\theta_1 = n\mu_{\Theta_1}(X, L);$$

the geometry of this equation is linked with that of the map $p : X \rightarrow Y$ [19, Section 4], where we have assumed θ_1 is the pullback of a Kähler metric from Y since the central charge is of map type.

This result can be seen as a differential-geometric counterpart to Proposition 2.13. When Z is actually nondegenerate, it follows that the ‘large volume limit’ of the Z -critical equation is the cscK equation, whereas on the algebro-geometric side, Proposition 2.13 shows that asymptotic Z -semistability implies K -semistability so that K -stability is the ‘large volume limit’ of asymptotic Z -stability. In order to more fully understand the links between the various concepts, we will later be interested in the analytic counterpart to K -semistability:

Definition 2.15. We say that (X, L) is *analytically K -semistable* if there is a test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) for which $(\mathcal{X}_0, \mathcal{L}_0)$ is a smooth polarised variety which admits a cscK metric.

It is conjectured that a K -semistable polarised variety admits a test configuration whose central fibre is K -polystable. The assumption of analytic K -semistability is thus a smoothness assumption since a smooth K -polystable polarised variety is itself expected to admit a cscK metric. It follows from work of Donaldson that analytically K -semistable varieties are actually K -semistable [25, Theorem 2].

3. Z -critical metrics on asymptotically Z -stable manifolds

Here, we prove our main result:

Theorem 3.1. *Let Z be an admissible central charge. Suppose that (X, L) is a polarised variety with discrete automorphism group which is analytically K -semistable, and suppose the deformation theory of its cscK degeneration is unobstructed. Then if (X, L) is asymptotically Z -stable, (X, L) admits Z_k -critical Kähler metrics for all $k \gg 0$.*

We will also state and prove a local converse, namely that existence implies stability in a local sense, later in Section 3.6. Here, we consider only the case that the central charge Z involves powers of K_X and no higher Chern classes, with the general case, in which the equation has a different flavour, being dealt with in Section 4. In comparison with the statement in the introduction, we are varying the central charge by k rather than scaling L ; these operations are clearly equivalent.

Unobstructedness of the deformation theory of the cscK degeneration of (X, L) will be used to ensure its Kuranishi space is smooth (as discussed in Section 3.4.6); this allows us to only consider genuine complex manifolds rather than almost complex manifolds in the analysis.

Admissibility requires three conditions. All of these conditions hold in the case of the deformed cscK equation described in Example 2.12. Firstly, we require that Z is nondegenerate, meaning the large

volume limit of the Z -critical equation is the cscK equation. Secondly, with the central charge given by

$$Z_k(X, L) = \sum_{l=0}^n \rho_l k^l \int_X L^l \left(\sum_{j=0}^n a_j K_X^j \right) \cdot \Theta,$$

we require that $\text{Re}(\rho_{n-1}) < 0, \text{Re}(\rho_{n-2}) > 0$ and $\text{Re}(\rho_{n-3}) = 0$. We also assume that $a_j = 1$ for all j for simplicity, though all that one needs is that the real parts are positive for $j = 0, 1, 2, 3$. These assumptions are used to control the behaviour of the linearisation of the equation. We expect that the condition on ρ_{n-3} can be removed.

The third condition concerns the form $\theta \in \Theta$. A basic technical assumption we make is that $\theta_2 = \theta_3 = 0$, though we also expect this assumption can be removed. We furthermore require that θ extends to a smooth, equivariant form on the test configuration $(\mathcal{X}, \mathcal{L})$ producing the cscK degeneration of (X, L) (which exists by analytic K-semistability), and also that θ extends to certain other deformations of $(\mathcal{X}_0, \mathcal{L}_0)$. More precisely, as we will recall in Section 3.4.6, the Kuranishi space of \mathcal{X}_0 admits an action of $\text{Aut}(\mathcal{X}_0, \mathcal{L}_0)$, and we require that θ extends smoothly to an equivariant form on the universal family over the Kuranishi space. The condition is modelled on the bundle situation [15], where the differential forms θ are forms on the base Y of the vector bundle E . Then if the polystable degeneration of E is F , there is still a map $F \rightarrow Y$, meaning one can still make sense of the relevant equation on F over Y .

3.1. Preliminaries on analytic Deligne pairings

As outlined in the Introduction, there are three steps to our work. The final step is to solve an abstract finite-dimensional problem in symplectic geometry, whereas the first two steps involve reducing to this finite-dimensional problem. A key tool for the first two steps is the theory of analytic Deligne pairings, established in [17, Section 4] and [57, Section 2.2], which give a direct approach to the properties of Deligne pairings in algebraic geometry. The additional flexibility of analytic Deligne pairings will allow us to include the extra forms θ into the theory, which do not fit into the usual algebro-geometric approach. Although the techniques developed in [17, 57] are essentially equivalent, our discussion is closer to that of Sjöström Dyrefelt [57].

The setup is simple case of the general theory, where we have a fixed smooth polarised variety; in general one considers holomorphic submersions. We thus let (X, L) be a smooth polarised variety of dimension n and suppose that $\eta_0, \dots, \eta_{n-p}$ are $n - p + 1$ closed $(1, 1)$ -forms on X . Any other forms $\eta'_j \in [\eta_j]$ are of the form $\eta'_j = \eta_j + i\partial\bar{\partial}\psi_j$ for some real-valued function ψ_j . We in addition suppose that θ is a closed real (p, p) -form on X which we will not be varied in our discussion and which has cohomology class $[\theta] = \Theta$. In our application, we will allow θ to be a closed complex (p, p) -form, but linearity of our constructions will allow us to reduce to the real case.

Definition 3.1. We define the *Deligne functional*, denoted

$$\langle \psi_0, \dots, \psi_{n-p}; \theta \rangle \in \mathbb{R},$$

by

$$\begin{aligned} \langle \psi_0, \dots, \psi_{n-p}; \theta \rangle &= \int_X \psi_0(\eta_1 + i\partial\bar{\partial}\psi_1) \wedge \dots \wedge (\eta_{n-p} + i\partial\bar{\partial}\psi_{n-p}) \wedge \theta \\ &+ \int_X \psi_1 \eta_0 \wedge (\eta_2 + i\partial\bar{\partial}\psi_2) \wedge \dots \wedge (\eta_{n-p} + i\partial\bar{\partial}\psi_{n-p}) \wedge \theta + \dots \\ &+ \int_X \psi_{n-p} \eta_0 \wedge \dots \wedge \eta_{n-p-1} \wedge \theta. \end{aligned}$$

The Deligne functional can be considered as an operator taking $n - p + 1$ functions to a real number. The definition is due to Sjöström Dyrefelt [57, Definition 2.1] and is implicit in [17, Section 4], in both

cases with $\theta = 0$. The inclusion of θ makes essentially no difference to the fundamental properties of the functional.

Proposition 3.2. *The Deligne functional $\langle \psi_0, \dots, \psi_{n-p}; \theta \rangle$ satisfies the following properties:*

- (i) *it is symmetric in the indices $0, 1, \dots, n - p$;*
- (ii) *it satisfies the ‘change of potential’ formula*

$$\langle \psi'_0, \dots, \psi'_{n-p}; \theta \rangle - \langle \psi_0, \dots, \psi_{n-p}; \theta \rangle = \int_X (\psi'_0 - \psi_0)(\eta_1 + i\partial\bar{\partial}\psi_1) \wedge \dots \wedge (\eta_m + i\partial\bar{\partial}\psi_{n-p}) \wedge \theta,$$

and analogous formulae hold when varying other ψ_j .

Proof. (i) This follows from an integration by parts formula when $\theta = 0$ [57, Proposition 2.3], and the proof in the general case is identical. The reason is that our form θ is fixed, so the fact that it is a form of higher degree is irrelevant.

(ii) This is immediate from the definition; this property is really the motivation for the chosen definition. Note that the statement when one changes any other ψ_j follows from the symmetry described as (i). □

We will be interested in the behaviour of Deligne functionals in families. The most basic property of these functionals in families is the following.

Proposition 3.3. *Suppose B is a complex manifold, and let $\pi : X \times B \rightarrow B$ be the projection, and write $\eta_0, \dots, \eta_{n-p}, \theta$ as the forms on $X \times B$ induced by pullback of the corresponding forms on X . Let $\psi_0, \dots, \psi_{n-p}$ be functions on $X \times B$, and denote by*

$$\langle \psi_0, \dots, \psi_{n-p}; \theta \rangle_B : B \rightarrow \mathbb{R}$$

the function of $b \in B$

$$\langle \psi_0, \dots, \psi_{n-p}; \theta \rangle_B(b) = \langle \psi_0|_{X \times \{b\}}, \dots, \psi_{n-p}|_{X \times \{b\}}; \theta|_{X \times \{b\}},$$

where this denotes the Deligne functional computed on the fibre $X \times \{b\}$ over $b \in B$. Then

$$\int_{X \times B/B} (\eta_0 + i\partial\bar{\partial}\psi_0) \wedge \dots \wedge (\eta_{n-p} + i\partial\bar{\partial}\psi_{n-p}) \wedge \theta = i\partial\bar{\partial} \langle \psi_0, \dots, \psi_{n-p}; \theta \rangle_B.$$

This result will produce Kähler potentials for natural Kähler metrics produced on holomorphic submersions via fibre integrals.

A closely related property of Deligne functionals allows the differential-geometric computation of intersection numbers on the total space of test configurations. To explain this, consider a test configuration $(\mathcal{X}, \mathcal{L}) \rightarrow \mathbb{C}$ with central fibre \mathcal{X}_0 smooth. It is equivalent to work with test configurations over twice the unit disc $2\Delta \subset \mathbb{C}$ (with the \mathbb{C}^* -action then meant only locally on 2Δ), and we will sometimes pass between the two conventions. The use of 2Δ is only for notational convenience to ensure $1 \in 2\Delta$. Fixing a fibre $\mathcal{X}_1 \cong X$, we obtain a form $\beta(t) \cdot \theta$ on $\mathcal{X} \setminus \mathcal{X}_0$ which we assume extends to a smooth form with cohomology class Θ on \mathcal{X} , where $\beta(t)$ denotes the \mathbb{C}^* -action on \mathcal{X} .

Let $\Omega_0, \Omega_1, \dots, \Omega_{n-p}$ be S^1 -invariant forms on \mathcal{X} with $[\Omega_0], [\Omega_1], \dots, [\Omega_{n-p}] \in \mathbb{C}^*$ -invariant cohomology classes on \mathcal{X} . Thus,

$$\beta(t)^* \Omega_j - \Omega_j = i\partial\bar{\partial}\psi_j^t$$

for some smooth family of functions ψ_j^t depending on t , with ψ_0 induced by the analogous procedure using $\omega_{\mathcal{X}}$. We next restrict ψ_j^t to our fixed fibre $\mathcal{X}_1 = X$. Set $\tau = -\log |t|^2$ so that $\tau \rightarrow \infty$ corresponds to $t \rightarrow 0$. The following then links the differential geometry with the intersection numbers of interest.

To link with the algebraic geometry to come, we assume $\Omega_j \in c_1(\mathcal{L}_j)$ for some line bundles \mathcal{L}_j on \mathcal{X} , though this is not essential.

Lemma 3.2 [57, Theorem 4.9][17, Theorem 1.4]. *We have*

$$\int_{\mathcal{X}} \mathcal{L}_0 \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-p} \cdot \Theta = \lim_{\tau \rightarrow \infty} \frac{d}{d\tau} \langle \psi_0^\tau, \dots, \psi_{n-p}^\tau; \theta \rangle (X).$$

Here, the intersection number on the left-hand side is computed over the compactification of the test configuration $\mathcal{X} \rightarrow \mathbb{P}^1$. The value on the right-hand side is the value of the Deligne functional on $X = \mathcal{X}_1$. This lemma is proven in [57, 17] only in the case $\theta = 0$, but as above the inclusion of the class Θ makes no difference to the proofs as θ extends smoothly to a \mathbb{C}^* -invariant form on \mathcal{X}_0 by assumption.

3.2. The Z-energy

We next fix a smooth polarised variety (X, L) . We fix the value k so that the central charge takes the form

$$Z(X, L) = \sum_{l=0}^n \rho_l \int_X L^l \cdot \left(\sum_{j=0}^n a_j K_X^j \right) \cdot \Theta.$$

We also fix a Kähler metric $\omega \in c_1(L)$ and denote by \mathcal{H}_ω the space of Kähler potentials with respect to ω . We then wish to define an energy functional

$$E_Z : \mathcal{H}_\omega \rightarrow \mathbb{R}$$

whose Euler–Lagrange equation is the Z-critical equation.

We proceed by first defining a functional

$$F_Z : \mathcal{H}_\omega \rightarrow \mathbb{C}$$

using the central charge and then define

$$E_Z(\psi) = \text{Im}(e^{-\varphi} F_Z(\psi)).$$

Our process is linear in the (n, n) -forms involved in the definition of $\tilde{Z}(\omega)$, so we fix a term $\int_X L^l \cdot K_X^j \cdot \Theta$, where we may assume Θ is a real cohomology class of degree $(n-l-j, n-l-j)$ again by linearity.

For this fixed term, we can use the theory of Deligne functionals to produce the desired functional

$$F_{Z,l} : \mathcal{H}_\omega \rightarrow \mathbb{R}.$$

Our reference metric ω induces a reference form $\text{Ric } \omega \in c_1(X)$. Any potential $\psi \in \mathcal{H}_\omega$ with associated Kähler metric $\omega_\psi = \omega + i\partial\bar{\partial}\psi$ induces a change in Ricci curvature

$$\text{Ric}(\omega_\psi) - \text{Ric}(\omega) = i\partial\bar{\partial} \log \left(\frac{\omega^n}{\omega_\psi^n} \right).$$

Thus, the theory of Deligne functionals over a point (i.e., taking the base B to be a point) produces a value

$$\frac{1}{l+1} \left\langle \overbrace{\psi, \dots, \psi}^{l+1 \text{ times}}, \overbrace{\log \left(\frac{\omega^n}{\omega_\psi^n} \right), \dots, \log \left(\frac{\omega^n}{\omega_\psi^n} \right)}^{j \text{ times}}; \theta \right\rangle \in \mathbb{R} \tag{3.1}$$

associated to our term $\int_X L^l \cdot K_X^j \cdot \Theta$ involved in the central charge. We emphasise that we are abusing notation slightly; θ is really only one component of the full representative of the unipotent class Θ . But by linearity, with real and imaginary terms handled separately, this produces the functional $E_Z : \mathcal{H}_\omega \rightarrow \mathbb{R}$, whose Euler–Lagrange equation we must calculate.

Remark 3.3. In the special case $\theta = 0$, the Deligne functional given by Equation (3.1) was introduced by Chen–Tian [10, Section 4] in relation to the Kähler–Ricci flow, where it was defined through its variation. Song–Weinkove later showed that these functionals can, again in the case $\theta = 0$, be obtained through Deligne pairings [58, Section 2.1]. Their work highlights the analytic significance of these functionals. Collins–Yau have introduced an energy functional designed to detect the existence of deformed Hermitian Yang–Mills connections on holomorphic line bundles [12, Section 2], and their functional bears some formal similarities with our Z -energy.

Definition 3.4. We define the Z -energy to be the functional $E_Z : \mathcal{H}_\omega \rightarrow \mathbb{R}$ associated to the central charge Z .

Remark 3.4. It is straightforward to check that $E_Z(\psi + c) = E_Z(\psi)$ so that one can view E_Z as a functional on Kähler metrics rather than Kähler potentials.

The most important aspect of the Z -energy is its Euler–Lagrange equation.

Proposition 3.5. Given a path of metrics $\psi_t \in \mathcal{H}_\omega$ with associated Kähler metric ω_t , we have

$$\frac{d}{dt} E_Z(\psi_t) = \int \dot{\psi}_t \operatorname{Im}(e^{-i\varphi} \tilde{Z}(\omega_t)) \omega_t^n.$$

Thus, the Euler–Lagrange equation for the Z -functional is the Z -critical equation.

Proof. By linearity, it suffices to calculate the variation of the operator $F_{Z,l} : \mathcal{H}_\omega \rightarrow \mathbb{R}$, given through Equation (3.1) as a Deligne pairing, along the path ψ_t . We will demonstrate that this variation is given by

$$\frac{d}{dt} F_{Z,l}(\psi_t) = \int_X \dot{\varphi}_t \left(\frac{\omega_t^l \wedge \operatorname{Ric} \omega_t^j \wedge \theta}{\omega_t^n} - \frac{j}{l+1} \Delta_t \left(\frac{\operatorname{Ric} \omega_t^{j-1} \wedge \omega_t^{l+1} \wedge \theta}{\omega_t} \right) \right) \omega_t^n,$$

which will imply the result we wish to prove since by definition of $\tilde{Z}(\omega_t)$ it is a sum of terms of the form

$$\tilde{Z}_l(\omega_t) = \frac{\omega_t^l \wedge \operatorname{Ric} \omega_t^j \wedge \theta}{\omega_t^n} - \frac{j}{l+1} \Delta_t \left(\frac{\operatorname{Ric} \omega_t^{j-1} \wedge \omega_t^{l+1} \wedge \theta}{\omega_t} \right).$$

The calculation from here is closely analogous to that of Song–Weinkove [58, Proposition 2.1], who proved the desired variational formula when $\theta = 0$. By the change of potential formula, our functional is given by

$$\begin{aligned} (l+1)F_{Z,l}(\psi) &= \sum_{m=0}^l \int_X \varphi \omega_\psi^m \wedge \operatorname{Ric} \omega^j \wedge \omega^{l-m} \wedge \theta \\ &\quad + \sum_{m=0}^{j-1} \int_X \log \left(\frac{\omega^n}{\omega_\psi^n} \right) \operatorname{Ric}(\omega_\psi)^m \wedge \operatorname{Ric} \omega^{j-m-1} \wedge \omega_\psi^{l+1} \wedge \theta. \end{aligned}$$

Differentiating along the path ψ_t gives

$$\begin{aligned}
 (l+1) \frac{d}{dt} F_{Z,l}(\psi_t) &= \sum_{m=0}^l \int_X \dot{\varphi}_t \omega_t^m \wedge \text{Ric } \omega^j \wedge \omega^{l-m} \wedge \theta \\
 &\quad + \sum_{m=0}^l m \int_X \varphi_t i \partial \bar{\partial} \dot{\psi}_t \omega_t^{m-1} \wedge \text{Ric } \omega^j \wedge \omega^{l-m} \wedge \theta \\
 &\quad - \sum_{m=0}^{j-1} \int_X \Delta_t \dot{\psi}_t \text{Ric}(\omega_t)^m \wedge \text{Ric } \omega^{j-m-1} \wedge \omega_t^{l+1} \wedge \theta \\
 &\quad - \sum_{m=0}^{j-1} m \int_X \log\left(\frac{\omega^n}{\omega_t^n}\right) i \partial \bar{\partial} \Delta_t \dot{\psi}_t \wedge \text{Ric } \omega_t^{m-1} \wedge \text{Ric } \omega^{j-m-1} \wedge \omega_t^{l+1} \wedge \theta \\
 &\quad + \sum_{m=0}^{j-1} (l+1) \int_X \log\left(\frac{\omega^n}{\omega_t^n}\right) \text{Ric } \omega_t^m \wedge \text{Ric } \omega^{j-m-1} \wedge i \partial \bar{\partial} \dot{\psi}_t \wedge \omega_t^l \wedge \theta,
 \end{aligned}$$

where Δ_t is the Laplacian with respect to the volume form ω_t , and where any term with negative exponent is taken to vanish. We note that this is self-adjoint with respect to ω_t^n . We use that

$$i \partial \bar{\partial} \dot{\psi}_t = \omega_t - \omega, \quad i \partial \bar{\partial} \log\left(\frac{\omega^n}{\omega_t^n}\right) = \text{Ric } \omega_t - \text{Ric } \omega$$

and the self-adjointness of the Laplacian just mentioned to obtain

$$\begin{aligned}
 (l+1) \frac{d}{dt} F_{Z,l}(\psi_t) &= \sum_{m=0}^l \int_X \dot{\varphi}_t \omega_t^m \wedge \text{Ric } \omega^j \wedge \omega^{l-m} \wedge \theta \\
 &\quad + \sum_{m=0}^l m \int_X \psi_t (\omega_t - \omega) \omega_t^{m-1} \wedge \text{Ric } \omega^j \wedge \omega^{l-m} \wedge \theta \\
 &\quad - \sum_{m=0}^{j-1} \int_X \dot{\psi}_t \Delta_t \left(\frac{\text{Ric}(\omega_t)^m \wedge \text{Ric } \omega^{j-m-1} \wedge \omega_t^{l+1} \wedge \theta}{\omega_t^n} \right) \omega_t^n \\
 &\quad - \sum_{m=0}^{j-1} m \int_X \dot{\psi}_t \Delta_t \left(\frac{(\text{Ric } \omega_t - \text{Ric } \omega) \wedge \text{Ric } \omega_t^{m-1} \wedge \text{Ric}(\omega)^{j-m-1} \wedge \omega_t^{l+1} \wedge \theta}{\omega_t^n} \right) \omega_t^n \\
 &\quad + \sum_{m=0}^{j-1} (l+1) \int_X \dot{\psi}_t (\text{Ric } \omega_t - \text{Ric } \omega) \wedge \text{Ric } \omega_t^m \wedge \text{Ric } \omega^{j-m-1} \wedge \omega_t^l \wedge \theta,
 \end{aligned}$$

where we use that θ is a closed form. We consider first the two terms involving Laplacians, which we see equal

$$\begin{aligned}
 &- \sum_{m=0}^{j-1} (m+1) \int_X \dot{\psi}_t \Delta_t \left(\frac{\text{Ric } \omega_t^m \wedge \text{Ric } \omega^{j-m-1} \wedge \omega_t^{l+1} \wedge \theta}{\omega_t^n} \right) \omega_t^n \\
 &+ \sum_{m=0}^{j-1} m \int_X \dot{\psi}_t \Delta_t \left(\frac{\text{Ric } \omega_t^{m-1} \wedge \text{Ric } \omega^{j-m} \wedge \omega_t^{l+1} \wedge \theta}{\omega_t^n} \right) \omega_t^n \\
 &= -j \int_X \dot{\psi}_t \Delta_t \left(\frac{\text{Ric } \omega_t^{j-1} \wedge \omega_t^{l+1} \wedge \theta}{\omega_t^n} \right) \omega_t^n.
 \end{aligned}$$

One similarly calculates that the remaining three terms sum to

$$(l + 1) \int_X \dot{\varphi}_t \omega_t^l \wedge \text{Ric} \omega_t^j \wedge \theta,$$

meaning that

$$\begin{aligned} \frac{d}{dt} F_{Z,l}(\psi_t) &= -\frac{j}{l+1} \int_X \dot{\psi}_t \Delta_t \left(\frac{\text{Ric}(\omega_t)^{j-1} \wedge \omega_t^l \wedge \theta}{\omega_t^n} \right) \omega_t^n + \int_X \dot{\psi}_t \omega_t^l \wedge \text{Ric} \omega_t^j \wedge \theta, \\ &= \int_X \dot{\psi}_t \left(\frac{\omega_t^l \wedge \text{Ric} \omega_t^j \wedge \theta}{\omega_t^n} - \frac{j}{l+1} \Delta_t \left(\frac{\text{Ric}(\omega_t)^{j-1} \wedge \omega_t^{l+1} \wedge \theta}{\omega_t^n} \right) \right) \omega_t^n, \end{aligned}$$

which is what we wanted to show. □

We now suppose that $(\mathcal{X}, \mathcal{L})$ is a test configuration for (X, L) with smooth central fibre (so that the total space is smooth), and with $\omega_{\mathcal{X}} \in c_1(\mathcal{L})$ a relatively Kähler S^1 -invariant metric. As will eventually be important, it follows by Ehresmann’s theorem that X_0 is diffeomorphic to X . The relatively Kähler metric $\omega_{\mathcal{X}}$ induces a Hermitian metric on the relative holomorphic tangent bundle $T_{\mathcal{X}/\mathbb{C}}$. Here, $T_{\mathcal{X}/\mathbb{C}}$ exists as the test configuration has smooth central fibre, meaning that $\pi : \mathcal{X} \rightarrow \mathbb{C}$ is a holomorphic submersion. This induces a metric on the relative anticanonical class $-K_{\mathcal{X}/\mathbb{C}}$ whose curvature we denote ρ . Following the process explained immediately before Lemma 3.2, we set

$$\beta(t)^* \omega_{\mathcal{X}} - \omega_{\mathcal{X}} = i\partial\bar{\partial}\psi_t.$$

Let $J_{\mathcal{V}}$ be the real holomorphic vector field inducing the S^1 -action on \mathcal{X} preserving $\omega_{\mathcal{X}}$, and define a function h on \mathcal{X} by

$$\mathcal{L}_{J_{\mathcal{V}}} \omega_{\mathcal{X}} = i\partial\bar{\partial}h$$

so that $\dot{\psi}_0 = h$. The form $\omega_{\mathcal{X}}$ restricts to an S^1 -invariant Kähler metric ω_0 on \mathcal{X}_0 .

Lemma 3.5 [33, Equation 2.1.4]. *The function h restricted to \mathcal{X}_0 is a Hamiltonian function with respect to ω_0 .*

Note that $\omega_{\mathcal{X}}$ is merely a Kähler form on each fibre, hence not actually a symplectic form on \mathcal{X} ; nevertheless, one could call h the Hamiltonian even in this situation. In what follows, we will also use the related property that

$$\frac{d}{dt} \beta(t)^* \omega_{\mathcal{X}} = i\partial\bar{\partial} \beta(t)^* h; \tag{3.2}$$

see [60, Example 4.26]. We can now relate the Z -energy to the algebro-geometric invariants of interest.

Proposition 3.6. *We have equalities*

$$\int_{\mathcal{X}_0} h \text{Im}(e^{-i\varphi} \tilde{Z}(\omega_0)) \omega_0^n = \lim_{\tau \rightarrow \infty} \frac{d}{d\tau} E_Z(\varphi_{\tau}) = \text{Im} \left(\frac{Z(\mathcal{X}, \mathcal{L})}{Z(X, L)} \right).$$

Proof. The second equality is an immediate consequence of our definition of E_Z through Deligne functionals and Lemma 3.2, using that

$$E_Z(\psi) = \text{Im}(e^{-i\varphi} F_Z(\psi)) = \text{Im} \left(\frac{F_Z(\psi)}{Z(X, L)} \right),$$

which is analogous to the fact used in Lemma 2.10. To prove the first equality, unravelling the definition of τ and the variational formula for E_Z proven in Proposition 3.5 with $\omega_t = \beta(t)^* \omega_{\mathcal{X}}|_{X=\mathcal{X}_t}$ we see that

$$\frac{d}{d\tau} E_Z(\varphi_\tau) = \int_X (\beta(t)^* h) \operatorname{Im}(e^{-\varphi} \tilde{Z}(\omega_t)) \omega_t^n,$$

where we have used that $\frac{d}{dt} \omega_t = i\partial\bar{\partial}\beta(t)^* h$ by Equation (3.2). But

$$\int_{\mathcal{X}_t} (\beta(t)^* h) \operatorname{Im}(e^{-i\varphi} \tilde{Z}(\omega_t)) \omega_t^n = \int_{\mathcal{X}_t} h \operatorname{Im}(e^{-i\varphi} \tilde{Z}(\omega_{\mathcal{X}}|_{\mathcal{X}_t})) \omega_{\mathcal{X}}|_{\mathcal{X}_t}^n,$$

which converges to $\int_{\mathcal{X}_0} h \operatorname{Im}(e^{-i\varphi} \tilde{Z}(\omega_0)) \omega_0^n$ as $t \rightarrow 0$, proving the result. □

This also produces an analogue of the classical Futaki invariant associated to a holomorphic vector field.

Corollary 3.7. *Suppose (X, L) admits a Z -critical Kähler metric, and suppose there is an S^1 -action on (X, L) . Then for any S^1 -invariant Kähler metric $\omega \in c_1(L)$ with associated Hamiltonian h we have*

$$\int_X h \operatorname{Im}(e^{-i\varphi} \tilde{Z}(\omega)) \omega^n = 0.$$

Proof. Note that a product test configuration, just as with any other test configuration, can be compactified to a family $(\mathcal{X}, \mathcal{L})$ over \mathbb{P}^1 . By the previous result, this integral is actually independent of $\omega \in c_1(L)$ as it equals

$$\int_X h \operatorname{Im}(e^{-i\varphi} \tilde{Z}(\omega)) \omega^n = \operatorname{Im} \left(\frac{Z(\mathcal{X}, \mathcal{L})}{Z(X, L)} \right),$$

which is patently independent of ω . But if ω' is the Z -critical Kähler metric, the corresponding integral on the left hand side clearly vanishes, as desired. □

3.3. Moment maps

3.3.1. Moment maps in finite dimensions

Many geometric equations have an interpretation through moment maps; this has been especially influential for the cscK equation. We will give two ways of viewing the Z -critical equation as a moment map. The first is a finite-dimensional geometric interpretation, on the base of a holomorphic submersion, while the second is closer in spirit to the infinite-dimensional viewpoint of Fujiki–Donaldson for the cscK equation [30, 23].

The setup is modelled on the situation of a test configuration $\pi : (\mathcal{X}, \mathcal{L}) \rightarrow \mathbb{C}$ for (X, L) with smooth central fibre. The properties of interest are firstly that there is an S^1 -action on both \mathbb{C} and $(\mathcal{X}, \mathcal{L})$, making π an S^1 -equivariant map, secondly that all fibres over the open dense orbit under the associated \mathbb{C}^* -action are isomorphic and thirdly that we may choose an S^1 -invariant relatively Kähler metric $\omega_{\mathcal{X}} \in c_1(\mathcal{L})$. If we had considered test configurations over the unit disc Δ , the same properties would be true with the \mathbb{C}^* -action meant only locally, in the sense that one only obtains an action induced by sufficiently small elements of the Lie algebra of \mathbb{C}^* .

More generally, we consider a holomorphic submersion $\pi : (\mathcal{X}, \mathcal{L}) \rightarrow B$ over a complex manifold B , with \mathcal{L} a relatively ample \mathbb{Q} -line bundle. We assume that B admits an effective action of a compact Lie group, which induces an effective local action of the complexification G of K . In addition, we assume that there is a K -action on $(\mathcal{X}, \mathcal{L})$ making π an equivariant map, and fix a K -invariant relatively Kähler metric $\omega_{\mathcal{X}} \in c_1(\mathcal{L})$. We lastly assume that there is an open dense orbit associated to G such that all fibres are isomorphic to (X, L) ; we denote this orbit as $\mathcal{X}^o \rightarrow B^o$. In practice, we will apply these results to the special case of an isotrivial family.

Let v be a holomorphic vector field on \mathcal{X} induced by an element of the Lie algebra \mathfrak{k} of K . Denote by h_v the function on \mathcal{X} defined by the equation

$$\mathcal{L}_{Jv}\omega_{\mathcal{X}} = i\partial\bar{\partial}h, \tag{3.3}$$

where J denotes the almost-complex structure of \mathcal{X} and the differentials on the right-hand side are also computed on \mathcal{X} . As in Section 3.1, we will refer to h as a Hamiltonian, even though $\omega_{\mathcal{X}}$ is only relatively Kähler. Note that h_v does restrict to a genuine Hamiltonian for v on the fibres over B on which v induces a holomorphic vector field; these are the fibres over points for which the corresponding vector field on B vanishes.

We now fix the input of the Z -critical equation. Setting $\varepsilon = k^{-1}$, our central charge can be written

$$Z_{\varepsilon}(X, L) = \sum_{l=0}^n \rho_l \varepsilon^{-l} \int_X L^l \cdot f(K_X) \cdot \Theta.$$

We have fixed a representative $\theta \in \Theta$, and we assume that the form $G.\theta$ defined on the dense orbit \mathcal{X}^o extends to a smooth form on \mathcal{X} itself and denote this form abusively by θ , which is automatically a G -invariant closed form on \mathcal{X} . The form $\omega_{\mathcal{X}}$ induces a metric on the relative holomorphic tangent bundle $T_{\mathcal{X}/B}$, and hence on its top exterior power $-K_{\mathcal{X}/B}$, and we denote the curvature of the latter metric as $\rho \in c_1(-K_{\mathcal{X}/B})$.

We associate to $Z_{\varepsilon}(X, L)$ an $(n+1, n+1)$ -form on \mathcal{X} as follows. We will define the $(n+1, n+1)$ -form on \mathcal{X} linearly in the terms of this expression, and hence it is sufficient to define an $(n+1, n+1)$ -form associated to a term of the form $\int_X L^l \cdot K_X^j \cdot \Theta$, to which we associate $\frac{1}{l+1} \omega_{\mathcal{X}}^{l+1} \wedge \rho^j \wedge \theta$. This induces a form $\tilde{Z}_{\varepsilon}(\mathcal{X}, \mathcal{L})$, and we set

$$\Omega_{\varepsilon} = \text{Im} \left(e^{-i\varphi_{\varepsilon}} \int_{\mathcal{X}/B} \tilde{Z}(\mathcal{X}, \mathcal{L}) \right) \tag{3.4}$$

to be the associated fibre integral. By general properties of fibre integrals, this produces a closed $(1, 1)$ -form on B . K -invariance of the forms on \mathcal{X} and of the map $\pi : \mathcal{X} \rightarrow B$ imply that Ω_{ε} is K -invariant. In general, the form Ω_{ε} may not be Kähler, which in addition requires positivity. In our applications, Ω_{ε} will, however, be Kähler for $0 < \varepsilon \ll 1$.

We let ω_b denote the restriction of $\omega_{\mathcal{X}}$ to the fibre \mathcal{X}_b over b and denote $\text{Im}(e^{-i\varphi_{\varepsilon}} \tilde{Z}_{\varepsilon}(\omega_b))$ the Z -critical operator computed on \mathcal{X}_b with respect to ω_b . We similarly set $h_{v,b}$ be the restriction of a Hamiltonian h_v to the fibre \mathcal{X}_b . Define a map $\mu_{\varepsilon} : B \rightarrow \mathfrak{k}^*$ by

$$\langle \mu_{\varepsilon}, v \rangle(b) = -\frac{1}{2} \int_{\mathcal{X}_b} h_{v,b} \text{Im}(e^{-i\varphi_{\varepsilon}} \tilde{Z}_{\varepsilon}(\omega_b)) \omega_b^n,$$

where $v \in \mathfrak{k}$ is viewed as inducing a holomorphic vector field on \mathcal{X} to induce the Hamiltonian h_v .

Theorem 3.6. μ_{ε} is a moment map with respect to the K -action on B and with respect to the form Ω_{ε} .

Here, we mean that the defining conditions of a moment map are satisfied, namely that

$$d\langle \mu_{\varepsilon}, v \rangle = -\iota_v \Omega_{\varepsilon},$$

and μ is K -equivariant when \mathfrak{k}^* is given the coadjoint action; in general we emphasise that Ω_{ε} is not actually a symplectic form (although for ε sufficiently small it will be in our applications, producing genuine moment maps). In the contraction $\iota_v \Omega_{\varepsilon}$, we view $v \in \mathfrak{k}$ as inducing a holomorphic vector field on B .

Proof. We first show that the equation $d\langle \mu_{\varepsilon}, v \rangle = -\iota_v \Omega_{\varepsilon}$ holds. Note that it is enough to show that this holds on the dense locus B^o since both sides of the equation extend continuously to B .

We fix a point $b \in B$ at which we wish to demonstrate the moment map equation and consider the orbit U of $b \in B$ under the G -action. We fix an isomorphism $(\mathcal{X}_b, \mathcal{L}_b) \cong (X, L)$ and simply write $\omega_b = \omega$. The G -action induces an isomorphism

$$(\mathcal{X}, \mathcal{L}) \cong (X \times U, L),$$

where $B^\circ \cong U \subset G$ is a submanifold. Since we only obtain a local action of G on B , U may not consist of all of G in general. The isomorphism $\mathcal{X} \cong X \times U$ is in addition compatible with the projections to B° . The relatively Kähler metric $\omega_{\mathcal{X}} \in c_1(\mathcal{L})$ thus induces a form $\omega_{X \times U}$ on $X \times U$ and we can define

$$E_Z : U \rightarrow \mathbb{R}$$

defined as the Z -energy with respect to the reference metric ω (or rather its pullback to $X \times U$) and the varying metric $\omega_{X \times U}$ on the fibres over U . Proposition 3.3 then implies that on $U \cong B^\circ$ we have

$$i\partial\bar{\partial}E_{Z_\varepsilon} = \Omega_\varepsilon. \tag{3.5}$$

By Proposition 3.5, the derivative of the Z -energy along any path $\omega_t = \omega + i\partial\bar{\partial}\psi_t$ satisfies

$$\frac{d}{dt}E_{Z_\varepsilon}(\psi_t) = \int_X \dot{\psi}_t \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\omega_t)) \omega_t^n.$$

Considering the path ω_t defined above, the defining property of the Hamiltonian h means that $\dot{\psi}_0 = h_{v,b}$ on $\mathcal{X}_b \cong X$. Thus,

$$\left. \frac{d}{dt} \right|_{t=0} \int_X \dot{\psi}_t \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}(\omega_t)) \omega_t^n = \int_X h_{v,b} \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}(\omega)) \omega^n.$$

But this is then all we need: By a standard calculation [59, Lemma 12]

$$\iota_v(i\partial\bar{\partial}E_{Z_\varepsilon}) = \frac{1}{2}d(Jv(E_{Z_\varepsilon})),$$

so it follows that

$$\iota_v(\Omega_\varepsilon) = \frac{1}{2}d(Jv(E_{Z_\varepsilon})) = \frac{1}{2}d\left(\frac{d}{dt}E_{Z_\varepsilon}(\exp(Jvt) \cdot p)\right) = \frac{1}{2}d\left(\frac{d}{dt}E_{Z_\varepsilon}(\psi_t)\right) = -d\langle \mu_\varepsilon, h \rangle, \tag{3.6}$$

proving the first defining property of a moment map with respect to v at the point b . But by continuity this implies that the same property holds on all of B .

What remains to prove is K -equivariance of μ_ε , which requires us to show for all $g \in K$

$$\langle \mu_\varepsilon(g \cdot b), v \rangle = \langle \mu_\varepsilon(g \cdot b), g^{-1} \cdot v \rangle,$$

where K acts on \mathfrak{k} by the adjoint action. However, the Hamiltonian on \mathcal{X} with respect to $g^{-1} \cdot v$ is simply the pullback g^*h_v , meaning $g^*h_{v,g(b)} = h_{g^{-1} \cdot v, b}$. Thus, using K -invariance of $\omega_{\mathcal{X}}$, the equality

$$\int_{\mathcal{X}_{g(b)}} h_{v,g(b)} \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\omega_{g(b)})) \omega_{g(b)}^n = \int_{\mathcal{X}_b} g^*h_{v,g(b)} \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\omega_b)) \omega_b^n$$

is enough to imply equivariance. □

Remark 3.7. All results in this section hold assuming less regularity than smoothness, for example considering L^2_k -Kähler metrics for k sufficiently large.

Remark 3.8. Our approach is partially inspired by an early use of Deligne pairings by Zhang, where he considered a family of Kähler metrics induced by embeddings of a fixed projective variety into projective space and where he showed that the existence of a balanced embedding is equivalent to Chow stability [70]. The approach above seems to be new even in the limiting case that $Z_\varepsilon(X, L) = \varepsilon^{-n} \int_X (iL^n - \varepsilon K_X \cdot L^{n-1})$, where this gives a new interpretation of the scalar curvature as a moment map.

3.3.2. Moment maps in infinite dimensions

We next demonstrate how the Z -critical equation appears as a moment map in infinite dimensions. While this is important conceptually, it will also be used in order to understand the linearisation of the Z -critical equation. Analogously to the Fujiki–Donaldson moment map interpretation of the cscK equation [30, 23], we fix a compact symplectic manifold (M, ω) and consider the space $\mathcal{J}(M, \omega)$ of almost complex structures compatible with ω . We refer to Scarpa [54, Section 1.3] and Gauduchon [33, Section 8] for a good exposition of this space its properties. $\mathcal{J}(M, \omega)$ naturally has the structure of an infinite-dimensional complex manifold; its tangent space at $J \in \mathcal{J}(M, \omega)$ is given by

$$T_J \mathcal{J}(M, \omega) = \{A : TM \rightarrow TM \mid AJ + JA = 0, \omega(\cdot, A\cdot) = \omega(A\cdot, \cdot)\},$$

with complex structure defined by $A \rightarrow JA$ on $T_J \mathcal{J}(M, \omega)$. At an almost complex structure J , the tangent space can be identified with $\Omega^{0,1}(TX^{1,0})$, the space of $(0, 1)$ -forms with values in holomorphic vector fields [54, p. 14].

We let \mathcal{G} denote the group of exact symplectomorphisms of (M, ω) , which acts naturally on $\mathcal{J}(M, \omega)$. The Lie algebra of \mathcal{G} can be identified with $C_0^\infty(M)$, the functions which integrate to zero, through the Hamiltonian construction. For $h \in C_0^\infty(M)$, we denote by v_h the associated Hamiltonian vector field. The infinitesimal action of \mathcal{G} is then given by

$$\begin{aligned} P : C_0^\infty(X, \mathbb{R}) &\rightarrow T_J \mathcal{J}(M, \omega), \\ Ph &= \mathcal{L}_{v_h} J. \end{aligned} \tag{3.7}$$

Under the identification of $T_J \mathcal{J}(M, \omega)$ with $\Omega^{0,1}(TX^{1,0})$, the operator P corresponds to the operator [54, Lemma 1.4.3]

$$\begin{aligned} \mathcal{D} : C_0^\infty(X, \mathbb{R}) &\rightarrow \Omega^{0,1}(TX^{1,0}), \\ \mathcal{D}h &= \bar{\partial} \nabla^{1,0} h. \end{aligned} \tag{3.8}$$

The operator \mathcal{D} plays a central role in the theory of cscK metrics. Note, for example, that its kernel consists of functions generating global holomorphic vector fields; these are called *holomorphy potentials*.

Now, let (X, L) be a smooth polarised variety with complex structure $J_X \in \mathcal{J}(M, \omega)$. We assume for the moment that $\text{Aut}(X, L)$ is trivial and will later consider the other case of interest for our main results, namely that $\text{Aut}(X, L)$ is finite. We denote by $\mathcal{J}_X(M, \omega) \subset \mathcal{J}(M, \omega)$ the set of $J' \in \mathcal{J}(M, \omega)$ such that there is a diffeomorphism γ , which lies in the connected component of the identity inside the space of diffeomorphisms of M , with $\gamma \cdot J' = J_X$. Thus, $\mathcal{J}_X(M, \omega)$ corresponds to complex structures producing manifolds biholomorphic to X . This space is discussed by Gauduchon [33, Section 8.1]; for us an important point will be that $\mathcal{J}_X(M, \omega)$ is actually a complex submanifold of $\mathcal{J}(M, \omega)$ [33, Proposition 8.2.3]. As in the work of Fujiki [30, Section 8], the space $\mathcal{J}(M, \omega)$ admits a universal family $(\mathcal{U}, \mathcal{L}_{\mathcal{U}}) \rightarrow \mathcal{J}(M, \omega)$ which hence restricts to a family $(\mathcal{U}, \mathcal{L}_{\mathcal{U}})$ over $\mathcal{J}_X(M, \omega)$. The fibre over a complex structure $J_b \in \mathcal{J}(M, \omega)$ is simply the complex manifold (M, J_b) .

We next induce a form $\theta_{\mathcal{U}}$ on $\mathcal{U} \rightarrow \mathcal{J}_X(M, \omega)$ associated to the form θ on \mathcal{U} , using the fact that each fibre of $\mathcal{U} \rightarrow \mathcal{J}_X(M, \omega)$ is isomorphic to X . For any $B \subset \mathcal{J}_X(M, \omega)$ a finite-dimensional complex submanifold, the Fischer–Grauert theorem produces an isomorphism $\mathcal{U}|_B \cong X \times B$ commuting with the

maps to B . One can extend this isomorphism to an isomorphism of line bundles

$$\Psi_B : (\mathcal{U}|_B, \mathcal{L}|_B) \cong (X, L) \times B, \tag{3.9}$$

perhaps after shrinking B [49, Lemma 5.10] (while the proof given by Newstead assumes algebraicity of B , it also holds in the holomorphic category [35, Lemma 6.3]). Then as we have assumed $\text{Aut}(X, L)$ is actually trivial, the isomorphism Ψ_B is actually unique. Pulling back θ on X via Ψ_B induces a closed form θ_B on $\mathcal{U}|_B$, and uniqueness then means that the forms θ_B glue to a closed form $\theta_{\mathcal{U}}$ on all of \mathcal{U} . Similarly, using these isomorphisms, the Z -energy induces a function

$$E_Z : \mathcal{J}_X(M, \omega) \rightarrow \mathbb{R} \tag{3.10}$$

after fixing the reference Kähler metric ω on X .

Denote by Ω_{ε} the family of closed $(1, 1)$ -forms on $\mathcal{J}_X(M, \omega)$ given by

$$\Omega_{\varepsilon} = \text{Im} \left(e^{-i\varphi_{\varepsilon}} \int_{\mathcal{U}|\mathcal{J}_X(M, \omega)} \tilde{Z}_{\varepsilon}(\mathcal{U}, \mathcal{L}_{\mathcal{U}}) \right), \tag{3.11}$$

where $\tilde{Z}_{\varepsilon}(\mathcal{U}, \mathcal{L}_{\mathcal{U}})$ is defined just as in Equation (3.4) using the relatively Kähler metric $\omega_{\mathcal{X}} \in c_1(\mathcal{L})$, the form $\rho \in c_1(-K_{\mathcal{U}|\mathcal{J}_X(M, \omega)})$ induced by the relatively Kähler metric $\omega_{\mathcal{X}}$ and $\theta_{\mathcal{U}}$. The forms Ω_{ε} are then closed \mathcal{G} -invariant $(1, 1)$ -forms which are not, however, positive in general.

The Z -critical operator can be viewed as a function

$$\text{Im}(e^{-i\varphi_{\varepsilon}} \tilde{Z}) : \mathcal{J}_X(M, \omega) \rightarrow C_0^{\infty}(X),$$

which we wish to demonstrate is a moment map with respect to the Ω_{ε} . Thus, we need to understand the behaviour of $\text{Im}(e^{-i\varphi_{\varepsilon}} \tilde{Z})$ under a change in complex structure. We will use a similar idea to Section 3.3.1, namely to realise the Z -energy as a Kähler potential, which requires us to relate the change in complex structure to the change in metric structure. Consider a path $J_t \in \mathcal{J}_X(M, \omega)$, and let $F_t \cdot J_t = J_X$ for F_t diffeomorphisms of X . Then we obtain a corresponding path of Kähler metrics $F_t^* \omega = \omega_t = \omega + i\partial\bar{\partial}\psi_t$ compatible with J_X . Then the key fact we need is that the path J_t satisfies [61, p. 1083]

$$\frac{d}{dt} \Big|_{t=0} J_t = JP\dot{\psi}_0. \tag{3.12}$$

Theorem 3.9. *The map*

$$\mu_{\varepsilon} = \text{Im}(e^{-i\varphi_{\varepsilon}} \tilde{Z}) : \mathcal{J}_X(M, \omega) \rightarrow C_0^{\infty}(X)$$

is a moment map for the \mathcal{G} -action on $\mathcal{J}_X(M, \omega)$ with respect to the forms Ω_{ε} .

Here, the statement means that the moment map condition is satisfied, note again that Ω_{ε} may not actually be positive (hence Kähler) in general.

Proof. Fix a point $b \in \mathcal{J}_X(M, \omega)$ at which we wish to demonstrate the moment map property, and let Ph be the tangent vector at b induced the element $h \in \text{Lie } \mathcal{G} \cong C_0^{\infty}(M)$. We show that for any finite-dimensional complex submanifold $B \subset \mathcal{J}_X(M, \omega)$ containing Ph , the moment map equality

$$-\iota_{Ph} \Omega_{\varepsilon} = d\langle \mu_{\varepsilon}, Ph \rangle$$

holds. The proof of this is essentially the same as that of Theorem 3.6.

Perhaps after shrinking B , the family $(\mathcal{U}_B, \mathcal{L}_B) \rightarrow B$ satisfies

$$(\mathcal{U}_B, \mathcal{L}_B) \cong (X, L) \times B \tag{3.13}$$

by the argument of Equation (3.9). We thus obtain a function

$$E_Z : B \rightarrow \mathbb{R}$$

by Equation 3.10, which by the argument of Theorem 3.6 satisfies

$$i\partial\bar{\partial}E_Z = \Omega_\varepsilon,$$

an equality of $(1, 1)$ -forms on B . Since this holds for each B , it also holds on $\mathcal{J}_X(M, \omega)$.

Consider a path $J_{b_t} \in B$ of almost complex structures such that the induced tangent vector at $t = 0$ is given by JPh . Then we obtain a corresponding path of Kähler metrics $\omega_t = \omega + i\partial\bar{\partial}\psi_t$ through the isomorphism of Equation (3.13), and Equation (3.12) implies that $\dot{\psi}_0 = h$. It follows that

$$\left. \frac{d}{dt} \right|_{t=0} E_Z(J_t) = \int_X h \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}(J_b)) \omega^n,$$

which means that as functions on $\mathcal{J}_X(M, \omega)$ we have

$$\langle dE_Z, JPh \rangle = \int_X h \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}(J_b)) \omega^n = -\langle \mu_\varepsilon(b), Ph \rangle.$$

Then the same argument as Equation (3.6) implies that

$$\iota_{Ph} \Omega_\varepsilon = \iota_{Ph} i\partial\bar{\partial}E_Z = -d\langle \mu_\varepsilon(b), Ph \rangle,$$

which proves the defining equation of the moment map.

The \mathcal{G} -action on $\operatorname{Lie}(\mathcal{G})$ is the adjoint action, which corresponds to pullback of Hamiltonians [54, Equation (1.5)]. Then equivariance follows by the same argument as Theorem 3.6. \square

Remark 3.10. Gauduchon has given another proof that the scalar curvature is a moment map on $\mathcal{J}_X(M, \omega)$ in a similar spirit, but using more direct properties of the Mabuchi functional rather than Deligne pairings [33, Proposition 8.2].

While positivity is not guaranteed for all ε , it will be important to have positivity for finite-dimensional submanifolds and for ε sufficiently small.

Proposition 3.8. *Let $B \subset \mathcal{J}_X(M, \omega)$ be a complex submanifold. Then Ω_ε restricts to a Kähler metric for all $0 < \varepsilon \ll 1$.*

Proof. Fujiki proves that the form

$$\Omega_0 = - \int_{\mathcal{U}/\mathcal{J}(M, \omega)} \rho \wedge \omega^n + \frac{n}{n+1} \mu(X, L) \int_{\mathcal{U}/\mathcal{J}(M, \omega)} \omega^{n+1}$$

is actually a Kähler metric on $\mathcal{J}(M, \omega)$ and agrees with the usual Kähler metric on $\mathcal{J}(M, \omega)$ used in the moment map interpretation of the scalar curvature on $\mathcal{J}(M, \omega)$ [30, Theorem 8.3]. Thus, since

$$Z_\varepsilon(X, L) = \varepsilon^{-n} \int_X (iL^n + \operatorname{Re}(\rho_{n-1}) \varepsilon K_X \cdot L^{n-1}) + O(\varepsilon^{-n+2}),$$

we have

$$\Omega_\varepsilon = \operatorname{Im} \left(e^{-i\varphi_\varepsilon} \int_{\mathcal{X}/B} \tilde{Z}(\mathcal{X}, \mathcal{L}) \right) = -\varepsilon \operatorname{Re}(\rho_{n-1}) n \Phi^* \Omega + O(\varepsilon^2),$$

which implies the result.

One can also prove Fujiki’s result, namely the equality of the fibre integral Ω_0 and the usual Kähler metric on $\mathcal{J}_X(M, \omega)$, directly, giving another proof. By [33, Equation 8.1.10], the tangent space $T_J \mathcal{J}_X(M, \omega)$ is spanned by elements of the form Ph, JPh , for $h \in C_0^\infty(M, \omega)$. But it follows from the argument of Theorem 3.9 that the moment map for the \mathcal{G} -action on $\mathcal{J}_X(M, \omega)$ is given by the scalar curvature. But then since

$$\iota_{Ph} \Omega_{\mathcal{J}} = \iota_{Ph} \Omega_0$$

for all $h \in C_0^\infty(M)$, it follows that the forms actually agree on $\mathcal{J}_X(M, \omega)$. □

In particular, if $B \subset \mathcal{J}_X(M, \omega)$ is a complex submanifold invariant under $K \subset \mathcal{G}$, we obtain a genuine sequence of moment maps μ_ε for $\varepsilon \ll 0$ with respect to genuine Kähler metrics Ω_ε .

Remark 3.11. In the case $\text{Aut}(X, L)$ is nontrivial, but still finite, we denote by $G = \text{Aut}(X, L)$, assume θ is G -invariant and work G -equivariantly. Let $\mathcal{J}_X(M, \omega)^G$ denote the fixed locus of the G -action on $\mathcal{J}_X(M, \omega)$. Then while the isomorphisms

$$(\mathcal{U}_B, \mathcal{L}_B) \cong (X, L) \times B$$

of Equation 3.13, which were used to construct the form $\theta_{\mathcal{U}}$ on \mathcal{U} and function E_Z on $\mathcal{J}_X(M, \omega)$ are no longer unique, they are unique up to the action of G . But since θ is G -invariant by assumption, working on $\mathcal{J}_X(M, \omega)^G$ instead allows us to construct functions E_Z on $\mathcal{J}_X(M, \omega)^G$ and a form $\theta_{\mathcal{U}^G}$ on $\mathcal{U}^G \rightarrow \mathcal{J}_X(M, \omega)^G$. The proof of the moment map property is then identical to the case G is trivial.

Remark 3.12. As with the previous section, all results in this section hold assuming less regularity than smoothness, for example considering L_k^2 -complex structures for k sufficiently large.

3.4. The core analytic argument

We next turn to analytic aspects of the Z -critical equation necessary to prove our main result, for which we assume Z is admissible. We assume here that (X, L) admits a cscK metric, and construct extremal solutions of the Z -critical equation for $k \gg 0$. In particular, this gives a general construction of Z -critical metrics. The reason we produce extremal solutions is that we allow (X, L) to have automorphisms; in the proof of our main theorem, we will apply these techniques to the cscK degeneration of the manifold of interest. At a key point in proving this result, we will assume that the manifold is a cscK degeneration of a polarised manifold with discrete automorphism group and will emphasise this point when it arises.

Theorem 3.13. *Suppose (X, L) admits a cscK metric and is a degeneration of a polarised manifold with discrete automorphism group. Then $c_1(L)$ admits solutions of the equation*

$$\bar{\partial} \nabla_\varepsilon^{1,0} (\text{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\omega_\varepsilon))) = 0$$

for all $0 < \varepsilon \ll 1$.

We call solutions of this equation Z -extremal metrics. Here, the gradient is defined using ω_ε , so the condition asks that $\text{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\omega_\varepsilon))$ is a holomorphy potential. In particular, if $\text{Aut}(X, L)$ is discrete, this gives a general construction of Z -critical Kähler metrics.

The difficulty in proving this result is that the Z -critical equation is a sixth-order PDE, while the cscK equation is fourth order. The basic idea to circumvent this is to use quantitative inverse function theorem, analogously to a strategy of Hashimoto for another problem in Kähler geometry [36]. This requires us to firstly construct approximate solutions of the Z -critical equation, which is straightforward. The main difficulty is then to understand the mapping properties of the linearised operator, which occupies much of the current section. When (X, L) admits automorphisms, the kernel of the linearised operator is nontrivial, which forces us to consider the more general Z -extremal equation, much as in the classical work of LeBrun–Simanca [41]. When applying this result to the case of an analytically K-semistable

manifold, we will then employ Kuranishi theory as in the fundamental work of Brönnle and Székelyhidi [6, 61] and will use the moment map property of the Z -critical equation and a version of the Kempf–Ness theorem to apply our assumption of asymptotic Z -stability.

We consider the Z -critical operator as an operator on the space of Kähler potentials with respect to a reference metric $\omega \in c_1(L)$

$$G_\varepsilon : \mathcal{H}_\omega \rightarrow \mathbb{R},$$

$$G_\varepsilon(\psi) = \text{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\omega + i\partial\bar{\partial}\psi)).$$

We set $\omega_\psi = \omega + i\partial\bar{\partial}\psi$. The main goal will be to understand the mapping properties of the linearisation of G_ε and variants of G_ε , in order to apply a quantitative version of the implicit function theorem.

We recall our central charge takes the form

$$Z_k(X, L) = \sum_{l=0}^n \rho_l \varepsilon^{-l} \int_X L^l \left(\sum_{j=1}^n a_j K_X^j \right) \cdot \Theta.$$

The simplest, but rather degenerate, case of this equation is when $a_j = 0$ for all $j \geq 2$, which means that the terms in the definition of the Z -critical equation involving the Laplacian vanishes; see Equation (2.1). In this case, for $\varepsilon \ll 1$ the equation is a fourth-order elliptic partial differential equation. In the general case which is of interest to us, the equation jumps from a fourth-order equation at $\varepsilon = 0$ to a sixth-order equation for $\varepsilon > 0$, which causes several additional analytic difficulties.

Lemma 3.14. *Suppose $\rho_{n-2} \neq 0$ and $a_2 \neq 0$. Then for all $0 < \varepsilon \ll 1$, the Z -critical equation is a sixth-order elliptic partial differential equation.*

Proof. Clearly, G_ε is a sixth-order partial differential operator as $\rho_{n-2} \neq 0$ and $a_2 \neq 0$, and we must show that it is elliptic, which means that we must show that its linearisation is elliptic. This is a condition on the highest-order derivatives, so we replace the Z -critical operator with the sum of the terms involving six derivatives. Since we are interested in the case $\varepsilon \ll 1$, we need only consider the lowest-order terms in ε . When one scales $0 \ll \varepsilon < 1$, the lowest-order term in ε is then

$$\psi \rightarrow c\Delta_\psi \left(\frac{\text{Ric } \omega_\psi \wedge \omega_\psi^{n-1}}{\omega_\psi^n} \right),$$

where Δ_ψ is the Laplacian with respect to ω_ψ and $c \neq 0$ since any forms involving the unipotent class Θ will be of higher order in ε . By the product rule, the linearisation of this operator along the path $t\psi$ is given by

$$\Delta^3\psi + \text{lower-order derivatives}$$

since the linearisation of the scalar curvature operator is given by

$$\left. \frac{d}{dt} \right|_{t=0} S(\omega + i\partial\bar{\partial}\psi) = \Delta^2\psi - S(\omega)\Delta\psi + n(n-1) \frac{i\partial\bar{\partial}\psi \wedge \text{Ric } \omega \wedge \omega^{n-2}}{\omega^n}.$$

This demonstrates ellipticity. □

Note that the condition $a_2 \neq 0$ is part of our hypothesis that Z is admissible, used to prove our main result.

3.4.1. Understanding the model operator

Let

$$\mathcal{F}_\varepsilon : C^\infty(X, \mathbb{R}) \rightarrow \mathbb{R}$$

denote the linearisation of the Z -critical operator G_ε . In order to understand the mapping properties of \mathcal{F}_ε , we will compare it to a simpler model operator. Much as with the linearisation of the scalar curvature, a key operator will be the operator

$$\mathcal{D}\psi = \bar{\partial}\nabla^{1,0}\psi,$$

as mentioned in Equation (3.8), whose kernel $\ker \mathcal{D}$ consists of functions inducing holomorphic vector fields on X . We denote the vector space of such functions, namely the holomorphy potentials, by \mathfrak{h} ; we include the constant functions in our definition. The space of holomorphy potentials of integral zero is isomorphic to the Lie algebra of the automorphism group $\text{Aut}(X, L)$. Letting \mathcal{D}^* be the L^2 -adjoint of \mathcal{D} with respect to the inner product induced by ω , the Lichnerowicz operator is given by $\mathcal{D}^*\mathcal{D}$; this is a fourth-order elliptic linear partial differential operator, whose kernel consists of holomorphy potentials [60, Definition 4.3]. It is then well-known that the linearisation of the scalar curvature at a cscK metric is given by $-\mathcal{D}^*\mathcal{D}$ [60, Lemma 4.4].

Another important term involved in the model operator is a sixth-order elliptic operator, defined as follows. As the vector bundle $TX^{1,0}$ is a holomorphic vector bundle, it admits a $\bar{\partial}$ -operator; we let $\bar{\partial}^*$ denote its L^2 -adjoint. We will then also consider the operator $\mathcal{D}^*\bar{\partial}^*\bar{\partial}\mathcal{D}$, which can also be written

$$\nabla^{1,0*}(\bar{\partial}^*\bar{\partial})^2\nabla^{1,0} = \nabla^{1,0*}\Delta_{\bar{\partial}}^2\nabla^{1,0},$$

where $\Delta_{\bar{\partial}}$ denotes the $\bar{\partial}$ -Laplacian. In particular, its symbol agrees with that of Δ^3 .

We will also need to consider two further operators H_1, H_2 , which are arbitrary self-adjoint operators satisfying for $j = 1, 2$

$$\int_X \gamma H_j \psi \omega^n = \int_X (\mathcal{D}\gamma, \mathcal{D}\psi)_{g_j} d\mu_j,$$

where each $d\mu_j$ is a smooth (n, n) -form and each

$$g_j : \Gamma(T^{1,0}X \otimes \Omega^{0,1}(X)) \otimes \Gamma(T^{1,0}X \otimes \Omega^{0,1}(X)) \rightarrow \mathbb{R}$$

is a smooth bilinear pairing but not necessarily a metric. Our model operator will then take the form

$$\mathcal{G}_\varepsilon = c_0\mathcal{D}^*\mathcal{D} + \varepsilon(c_1\mathcal{D}^*\bar{\partial}^*\bar{\partial}\mathcal{D} + H_1) + \varepsilon^2(c_2\mathcal{D}^*\bar{\partial}^*\bar{\partial}\mathcal{D} + H_2), \tag{3.14}$$

where c_0 and c_1 are strictly positive. Note that this is a self-adjoint elliptic operator for ε sufficiently small, as its symbol agrees with that of $\varepsilon c_1\Delta^3 + \varepsilon^2 c_2\Delta^3$, which is elliptic for ε sufficiently small since $c_1 > 0$. As we explain in Remark 3.16, the ε^2 -term is included as the estimates we prove will only allow us to perturb the operator by an $O(\varepsilon^3)$ -term while retaining the relevant mapping properties.

We now work with Sobolev spaces L^2_k for some large k . We let $\mathfrak{h}^2_{k,\perp}$ denote the L^2 -orthogonal complement of the holomorphy potentials inside L^2_k . Note that the holomorphy potentials themselves are actually smooth, being the kernel of the elliptic operator $\mathcal{D}^*\mathcal{D}$, but we will sometimes also denote the space of holomorphy potentials as \mathfrak{h}^2_k when considered as a subspace of L^2_k .

Lemma 3.15. *There is a constant $c > 0$ such that for all sufficiently small ε and for all $\psi \in \mathfrak{h}^2_{k,\perp}$ we have*

$$\langle \psi, \mathcal{G}_\varepsilon \psi \rangle_{L^2} \geq c \|\psi\|_{L^2}^2.$$

Furthermore, the kernel of \mathcal{G}_ε consists of holomorphy potentials.

Proof. We first consider the operator

$$c_0\mathcal{D}^*\mathcal{D} + \varepsilon H_1 + \varepsilon^2 H_2.$$

The desired bound for the operator $\mathcal{D}^*\mathcal{D}$ is well-known: There is a constant c' such that for all $\psi \in \mathfrak{F}_{k,\perp}^2$ we have

$$\langle \psi, \mathcal{D}^*\mathcal{D}\psi \rangle_{L^2} \geq c' \|\psi\|_{L^2}^2;$$

see, for example, Brönnle [7, Lemma 37]. We can obtain uniform bounds for $j = 1, 2$

$$-C_1(\mathcal{D}\gamma, \mathcal{D}\psi)_\omega \leq (\mathcal{D}\gamma, \mathcal{D}\psi)_{g_j} \leq C_j(\mathcal{D}\gamma, \mathcal{D}\psi)_\omega$$

for some $C_j > 0$, independent of ψ, γ and hence can obtain uniform bounds for some possibly different C_j

$$-C_j \int_X (\mathcal{D}\gamma, \mathcal{D}\psi)_\omega \omega^n \leq \int_X (\mathcal{D}\gamma, \mathcal{D}\psi)_{g_j} d\mu_j \leq C_j \int_X (\mathcal{D}\gamma, \mathcal{D}\psi)_\omega \omega^n.$$

Here, we view ω as inducing a metric on $TX^{1,0} \otimes \Omega^{0,1}$. It follows that for ε sufficiently small we have a bound

$$\langle \psi, c_0\mathcal{D}^*\mathcal{D}\psi + \varepsilon H_1 + \varepsilon H_2\psi \rangle_{L^2} \geq c \|\psi\|_{L^2}^2$$

for some $c > 0$.

The remaining terms are nonnegative for ε sufficiently small. Indeed, for ε sufficiently small the coefficient $\varepsilon c_1 + \varepsilon^2 c_2$ is positive and

$$\langle \psi, (\varepsilon c_1 + \varepsilon^2 c_2)\mathcal{D}^*\bar{\partial}^*\bar{\partial}\mathcal{D}\psi \rangle_{L^2} = (\varepsilon c_1 + \varepsilon^2 c_2) \|\bar{\partial}^*\mathcal{D}\psi\|_{L^2}^2 \geq 0.$$

It follows that

$$\langle \psi, \mathcal{G}_\varepsilon\psi \rangle_{L^2} \geq c \|\psi\|_{L^2}^2,$$

as required.

What remains is to characterise the kernel of \mathcal{G}_ε . Note that certainly $\mathfrak{F} \subset \ker \mathcal{G}_\varepsilon$ since $\mathfrak{F} = \ker \mathcal{D}$. Otherwise, we may write $\psi \in L_k^2$ as $\psi = \psi_{\mathfrak{F}_k^2} + \psi_{\mathfrak{F}_{k,\perp}^2}$ where $\psi_{\mathfrak{F}_k^2} \in \mathfrak{F}_k^2$ and $\psi_{\mathfrak{F}_{k,\perp}^2} \in \mathfrak{F}_{k,\perp}^2$ are L^2 -orthogonal and we may assume $\psi_{\mathfrak{F}_{k,\perp}^2} \neq 0$, and we see that

$$\langle \psi, \mathcal{G}_\varepsilon\psi \rangle_{L^2} = \langle \psi_{\mathfrak{F}_{k,\perp}^2}, \mathcal{G}_\varepsilon\psi_{\mathfrak{F}_{k,\perp}^2} \rangle_{L^2} \geq c \|\psi_{\mathfrak{F}_{k,\perp}^2}\|_{L^2}^2 > 0,$$

where we have used that

$$\langle \psi_{\mathfrak{F}_k^2}, \mathcal{G}_\varepsilon\psi_{\mathfrak{F}_k^2} \rangle = 0$$

since \mathcal{G}_ε is self-adjoint and $\mathcal{G}_\varepsilon\psi_{\mathfrak{F}_k^2} = 0$. □

Corollary 3.9. *For sufficiently small ε , the operator*

$$\mathcal{G}_\varepsilon : \mathfrak{F}_{k,\perp}^2 \rightarrow \mathfrak{F}_{k-6,\perp}^2$$

is an isomorphism. Furthermore, the induced map

$$\begin{aligned} \hat{\mathcal{G}}_\varepsilon : L_k^2 \times \mathfrak{F} &\rightarrow L_{k-6}^2, \\ (\psi, h) &\rightarrow \mathcal{G}_\varepsilon\psi + h \end{aligned}$$

is surjective and admits a right inverse.

Proof. We first show that \mathcal{G}_ε does actually send $\mathfrak{f}_{k,\perp}^2$ to $\mathfrak{f}_{k-6,\perp}^2$. In fact, for any $\psi \in L_k^2$ and any $h \in \mathfrak{k}$ we have

$$\langle h, \mathcal{G}_\varepsilon \psi \rangle_{L^2} = 0$$

again by self-adjointness of \mathcal{G}_ε . Since

$$\mathcal{G}_\varepsilon : \mathfrak{f}_{k,\perp}^2 \rightarrow \mathfrak{f}_{k-6,\perp}^2$$

has trivial kernel by Lemma 3.15, it is a self-adjoint elliptic partial differential operator with trivial kernel, hence is an isomorphism by the Fredholm alternative.

Surjectivity of the induced map $\hat{\mathcal{G}}_\varepsilon : L_k^2 \times \mathfrak{k} \rightarrow L_{k-6}^2$, is an immediate consequence, while a right inverse can be constructed explicitly. Indeed, since the operator $\mathcal{G}_\varepsilon : \mathfrak{f}_{k,\perp}^2 \rightarrow \mathfrak{f}_{k-6,\perp}^2$ is an isomorphism, it admits some inverse $\mathcal{G}_\varepsilon^{-1} : \mathfrak{f}_{k-6,\perp}^2 \rightarrow \mathfrak{f}_{k,\perp}^2$. Write $\psi \in \mathfrak{f}_{k-6,\perp}^2$ as ψ as $\psi_{\mathfrak{f}_{k-6}^2} + \psi_{\mathfrak{f}_{k-6,\perp}^2}$ where $\psi_{\mathfrak{f}_{k-6}^2} \in \mathfrak{f}_{k-6}^2$ and $\psi_{\mathfrak{f}_{k-6,\perp}^2} \in \mathfrak{f}_{k-6,\perp}^2$ are L^2 -orthogonal. Note that $\psi_{\mathfrak{f}_{k-6}^2}$ is actually smooth as it is a holomorphy potential. Then a right inverse is given by

$$\mathcal{M}_\varepsilon(\psi) = (\mathcal{G}_\varepsilon^{-1} \psi_{\mathfrak{f}_{k-6}^2}, \psi_{\mathfrak{f}_{k-6,\perp}^2}). \tag{3.15}$$

□

We next obtain an operator norm of the inverse operator $\mathcal{G}_\varepsilon^{-1} : \mathfrak{f}_{k,\perp}^2 \rightarrow \mathfrak{f}_{k-6,\perp}^2$. We will use elliptic regularity estimates for this, so it is more convenient to consider the rescaled operator $\varepsilon^{-1} \mathcal{G}_\varepsilon$ so that the ellipticity constants are actually uniformly bounded in ε ; here, we recall that ellipticity follows from the fact that the sixth-order coefficient of \mathcal{G}_ε is $(\varepsilon c_1 + \varepsilon^2 c_2) \Delta^3$, where we have assumed $c_1 > 0$, so scaling by ε^{-1} gives a family of operators whose ellipticity constants are actually bounded independently of ε .

Proposition 3.10 [62, Chapter 5, Theorem 11.1]. *There is a constant $c > 0$ such that for any $\psi \in \mathfrak{f}_{k-6,\perp}^2$ and for all sufficiently small ε there is a bound of the form*

$$\|(\varepsilon^{-1} \mathcal{G}_\varepsilon)^{-1} \psi\|_{L_k^2} \leq c \varepsilon^{-1} \left(\|(\varepsilon^{-1} \mathcal{G}_\varepsilon)^{-1} \psi\|_{L^2} + \|\psi\|_{L_{k-6}^2} \right).$$

The point here is that our model operator $(\varepsilon^{-1} \mathcal{G}_\varepsilon)^{-1}$ has uniformly bounded ellipticity constants, but the norm of the coefficients of the equation are actually only bounded uniformly by $C\varepsilon^{-1}$ for some constant C and hence are blowing up as $\varepsilon \rightarrow 0$. Explicitly, the term which is blowing up is the leading term $\varepsilon^{-1} \mathcal{D}^* \mathcal{D}$. In this situation, one obtains an elliptic regularity estimate where the coefficient in the bound is $c\varepsilon^{-1}$. We learned that such an elliptic regularity estimate holds from an observation of Hashimoto for general elliptic operators [36, p. 800]; the dependence of the coefficient in the bound on the norm of the coefficients is standard for second-order elliptic operators [34, p. 92].

Corollary 3.11. *There is a bound of the form*

$$\|\mathcal{G}_\varepsilon^{-1}\|_{op} \leq C\varepsilon^{-2}$$

for the operator $\mathcal{G}_\varepsilon^{-1} : \mathfrak{f}_{k-6,\perp}^2 \rightarrow \mathfrak{f}_{k,\perp}^2$, for some $C > 0$.

Proof. Let $\psi \in \mathfrak{f}_{k-6,\perp}^2$, and set $\gamma = \mathcal{G}_\varepsilon^{-1} \psi$, so that $\mathcal{G}_\varepsilon \gamma = \psi$. The elliptic regularity estimate gives

$$\frac{\|(\varepsilon^{-1} \mathcal{G}_\varepsilon)^{-1} \psi\|_{L_k^2}}{\|\psi\|_{L_{k-6}^2}} \leq c\varepsilon^{-1} + c\varepsilon^{-1} \frac{\|(\varepsilon^{-1} \mathcal{G}_\varepsilon)^{-1} \psi\|_{L^2}}{\|\psi\|_{L_{k-6}^2}} = c\varepsilon^{-1} + c \frac{\|\mathcal{G}_\varepsilon^{-1} \psi\|_{L^2}}{\|\psi\|_{L_{k-6}^2}}.$$

By Cauchy–Schwarz, we have

$$\|\gamma\|_{L^2} \|\mathcal{G}_\varepsilon \gamma\|_{L^2} \geq \langle \gamma, \mathcal{G}_\varepsilon \gamma \rangle_{L^2},$$

so the bound

$$\langle \gamma, \mathcal{G}_\varepsilon \gamma \rangle_{L^2} \geq \tilde{c} \|\gamma\|_{L^2}^2$$

for some $\tilde{c} > 0$ given by Lemma 3.15 implies

$$\|\mathcal{G}_\varepsilon \gamma\|_{L^2} \geq \tilde{c} \|\gamma\|_{L^2}.$$

Thus,

$$\frac{\|\mathcal{G}_\varepsilon^{-1} \psi\|_{L^2}}{\|\psi\|_{L^2_{k-6}}} \leq \frac{\|\mathcal{G}_\varepsilon^{-1} \psi\|_{L^2}}{\|\psi\|_{L^2}} = \frac{\|\gamma\|_{L^2}}{\|\mathcal{G}_\varepsilon \gamma\|_{L^2}} \leq \tilde{c}^{-1}.$$

It follows that

$$\frac{\|(\varepsilon^{-1} \mathcal{G}_\varepsilon)^{-1} \psi\|_{L^2_k}}{\|\psi\|_{L^2_{k-6}}} \leq c\varepsilon^{-1} + c(\tilde{c}^{-1}) \leq C\varepsilon^{-1}$$

for ε sufficiently small and some $C > 0$, as required. □

Recall that a right inverse to the induced map

$$\begin{aligned} \hat{\mathcal{G}}_\varepsilon : L^2_k \times \mathfrak{k} &\rightarrow L^2_{k-6}, \\ (\psi, h) &\rightarrow \mathcal{G}_\varepsilon \psi + h \end{aligned}$$

is given through Equation (3.15) by

$$\mathcal{M}_\varepsilon(\psi) = (\mathcal{G}_\varepsilon^{-1} \psi_{\mathfrak{k}^2_{k-6}}, \psi_{\mathfrak{k}^2_{k-6}}),$$

where $\psi_{\mathfrak{k}^2_{k-6}} \in \mathfrak{k}$ is the L^2 -projection of ψ onto \mathfrak{k} .

Corollary 3.12. *There is a bound on the operator norm of \mathcal{M}_ε of the form*

$$\|\mathcal{M}_\varepsilon^{-1}\|_{op} \leq C\varepsilon^{-2}$$

for some $C > 0$.

Proof. The operator $\psi \rightarrow \psi_{\mathfrak{k}^2_{k-6}}$ has operator norm bounded independently of ε , so this is a direct consequence of Corollary 3.11. □

We will eventually be interested in perturbations of $\hat{\mathcal{G}}_\varepsilon$. The following is then a consequence of standard linear algebra (see for example [7, Lemma 4.3] for the result in linear algebra).

Corollary 3.13. *Suppose $L_\varepsilon : L^2_k \rightarrow L^2_{k-6}$ is a sequence of bounded operators with $\|L_\varepsilon\|_{op} \leq K$ for some K independent of ε . Then for all sufficiently small ε the operator*

$$(\psi, h) \rightarrow \hat{\mathcal{G}}_\varepsilon \psi + \varepsilon^3 L_\varepsilon \psi + h$$

is surjective and admits a right inverse $\tilde{\mathcal{M}}_\varepsilon$. Moreover, there is a constant $C > 0$ such that

$$\|\tilde{\mathcal{M}}_\varepsilon^{-1}\|_{op} \leq C\varepsilon^{-2}.$$

Remark 3.16. This result is the reason we must include the ε^2 term in our model operator: Our bound on the operator norm of the right inverse means we can only add additional terms at order ε^3 and retain the desired mapping properties.

3.4.2. The approximate solution

We now assume that $\omega \in c_1(L)$ is cscK. Lemma 2.13 then implies that we have

$$\text{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\omega)) = O(\varepsilon^2).$$

In order for our model linear operator to be a good approximation of the genuine linearised operator, we will need to consider a better approximation to a Z -critical Kähler metric. Since we are considering the general case when the Lichnerowicz operator $\mathcal{D}^*\mathcal{D}$ may have nontrivial kernel, or equivalently the case when $\text{Aut}(X, L)$ may not be discrete, rather than finding approximate Z -critical Kähler metrics, we will instead try to find a ω_ε approximately solving the condition that

$$\text{Im}(e^{-\varphi_\varepsilon} \tilde{Z}_\varepsilon(\omega_\varepsilon)) \in \ker \mathcal{D}_\varepsilon, \tag{3.16}$$

where $\mathcal{D}_\varepsilon = \bar{\partial}\nabla_\varepsilon^{1,0}$ is defined using ω_ε . That is to say, the function $\text{Im}(e^{-\varphi_\varepsilon} \tilde{Z}_\varepsilon(\omega_\varepsilon))$ is a holomorphy potential with respect to ω_ε . To this end, we recall that if v is a Kähler potential and if h is the holomorphy potential with respect to ω for some holomorphic vector field, then the function

$$h + \frac{1}{2}\langle \nabla v, \nabla h \rangle \tag{3.17}$$

is the holomorphy potential with respect to the Kähler metric $\omega_v = \omega + i\bar{\partial}\bar{\partial}v$ (see, for example, [59, Lemma 12]).

Analogously to Corollary 3.9, the operator

$$\begin{aligned} L_k^2 \times \mathfrak{k} &\rightarrow L_{k-4}^2, \\ (\psi, h) &\rightarrow \mathcal{D}^*\mathcal{D}\psi + h \end{aligned}$$

is surjective. Although we have worked in Sobolev spaces, since the operator is elliptic, the same holds for smooth functions. Thus, given $e \in C^\infty(X)$, there is a pair (ψ, h) with

$$\mathcal{D}^*\mathcal{D}\psi + h = e. \tag{3.18}$$

Lemma 3.17. *Suppose ω is a cscK metric. Then for any fixed m there is a sequence ψ_j and holomorphy potentials h_j such that*

$$\text{Im}\left(e^{-i\varphi_\varepsilon} \tilde{Z}\left(\omega + \sum_{j=1}^m \varepsilon^j i\bar{\partial}\bar{\partial}\psi_j\right)\right) = \sum_{j=2}^{m+1} \varepsilon^2 \left(h_j + \frac{1}{2}\left\langle h_j, \sum_{i=1}^m \varepsilon^i \psi_i \right\rangle\right) + O(\varepsilon^{m+2}).$$

These are approximate solutions to Equation (3.16).

Proof. The linearisation of the scalar curvature at a cscK metric is the operator $-\mathcal{D}^*\mathcal{D}$ [60, Lemma 4.4]. As we have assumed ω is cscK, we have

$$\text{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\omega)) = e_2\varepsilon^2 + O(\varepsilon^3).$$

By right invertibility of the Lichnerowicz operator there is a function ψ_2 and a holomorphy potential $h_2 \in \mathfrak{k}$ such that

$$(\text{Re}(\rho_{n-1})L^n)\mathcal{D}^*\mathcal{D}\psi_1 = e_2 - h_2.$$

Since $\mathcal{F}_\varepsilon = \varepsilon(\text{Re}(\rho_{n-1})L^n)\mathcal{D}^*\mathcal{D} + O(\varepsilon^2)$, it follows that

$$\text{Im}\left(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon\left(\omega + \varepsilon\bar{\partial}\bar{\partial}\psi_1\right)\right) = h_2\varepsilon^2 + O(\varepsilon^3).$$

Next, consider the error term

$$\operatorname{Im}\left(e^{-i\varphi_\varepsilon}\tilde{Z}_\varepsilon\left(\omega + \varepsilon\partial\bar{\partial}\psi_1\right)\right) - \varepsilon^2\left(h_2 + \frac{1}{2}\langle\nabla h_2, \nabla\varepsilon\psi_1\rangle\right) = e_3\varepsilon^3.$$

We continue by applying Equation (3.18) to find a function ψ_2 and a holomorphy potential h_3 such that

$$(\operatorname{Re}(\rho_{n-1})L^n)\mathcal{D}^*\mathcal{D}\psi_2 = e_2 - h_3.$$

Again, since the leading order linear operator is $(\operatorname{Re}(\rho_{n-1})L^n)\mathcal{D}^*\mathcal{D}$, it follows that

$$\operatorname{Im}\left(e^{-i\varphi_\varepsilon}\tilde{Z}_\varepsilon\left(\omega + i\partial\bar{\partial}(\varepsilon\psi_1 + \varepsilon^2\psi_2)\right)\right) = \left(h_2 + \frac{1}{2}\langle\nabla h_2, \nabla\varepsilon\psi_1\rangle\right)\varepsilon^2 + h_3\varepsilon^3 + O(\varepsilon^4).$$

In particular,

$$\operatorname{Im}\left(e^{-i\varphi_\varepsilon}\tilde{Z}_\varepsilon\left(\omega + i\partial\bar{\partial}(\varepsilon\psi_1 + \varepsilon^2\psi_2)\right)\right) = \sum_{j=2}^3 \varepsilon^2\left(h_j + \frac{1}{2}\left\langle h_j, \sum_{l=1}^2 \varepsilon^l\psi_l \right\rangle\right) + O(\varepsilon^4).$$

Iterating this process gives the result. □

We will only require the approximate solution

$$\omega_\varepsilon = \omega + \sum_{j=1}^3 \varepsilon^j i\partial\bar{\partial}\psi_j, \tag{3.19}$$

which satisfies

$$\operatorname{Im}\left(e^{-i\varphi_\varepsilon}\tilde{Z}_\varepsilon\left(\omega + \sum_{j=1}^3 \varepsilon^j i\partial\bar{\partial}\psi_j\right)\right) = \sum_{j=1}^3 \varepsilon^2\left(h_j + \frac{1}{2}\left\langle\nabla h_j, \nabla\left(\sum_{i=1}^3 \varepsilon^i\psi_i\right)\right\rangle\right) + O(\varepsilon^5).$$

We then set

$$\gamma_\varepsilon = \sum_{j=1}^3 \varepsilon^j i\partial\bar{\partial}\psi_j,$$

so that if h is a holomorphy potential with respect to ω , then $h + \frac{1}{2}\langle\nabla h, \nabla\gamma_\varepsilon\rangle$ is a holomorphy potential with respect to ω_ε by Equation (3.17).

We return to the model operator \mathcal{G}_ε , however, now defined with respect to the approximate solution ω_ε . In order to understand its properties, for clarity we consider the Kähler metric ω_δ the approximate solution to order $O(\delta^5)$ given by Equation (3.19) (namely, we replace ε with δ). Denote by \mathfrak{f}_δ the space of holomorphy potentials with respect to ω_δ . Then the results we have already established imply that for each fixed δ , the operator

$$\begin{aligned} \hat{\mathcal{G}}_{\varepsilon,\delta} : L_k^2 \times \mathfrak{f}_\delta &\rightarrow L_{k-6}^2, \\ (\psi, h) &\rightarrow \mathcal{G}_{\varepsilon,\delta}\psi + h \end{aligned}$$

is surjective for ε sufficiently small.

We claim that one can take the ε for which surjectivity of $\hat{\mathcal{G}}_{\varepsilon,\delta}$ holds to be independent of δ for δ sufficiently small. More precisely, we claim that there is an ε_0 and a δ_0 such that $\hat{\mathcal{G}}_{\varepsilon,\delta}$ is surjective for

all $\delta \leq \delta_0$ and $\varepsilon \leq \varepsilon_0$. But this follows since in the ‘eigenvalue bound’ of Lemma 3.15

$$\langle \psi, \mathcal{G}_{\varepsilon, \delta} \psi \rangle_{L^2} \geq c_\delta \|\psi\|_{L^2}^2,$$

for ψ orthogonal to \mathfrak{k}_δ , the value c_δ is actually continuous in δ . Similar continuity statements in δ then further imply that the right inverse $\mathcal{M}_{\varepsilon, \delta} : L^2_{k-6} \rightarrow L^2_k \times \mathfrak{k}_\delta$ has operator norm which satisfies a uniform bound

$$\|\mathcal{M}_{\varepsilon, \delta}\|_{op} \leq C\varepsilon^{-2},$$

where C is independent of both δ and ε . Here, the continuity used is in the elliptic regularity estimate of Proposition 3.10. It follows that we can take $\delta = \varepsilon$ and obtain a bound with respect to the approximate solution ω_ε . We will rephrase this in a form in which we will use these results.

Corollary 3.14. *Denote by \mathcal{G}_ε model operator with respect to the approximate solution ω_ε . Then the operator*

$$\begin{aligned} \tilde{\mathcal{G}}_\varepsilon : L^2_k \times \mathfrak{k}_\varepsilon &\rightarrow L^2_{k-6}, \\ (\psi, h) &\rightarrow \mathcal{G}_\varepsilon \psi + h + \frac{1}{2} \langle \nabla h, \nabla \gamma_\varepsilon \rangle \end{aligned}$$

is surjective and admits a right inverse $\tilde{\mathcal{M}}_\varepsilon$. There is a bound on the operator norm of $\tilde{\mathcal{M}}_\varepsilon$ of the form $\|\tilde{\mathcal{M}}_\varepsilon\|_{op} \leq C\varepsilon^{-2}$.

Thus, if $L_\varepsilon : L^2_k \rightarrow L^2_{k-6}$ is a sequence of operators satisfying a uniform bound $\|L_\varepsilon\|_{op} \leq K$ independent of ε , then the operator

$$(\psi, h) \rightarrow \mathcal{G}_\varepsilon \psi + h + \frac{1}{2} \langle \nabla h, \nabla \gamma_\varepsilon \rangle + \varepsilon^3 L_\varepsilon$$

is surjective and right invertible. The resulting right inverse also has operator norm satisfying a uniform bound by $C'\varepsilon^{-2}$ for some $C' > 0$.

Proof. We first consider the operator $\tilde{\mathcal{G}}_\varepsilon$ itself. In comparison to the discussion immediately preceding the statement, the only difference is in the range of the operator. The discussion involves \mathfrak{k}_ε rather than \mathfrak{k} itself. But if $h \in \mathfrak{k}$, then $h + \frac{1}{2} \langle \nabla h, \nabla \gamma_\varepsilon \rangle \in \mathfrak{k}_\varepsilon$. So the statement of the corollary is simply a rephrasing of the discussion. The statements about perturbations are consequences of linear algebra as in Corollary 3.13. □

3.4.3. Understanding the expansion of the operator

We next consider some general aspects of the structure of the Z -critical equation. We will consider its expansion in powers of ε , and to match with what we have considered it will be convenient to consider the ‘rescaled’ equation

$$-\varepsilon^{-1} \operatorname{Im} \left(\frac{\tilde{Z}_\varepsilon(\omega)}{Z_\varepsilon(X, L)} \right) = \operatorname{Re}(\rho_{n-1}) L^n S(\omega) + O(\varepsilon)$$

so that if ω is a cscK metric its linearisation takes the form $-\operatorname{Re}(\rho_{n-1}) L^n \mathcal{D}^* \mathcal{D} + O(\varepsilon)$. We will be interested in understanding the terms of order ε and ε^2 ; controlling these will allow us to see the full linearised operator as a perturbation of the sum involving only terms of order up to ε^2 which will be sufficiently by Corollary 3.14. We will begin only by considering ω and will then later consider the approximate solution ω_ε .

We use our assumptions that:

- (i) $\theta_1 = 0 = \theta_2 = \theta_3 = 0$. The condition on θ_1 is used so that the leading order term in the expansion is the scalar curvature, rather than the twisted scalar curvature, while the conditions on θ_2 and θ_3

are of a more technical nature and allow us to understand the ε^2 -term of the linearised operator. We expect that the conditions on θ_2 and θ_3 can be removed.

- (ii) $\text{Re}(\rho_{n-1}) < 0$, $\text{Re}(\rho_{n-2}) > 0$ and $\text{Re}(\rho_{n-3}) = 0$. The condition on $\text{Re}(\rho_{n-1})$ is essentially a sign convention, what is really needed is that these two real parts have opposite sign. This is essential to the analysis and is used in the L^2 -bound for the model operator proved in Lemma 3.15. The condition on $\text{Re}(\rho_{n-3})$ is a technical assumption which we expect can be removed.

As in Lemma 2.13 we write $Z_\varepsilon(X, L) = r_\varepsilon e^{i\varphi_\varepsilon}$, so that

$$\begin{aligned} \text{Im}(e^{-i\varphi_\varepsilon(X,L)} \tilde{Z}_\varepsilon(\omega)) &= r_\varepsilon(X, L) \text{Im}\left(\frac{\tilde{Z}_\varepsilon(\omega)}{Z_\varepsilon(X, L)}\right), \\ &= r_\varepsilon(X, L) \frac{\text{Im} \tilde{Z}_\varepsilon(\omega) \text{Re} Z_\varepsilon(X, L) - \text{Re} \tilde{Z}_\varepsilon(\omega) \text{Im} Z_\varepsilon(X, L)}{\text{Re} Z_\varepsilon(X, L)^2 + \text{Im} Z_\varepsilon(X, L)^2}, \end{aligned}$$

where we recall

$$\begin{aligned} Z_\varepsilon(X, L) &= iL^n \varepsilon^{-n} + \rho_{n-1} L^{n-1} \cdot K_X \varepsilon^{-n+1} + \rho_{n-2} L^{n-2} \cdot K_X^2 \varepsilon^{-n+2} + \dots, \\ \tilde{Z}_\varepsilon(\omega) &= i - \rho_{n-1} \frac{\text{Ric} \omega \wedge \omega^{n-1}}{\omega^n} \varepsilon + O(\varepsilon^2). \end{aligned}$$

Here, we have used our assumptions

Our equation takes the form

$$\text{Im}\left(\frac{\tilde{Z}_\varepsilon(\omega)}{Z_\varepsilon(X, L)}\right) = \frac{\text{Im} \tilde{Z}_\varepsilon(\omega) \text{Re} Z_\varepsilon(X, L) - \text{Re} \tilde{Z}_\varepsilon(\omega) \text{Im} Z_\varepsilon(X, L)}{\text{Re} Z_\varepsilon(X, L)^2 + \text{Im} Z_\varepsilon(X, L)^2},$$

where explicitly

$$\begin{aligned} Z_\varepsilon(X, L) &= iL^n \varepsilon^{-n} + \rho_{n-1} \alpha_1 \varepsilon^{-n+1} + \rho_{n-2} \alpha_2 \varepsilon^{-n+2} + \rho_{n-1} \alpha_3 \varepsilon^{-n+3} + O(\varepsilon^{-n+4}), \\ \tilde{Z}_\varepsilon(\omega) &= i + \rho_{n-1} \tilde{\alpha}_1 \varepsilon + \rho_{n-2} \tilde{\alpha}_2 \varepsilon^2 + \rho_{n-3} \tilde{\alpha}_3 \varepsilon^3 + O(\varepsilon^4), \end{aligned}$$

and where $\alpha_1 = L^{n-1} \cdot K_X$, $\alpha_2 = L^{n-2} \cdot K_X^2$, $\alpha_3 = L^{n-3} \cdot K_X^3$, while

$$\begin{aligned} \tilde{\alpha}_1 &= -\frac{\text{Ric} \omega \wedge \omega^{n-1}}{\omega^n}, & \tilde{\alpha}_2 &= \frac{\text{Ric} \omega^2 \wedge \omega^{n-2}}{\omega^n} - \frac{2}{n-1} \Delta \frac{\text{Ric} \omega \wedge \omega^{n-1}}{\omega^n}, \\ \tilde{\alpha}_3 &= -\frac{\text{Ric} \omega^3 \wedge \omega^{n-3}}{\omega^n} + \frac{3}{n-2} \Delta \frac{\text{Ric} \omega^2 \wedge \omega^{n-2}}{\omega^n}. \end{aligned}$$

The factor

$$\frac{r_\varepsilon(X, L)}{\text{Re} Z_\varepsilon(X, L)^2 + \text{Im} Z_\varepsilon(X, L)^2}$$

plays only a minor role in our expansion of $\text{Im}\left(\frac{\tilde{Z}_\varepsilon(\omega)}{Z_\varepsilon(X,L)}\right)$. Indeed, we will have good control over the leading order two terms in ε , while the third-order (for our rescaled equation) ε^2 term will require the most care to manage. So we can ignore this factor in controlling the linearisation. In addition, all relevant terms below have a uniform factor of L^n arising from the leading order term of the expansion $Z_\varepsilon(X, L) = iL^n \varepsilon^{-n} + \dots$, and we also omit this uniform factor. Thus, we need only understand the leading order three terms in the expansion of

$$\text{Im} \tilde{Z}_\varepsilon(\omega) \text{Re} Z_\varepsilon(X, L) - \text{Re} \tilde{Z}_\varepsilon(\omega) \text{Im} Z_\varepsilon(X, L).$$

Recall that we have assumed $\theta_1 = \theta_2 = \theta_3 = 0$.

We see that the leading order term is

$$\varepsilon^{-n+1} \operatorname{Re}(\rho_{n-1}) \left(L^{n-1} \cdot K_X + \frac{\operatorname{Ric} \omega \wedge \omega^{n-1}}{\omega^n} \right).$$

For the ε^{-n+2} -term, we will for the moment only be interested in the degree six operator, which we see is given by

$$-\varepsilon^{-n+2} \frac{2 \operatorname{Re}(\rho_{n-2})}{n-1} \Delta \left(\frac{\operatorname{Ric} \omega \wedge \omega^{n-1}}{\omega^n} \right).$$

For the ε^{-n+3} -term, we see that the sixth-order component is given by, for some topological constant c

$$-\frac{3 \operatorname{Re}(\rho_{n-3})}{n-2} \Delta \left(\frac{\operatorname{Ric} \omega^2 \wedge \omega^{n-2}}{\omega^n} \right) + c \operatorname{Im}(\rho_{n-2}) \Delta \left(\frac{\operatorname{Ric} \omega \wedge \omega^{n-1}}{\omega^n} \right). \tag{3.20}$$

In particular, if $\operatorname{Re}(\rho_{n-3}) = 0$, the first of these two terms vanishes.

While we have considered ω rather than the approximate solution $\omega_\varepsilon = \omega + i\partial\bar{\partial}\gamma_\varepsilon$, essentially the same statements hold using ω_ε . If we write $\alpha_{j,\varepsilon}$ for the coefficients of ε^j in $\tilde{Z}_\varepsilon(\omega_\varepsilon)$, then we still have

$$\tilde{Z}_\varepsilon(\omega_\varepsilon) = i + \rho_{n-1} \tilde{\alpha}_{1,\varepsilon} \varepsilon + \rho_{n-2} \tilde{\alpha}_{2,\varepsilon} \varepsilon^2 + \rho_{n-3} \tilde{\alpha}_{3,\varepsilon} \varepsilon^3 + O(\varepsilon^4),$$

implying the linearisation has similar properties up to order ε^4 but for example with the leading order term replaced with

$$\varepsilon^{-n+1} \operatorname{Re}(\rho_{n-1}) \left(L^{n-1} \cdot K_X + \frac{\operatorname{Ric} \omega_\varepsilon \wedge \omega_\varepsilon^{n-1}}{\omega_\varepsilon^n} \right).$$

3.4.4. Properties of the linearisation

We now turn to the linearisation of the Z-critical equation. The aim is to compare the linearisation at the approximate solution $\omega + i\partial\bar{\partial}\gamma_\varepsilon$ to the model operator \mathcal{G}_ε and in particular to use Corollary 3.14 to infer properties of the genuine linearised operator.

We begin with a general result. We fix a K -equivariant Kähler metric $\omega \in c_1(L)$, not assumed to be cscK and denote by \mathcal{F}_ε the linearisation of the operator

$$\psi \rightarrow \operatorname{Im} \left(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon \left(\omega + i\partial\bar{\partial}\psi \right) \right).$$

Denote also \mathfrak{k} the space of holomorphy potentials with respect to ω .

Proposition 3.15. *For all $0 < \varepsilon \ll 1$, the map*

$$\begin{aligned} \hat{\mathcal{F}}_\varepsilon : L_k^2 \times \mathfrak{k} &\rightarrow L_{k-6}^2, \\ (\psi, h) &\rightarrow \mathcal{F}_\varepsilon \psi - \langle \nabla \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\omega)), \nabla \psi \rangle + h \end{aligned}$$

is surjective. In addition exists a right inverse $\hat{\mathcal{P}}_\varepsilon$ of $\hat{\mathcal{F}}_\varepsilon$ whose operator norm satisfies a bound of the form $\|\hat{\mathcal{P}}_\varepsilon\|_{op} \leq C\varepsilon^{-2}$.

We recall our assumption, which will be used in the proof, that (X, L) is a degeneration of a polarised manifold with discrete automorphism group.

Remark 3.18. To compare Proposition 3.15 to a well-known result in Kähler geometry, recall that the scalar curvature operator $\psi \rightarrow S(\omega + i\partial\bar{\partial}\psi)$ has linearisation [60, Lemma 4.4]

$$\psi \rightarrow -\mathcal{D}^* \mathcal{D} \psi + \langle \nabla S(\omega), \nabla \psi \rangle,$$

so subtracting $\langle \nabla S(\omega), \nabla \psi \rangle$ leads to an operator whose kernel is precisely given by \mathfrak{k} . Thus, adding h leads to a surjective operator, mirroring Proposition 3.15.

The proof will use the moment map techniques developed in Section 3.3.2. We continue to denote by $\mathcal{J}_X(M, \omega)$ the space of complex structures biholomorphic to the reference complex structure J and recall the closed $(1, 1)$ -forms Ω_ε defined on $\mathcal{J}_X(M, \omega)$ through Equation (3.11). Any functions $u, v \in C^\infty(X, \mathbb{R})$ induce tangent vectors on $\mathcal{J}_X(M, \omega)$ through the assignment $u \rightarrow Pu$ of Equation (3.7); the same as true for functions in L^2_k . As in Section 3.4.6, this process can be integrated, associating to ψ a new complex structure $F_\psi(J)$. We will use that the differential of the map $\psi \rightarrow F_\psi(J)$ at $\psi = 0$ is [61, Equation 3]

$$\psi \rightarrow JP(\psi).$$

Proof of Proposition 3.15. We use many of the ideas of Section 3.3 to understand the general properties of the linearised operator. Consider $\omega_t = \omega + ti\partial\bar{\partial}v$ so that the derivative of

$$\int_X u \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\omega_t)) \omega_t^n$$

is given by

$$\frac{d}{dt} \int_X u \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\omega_t)) \omega_t^n = \int_X u \mathcal{F}_\varepsilon v \omega^n + \int_X u \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\omega)) \Delta v \omega^n. \tag{3.21}$$

We are interested in the first of these terms, but the advantage of this perspective is that from the proof of Theorem 3.9 we know that for each t

$$\left. \frac{d}{ds} \right|_{s=0} E_Z(tv + su) = \int_X u \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}(\omega_t)) \omega_t^n,$$

so that

$$\left. \frac{d^2}{dt ds} \right|_{s,t=0} E_Z(tv + su) = \int_X u \mathcal{F}_\varepsilon v \omega^n + \int_X u \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\omega)) \Delta v \omega^n.$$

It follows that the integral on the right-hand side, considered as a pairing on functions, is actually symmetric.

We need to identify the ε^2 and ε^3 terms in the expansion of \mathcal{F}_ε in order to compare it to the model operator \mathcal{G}_ε . For this, we will link with the space $\mathcal{J}_X(M, \omega)$ and the moment map interpretation of the Z -critical equation established in Section 3.3. We first consider the case $\operatorname{Aut}(X, L)$ is discrete, which allows us to use the results of Section 3.3, which were proven under that assumption. Our functions u, v can be viewed as inducing tangent vectors to $\mathcal{J}_X(M, \omega)$ at the point J_X and we see from Equation (3.6) that

$$\Omega_\varepsilon(Pu, JPv) = \left. \frac{d}{dt} \right|_{t=0} \int_X u \operatorname{Im}(e^{-\varphi_\varepsilon} \tilde{Z}_\varepsilon(J_t)) \omega_t^n, \tag{3.22}$$

where we emphasise that we take the perspective that the complex structure is changing but the symplectic form ω is fixed.

We next compare this to the linearisation with fixed complex structure and varying symplectic structure. Let f_t be the diffeomorphisms of X such that $f_t^* \omega_t = \omega$ and $f_t \cdot J = J$. Then $f_t^* \omega_t^n = \omega^n$, while $f_t^* \operatorname{Im}(e^{-\varphi_\varepsilon} \tilde{Z}_\varepsilon(\omega_t)) = \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(J_t))$. We also need to understand the infinitesimal change in u as we pull back along f_t , for which we need to understand the construction of f_t in more detail. As we only need to understand the infinitesimal construction of f_t near $t = 0$, it suffices to note that f_t is given

by taking the gradient flow along a path of vector fields v_t on X such that v_0 is the Hamiltonian vector field associated with the function v . Thus, the infinitesimal change in u is simply the Lie derivative

$$\mathcal{L}_{v_0}u = \langle \nabla u, \nabla v \rangle,$$

where we have used the relationship between the Poisson bracket of functions (that is, the pairing of the induced Hamiltonian vector fields with respect to ω) and the inner products of the Riemannian gradients. That is,

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \int_X u \operatorname{Im}(e^{-\varphi_\varepsilon} \tilde{Z}_\varepsilon(J_t))\omega^n &= \frac{d}{dt}\Big|_{t=0} \int_X u \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\omega_t))\omega_t^n \\ &\quad - \int_X \langle \nabla u, \nabla v \rangle \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\omega))\omega^n. \end{aligned}$$

We now use Equation (3.21), from which it follows that

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \int_X u \operatorname{Im}(e^{-\varphi_\varepsilon} \tilde{Z}_\varepsilon(J_t))\omega^n &= \int_X u \mathcal{F}_\varepsilon v \omega^n \\ &\quad + \int_X u \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\omega))\Delta v \omega^n - \int_X \langle \nabla u, \nabla v \rangle \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\omega))\omega^n. \end{aligned}$$

Since the final two terms on the right-hand side sum to $-\int_X u \langle \nabla \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\omega)), \nabla v \rangle \omega^n$, we have

$$\begin{aligned} \Omega_\varepsilon(Pu, JPv) &= \frac{d}{dt}\Big|_{t=0} \int_X u \operatorname{Im}(e^{-\varphi_\varepsilon} \tilde{Z}_\varepsilon(J_t))\omega^n \\ &= \int_X u \mathcal{F}_\varepsilon v \omega^n - \int_X u \langle \nabla \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\omega)), \nabla v \rangle \omega^n. \end{aligned}$$

Thus, the operator

$$(u, v) \rightarrow \int_X u(\mathcal{F}_\varepsilon v - \langle \nabla \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\omega)), \nabla v \rangle)\omega^n \tag{3.23}$$

is a self-adjoint operator which only depends on Pu, Pv . As this is true for all ε , it is true for each term in the associated expansion in powers of ε .

When $\operatorname{Aut}(X, L)$ is not discrete, we use the key assumption that (X, L) is a degeneration of a polarised manifold with discrete automorphism group. That is, (X, L) is the central fibre of a test configuration for a polarised manifold with discrete automorphism group (to compare with our previous notation, we are considering (X, L) to be what was previously denoted $(\mathcal{X}_0, \mathcal{L}_0)$). Thus, we obtain a family J_t of complex structures on the fixed underlying smooth manifold M converging to J_0 , the complex structure inducing X . Since the linearisation satisfies Equation (3.23) for each t , the same equation holds at $t = 0$. In particular self-adjointness, and dependence only on Pu, Pv hold also with respect to J_0 as well.

We use the results of Section 3.4.3 to identify the $\varepsilon, \varepsilon^2$ and ε^3 terms in the expansion of the operator

$$v \rightarrow \mathcal{F}_\varepsilon v - \langle \nabla \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\omega)), \nabla v \rangle,$$

in order to compare them to the model operator. By what we have just proven, this operator must be self-adjoint, and the pairing

$$(u, v) \rightarrow \int_X u(\mathcal{F}_\varepsilon v - \langle \nabla \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\omega)), \nabla v \rangle)\omega^n$$

can only depend on $\mathcal{D}u$ and $\mathcal{D}v$, due to the identification of Equation (3.8).

The leading order ε -term is given by $-\operatorname{Re}(\rho_{n-1})\mathcal{D}^*\mathcal{D}$ since the leading order ε -term in the expansion of $\operatorname{Im}(e^{-\varphi_\varepsilon}\tilde{Z}_\varepsilon(\omega))$ is simply the scalar curvature. The sixth-order operator in the ε^2 -term arises from linearising $\frac{2\operatorname{Re}(\rho_{n-1})}{n(n-1)}\Delta S(\omega)$, meaning that the linearisation inherits a term of the form $-\frac{2\operatorname{Re}(\rho_{n-1})}{n(n-1)}\Delta\mathcal{D}^*\mathcal{D}$. As we know the ε^2 -term only depends on $\mathcal{D}u, \mathcal{D}v$, the difference between the ε^2 -term and $-\frac{2\operatorname{Re}(\rho_{n-1})}{n(n-1)}(\bar{\partial}^*\mathcal{D})^*\bar{\partial}^*\mathcal{D}$ must be a fourth-order operator depending only on $\mathcal{D}u, \mathcal{D}v$ as both are of the form $-\frac{2\operatorname{Re}(\rho_{n-1})}{n(n-1)}\Delta^3$ plus some fourth-order operator. In particular, the ε^2 -term must be of the form $c_1\mathcal{D}^*\bar{\partial}^*\bar{\partial}\mathcal{D} + H_1$, where

$$\int_X uH_1v\omega^n = \int_X (\mathcal{D}u, \mathcal{D}v)_{g_1}d\mu_1,$$

and where $d\mu_1$ is a smooth (n, n) -form and

$$g_1 : \Gamma(T^{1,0}X \otimes \Omega^{0,1}(X)) \otimes \Gamma(T^{1,0}X \otimes \Omega^{0,1}(X)) \rightarrow \mathbb{R}$$

is a smooth bilinear pairing but not necessarily a metric. In particular, this is of the same form as the ε^2 -term of our model operator of Equation (3.14) computed with respect to ω .

We finally show that the ε^3 -term of our linearisation takes the same form as the model operator, for which we use that $\operatorname{Re}(\rho_{n-3}) = 0$ and $\theta_3 = 0$. From Equation (3.20), it follows that the only sixth-order term arises from linearising a multiple of $\operatorname{Im}(\rho_{n-2})\Delta S(\omega)$, which contributes one term which is involved in the ε^3 -term of the model operator. The remaining order terms are fourth order and so again are given by some H_2 of the same form as H_1 .

What we have demonstrated is that the linearised operator agrees with the model operator to order ε^3 . In particular Corollary 3.13 applies to give the statement of the Proposition. \square

In general, we wish to solve the equation

$$\operatorname{Im}\left(e^{-i\varphi_\varepsilon}\tilde{Z}_\varepsilon\left(\omega + i\partial\bar{\partial}\psi\right)\right) - f - \frac{1}{2}\langle\nabla\psi, \nabla f\rangle = 0,$$

for $f \in \mathfrak{k}$ and ψ a Kähler potential. The linearisation of this operator is given by

$$dS_{0,f}(\psi, h) = \mathcal{F}_\varepsilon\psi - \frac{1}{2}\langle\nabla\psi, \nabla f\rangle - h.$$

The following is an immediate consequence of Proposition 3.15.

Corollary 3.16. *For all $0 < \varepsilon \ll 1$, the operator*

$$(\psi, h) \rightarrow dS_{0,f}(\psi, h) + \frac{1}{2}\langle\nabla\psi, \nabla(f - 2\operatorname{Im}(e^{-i\varphi_\varepsilon}\tilde{Z}_\varepsilon(\omega)))\rangle$$

is surjective, admits a right inverse and the operator norm of the inverse is bounded by $C\varepsilon^{-2}$ for some $C > 0$.

Here, h and f are holomorphy potentials with respect to ω , which was arbitrary. We apply this to the approximate solutions ω_ε constructed in Lemma 3.17. Rescaling the holomorphy potentials by a factor of two, ω_ε satisfies

$$\operatorname{Im}(e^{-i\varphi_\varepsilon}\tilde{Z}(\omega_\varepsilon)) - \frac{1}{2}f_\varepsilon = O(\varepsilon^5),$$

where the $f_\varepsilon \in \mathfrak{k}_\varepsilon$, hence the term

$$\frac{1}{2}\langle\nabla\psi, \nabla(f - 2\operatorname{Im}(e^{-i\varphi_\varepsilon}\tilde{Z}_\varepsilon(\omega)))\rangle = O(\varepsilon^5)$$

is of high order in ε . In particular this term does not affect the mapping properties of the linearised operator. The following is then the statement of ultimate interest from the present section.

Corollary 3.17. *The linearisation dS computed at the approximate solution ω_ε is surjective, and right invertible. Moreover its right inverse has operator norm bounded by $C\varepsilon^{-2}$ for some $C > 0$.*

3.4.5. Applying the quantitative inverse function theorem

We can now construct Z -critical Kähler metrics in the large volume limit, as well as their extremal analogue. We continue with the notation and hypotheses of the previous sections.

Theorem 3.19. *Suppose (X, L) admits a cscK metric ω and is a degeneration of a polarised manifold with discrete automorphism group. Then (X, L) admits solutions $\tilde{\omega}_\varepsilon$ to the equation*

$$\mathrm{Im}(e^{-\varphi_\varepsilon} \tilde{Z}_\varepsilon(\tilde{\omega}_\varepsilon)) \in \mathfrak{k}_\varepsilon,$$

where \mathfrak{k}_ε denotes the space of holomorphy potentials with respect to $\tilde{\omega}_\varepsilon$.

This result proves the existence of the analogue of extremal Z -critical Kähler metrics. It is a straightforward consequence that (X, L) admits Z_ε -critical Kähler metrics if and only if the analogue of the Futaki invariant described in Proposition 3.6 vanishes for all holomorphic vector fields. In the discrete automorphism group case, this produces the following.

Corollary 3.18. *Suppose (X, L) has discrete automorphism group and admits a cscK metric. Then (X, L) admits Z_ε -critical Kähler metrics for all $\varepsilon \ll 1$.*

To prove these results, we will apply the quantitative implicit function theorem:

Theorem 3.20 [6, Theorem 4.1]. *Let $G : B_1 \rightarrow B_2$ be a differentiable map between Banach spaces, whose derivative at $0 \in B_1$ is surjective with right inverse P . Let*

- (i) δ' be the radius of the closed ball in B_1 around the origin on which $G - dG$ is Lipschitz with Lipschitz constant $1/(2\|P\|)$, where we use the operator norm;
- (ii) $\delta = \delta'/(2\|P\|)$.

Then whenever $y \in B_2$ satisfies $\|y - G(0)\| < \delta$, there is an $x \in B_1$ such that $G(x) = y$.

Denote by G_ε the operator

$$G_\varepsilon(\psi) = \mathrm{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\omega_\varepsilon + i\partial\bar{\partial}\psi)).$$

Then the linearisation of the map $\tilde{G}_\varepsilon : L_k^2 \times \mathfrak{k} \rightarrow L_{k-6}^2$ defined by

$$(\psi, h) \rightarrow G_\varepsilon\psi - h - \frac{1}{2}\langle \nabla h, \nabla \gamma_\varepsilon \rangle$$

is the map $\tilde{\mathcal{F}}_\varepsilon : L_k^2 \times \mathfrak{k} \rightarrow L_{k-6}^2$ defined by

$$(\psi, h) \rightarrow \mathcal{F}_\varepsilon\psi - h - \frac{1}{2}\langle \nabla h, \nabla \gamma_\varepsilon \rangle$$

since the terms not involving G_ε are actually linear in both factors. Corollary 3.17 then implies that the linearisation of \tilde{G}_ε is surjective and admits a right inverse and moreover provides a uniform bound on the operator norm of this right inverse in terms of a constant multiple of ε^{-2} .

To apply Theorem 3.20, we thus need to obtain a bound on the operator norm of the operators $\tilde{G}_\varepsilon - d\tilde{G}_\varepsilon$. Denote $\mathcal{N}_\varepsilon = \tilde{G}_\varepsilon - \tilde{\mathcal{F}}_\varepsilon$ the nonlinear terms of the Z -critical operator, calculated with respect to the approximate solution ω_ε .

Lemma 3.21. For all ε sufficiently small, there are constants $c, C > 0$ such that for all sufficiently small ε , if $\psi, \psi' \in L^2_k(X, \mathbb{R})$ satisfy $\|\psi\|_{L^2_k}, \|\psi'\|_{L^2_k} \leq c$, then

$$\|\mathcal{N}_\varepsilon(\psi) - \mathcal{N}_\varepsilon(\psi')\|_{L^2_{k-6}} \leq C(\|\psi\|_{L^2_k} + \|\psi'\|_{L^2_k})\|\psi - \psi'\|_{L^2_k}.$$

Proof. Since the two terms involving the Hamiltonian h in \tilde{G} are actually linear in h and ψ , we may replace $\mathcal{N}_\varepsilon(\psi) - \mathcal{N}_\varepsilon(\psi')$ with the terms only involving $G_\varepsilon(\psi)$.

The proof is then similar to a situation considered by Fine [28, Lemma 7.1], and is a straightforward consequence of the mean value theorem, which gives a bound

$$\|\mathcal{N}_\varepsilon(\psi) - \mathcal{N}_\varepsilon(\psi')\|_{L^2_{k-6}} \leq \sup_{\chi_t} \|(D\mathcal{N}_\varepsilon)_{\chi_t}\|_{op} \|\psi - \psi'\|_{L^2_k},$$

where $\chi_t = t\psi + (1 - t)\psi'$ and $t \in [0, 1]$. But

$$(D\mathcal{N}_\varepsilon)_{\chi_t} = \mathcal{F}_{\varepsilon, \chi_t} - \mathcal{F}_{\varepsilon, m},$$

where $\mathcal{F}_{\varepsilon, \chi_t}$ is the linearisation of the Z_ε -critical operator at $\omega_\varepsilon + i\partial\bar{\partial}\chi$. So we seek a bound on the difference of the linearisations when we change the Kähler potential, but for $\varepsilon \ll 1$ this can be bounded by

$$\|\mathcal{F}_{\varepsilon, \chi_t} - \mathcal{F}_\varepsilon\|_{op} \leq c'\|\chi\|_{L^2_k},$$

where c' is independent of ε , which completes the proof. □

Remark 3.22. In fact, as explained by Fine [28, Section 2.2 and Lemma 8.10], the above proof applies very generally, even varying in addition the complex structure. In the case the complex structure is varying, one obtains a bound where $\|\psi\|_{L^2_k} + \|\psi'\|_{L^2_k}$ is replaced by the norm of the difference $(J, \psi) - (J', \psi')$ [28, Lemma 2.10], so the constant obtained can be taken to be continuous when varying the complex structure. Fine explains this for the linearisation of the scalar curvature, but all that is needed is that the operator in question is a polynomial operator in the curvature tensor, which is true for \tilde{Z}_ε and which implies the same result for $\text{Im}(e^{-\varphi_\varepsilon}\tilde{Z}_\varepsilon)$.

This is everything needed to apply the quantitative inverse function theorem, as in Fine [28, Proof of Theorem 1.1].

Proof of Theorem 3.19. We consider the approximate solution ω_ε which satisfies $\text{Im}(e^{-\varphi_\varepsilon}\tilde{Z}_\varepsilon(\omega_\varepsilon)) = O(\varepsilon^5)$. We note that as all of the input is invariant under a maximal compact torus $\text{Aut}(X, L)$, the output produced will also be invariant. There are three ingredients which we have established necessary to apply the implicit function theorem:

- (i) Since we are considering the approximate solution, we have $\|G_\varepsilon(0)\| = O(\varepsilon^5)$.
- (ii) Next, note that the operator $\tilde{\mathcal{F}}_\varepsilon$ is an surjective for ε small and the right inverse \tilde{P}_ε satisfies

$$\|\tilde{P}_\varepsilon\|_{op} \leq \varepsilon^{-2}K_1$$

by Corollary 3.17.

- (iii) Finally, note that there is a constant M such that for all sufficiently small κ , the operator $\tilde{G}_\varepsilon - D\tilde{G}_\varepsilon$ is Lipschitz with constant κ on $B_{M\kappa}$.

The second and third of these imply that the radius δ'_ε of the ball around the origin on which $\tilde{G}_\varepsilon - D\tilde{G}_\varepsilon$ is Lipschitz with constant $(2\|\tilde{P}_\varepsilon\|)^{-1}$ is bounded below by $C\varepsilon^2$ for a positive constant C . From the statement of the quantitative inverse function theorem, the radius δ_ε of interest is defined by

$$\delta_\varepsilon = \delta'_\varepsilon(2\|\tilde{P}_\varepsilon\|)^{-1},$$

meaning that δ_ε is bounded below by $C^2\varepsilon^4$. It follows that for $\varepsilon \ll 1$, if $\|\tilde{G}_\varepsilon(0)\| < C^2\varepsilon^4$, then there is a solution to the equation $G_\varepsilon(0)$, which is what we wanted to produce. Thus, as our approximate solutions satisfy $\|G_\varepsilon(0)\| = O(\varepsilon^5)$, for $0 < \varepsilon \ll 1$, the proof is complete. Note that this produces solutions in some Sobolev space, but elliptic regularity produces smooth solutions as our equation is elliptic for sufficiently small ε by Lemma 3.14.

Finally, while our definition of a Z -critical Kähler metric requires the positivity condition $\text{Re}(e^{-i\varphi_\varepsilon(X,L)}\tilde{Z}_\varepsilon(\omega)) > 0$, one calculates that this is automatic for $0 < \varepsilon \ll 1$. □

3.4.6. Analysis over the Kuranishi space

We next apply the quantitative implicit function theorem in a similar manner in *families*. Recall our polarised manifold of interest (X, L) is analytically K -semistable so that it degenerates to a cscK manifold (X_0, L_0) . We will be interested in the Kuranishi space of (X_0, L_0) , which captures all deformations of (X_0, L_0) . Our setup and discussion is based on that of Székelyhidi [61, Section 3], to which we refer for more details (see Inoue [39, Section 3.2] for another clear exposition). We denote by (M, ω) the underlying symplectic manifold of (X_0, ω) , with ω the cscK metric and denote by $\mathcal{J}(M, \omega)$ the space of almost complex structures on M compatible with ω . As in the work of Székelyhidi, using the operator $Ph = \bar{\partial}\nabla^{1,0}h$ of Equation (3.8) (computed on X_0) we denote

$$\tilde{H}^1 = \{\alpha \in T_{J_0}\mathcal{J} : P^*\alpha = \bar{\partial}\alpha = 0\};$$

this is a finite-dimensional vector space as it is the kernel of the elliptic operator $P^*P + \bar{\partial}^*\bar{\partial}$ and is the first cohomology of the elliptic complex [61, Equation (4)]

$$C_0^\infty(M, \mathbb{C}) \rightarrow T_{J_0}\mathcal{J}(M, \omega) \rightarrow \Omega^{0,2}(M),$$

where the first morphism is given by P and the second morphism is given by the $\bar{\partial}$ -operator (again both associated to the complex manifold X_0). In the following, we assume that the deformation theory of (X_0, L_0) is *unobstructed*, in the sense that the second cohomology of this complex vanishes.

Denote by K the stabiliser of J_0 under the action of \mathcal{G} so that K is the group of biholomorphisms of (X_0, L_0) preserving the Kähler metric ω and the complexification $K^\mathbb{C}$ equals $\text{Aut}(X_0, L_0)$ by a result of Matsushima [33, Theorem 3.5.1]. The vector space \tilde{H}^1 admits a linear K -action.

Note that any holomorphic map $q : B \rightarrow \mathcal{J}(M, \omega)$ from a complex manifold B and with image lying in the space of integrable complex structures produces a family of complex manifolds $\mathcal{X} \rightarrow B$ where the fibre is given by $\mathcal{X}_b = (M, J_{q(b)})$. Fixing a point $b \in B$, recall that we say that \mathcal{X} is a *versal deformation space* for \mathcal{X}_b if every other holomorphic family $\mathcal{U} \rightarrow B'$ with $\mathcal{U}_{b'} \cong \mathcal{X}_b$ is locally the pullback of \mathcal{X} through some holomorphic map $B' \rightarrow B$. We recall Kuranishi’s result:

Theorem 3.23 [61, Proposition 7][9, Lemma 6.1]. *There is an open neighbourhood $B \subset \tilde{H}^1$ of the origin and an embedding*

$$\Phi : B \rightarrow \mathcal{J}(M, \omega)$$

with $\Phi(0) = J_0$ and which produces a versal deformation space for X_0 . Points in B inside the same $K^\mathbb{C}$ -orbit correspond to biholomorphic complex manifolds. The universal family $\mathcal{X} \rightarrow B$ admits a holomorphic line bundle \mathcal{L} and a holomorphic K -action making $\mathcal{X} \rightarrow B$ a K -equivariant map. The form ω induces a K -invariant relatively Kähler metric which we denote $\omega_\mathcal{X} \in c_1(\mathcal{L})$.

Unobstructedness allows us to assume that B is an open neighbourhood of the origin, hence smooth. By construction, the underlying smooth manifold M of X and symplectic form are fixed in the Kuranishi family, while the complex structure varies. Thus, we may smoothly write $\mathcal{X} = B \times M$, with \mathcal{X} admitting a K -invariant relatively Kähler metric $\omega_\mathcal{X} \in c_1(\mathcal{L})$ which restricts to the cscK metric on X_0 . The Kuranishi family is precisely the pullback of the universal family over $\mathcal{J}(M, \omega)$.

Remark 3.24. When $\text{Aut}(X, L)$ is discrete but not finite, we have assumed that the test configuration producing the cscK degeneration of (X, L) is $\text{Aut}(X, L)$ -equivariant, producing an $\text{Aut}(X, L)$ -action on (X_0, L_0) since by Remark 3.11 this was used in our proof of the moment map interpretation of the equation (and hence in understanding the linearisation of the equation). In this case, we use the equivariant Kuranishi family as in the work of Inoue [39, Section 3.2], which has a universal family admitting an $\text{Aut}(X, L)$ -action, and which has the property that maps to the equivariant Kuranishi space correspond to deformations of (X_0, L_0) which are $\text{Aut}(X, L)$ -equivariant.

We now apply the analysis to the entire Kuranishi space. We perturb the initial relatively Kähler metric $\omega_{\mathcal{X}}$ in such a way as to allow us to later employ the moment map property of the Z -critical operator. To do so, we define a finite-dimensional function space depending on $b \in B$, which as a vector space will be isomorphic to the Lie algebra \mathfrak{k} of K for each $b \in B$ (perhaps after shrinking B). The definition of the function space is motivated by the moment map interpretation of the Z -critical operator given in Section 3.3.1, and implicitly appears there.

For each $v \in \mathfrak{k}$, much as in Equation (3.3), to the relatively Kähler metric $\omega_{\mathcal{X}}$ we associate the function $h_{\mathcal{X},v} \in C^\infty(\mathcal{X})$ satisfying

$$\mathcal{L}_{J_{\mathcal{X}}v}\omega_{\mathcal{X}} = i\partial\bar{\partial}h_{\mathcal{X},v},$$

where $J_{\mathcal{X}}$ is the almost complex structure of \mathcal{X} . Such a $h_{\mathcal{X},v}$ is unique up to the addition of a function f satisfying $i\partial\bar{\partial}f = 0$; as such, f is constant along the fibres of the Kuranishi family (as its fibres are compact), hence f is the pullback of a function $f_B \in C^\infty(B)$ satisfying $i\partial_B\bar{\partial}_B f_B = 0$. The choice of such a function f_B will not affect the definition of the function space of interest to us, so any choice suffices.

Through the identification $\mathcal{X} = B \times M$, by pullback any (K -invariant) function $\psi \in C^\infty(M, \mathbb{R})$ induces a (K -invariant) function on \mathcal{X} which we still denote $\psi \in C^\infty(\mathcal{X})$. If we change $\omega_{\mathcal{X}}$ to a new K -invariant relatively Kähler metric $\omega_{\omega_{\mathcal{X}}} + i\partial\bar{\partial}\psi$ (so that we assume ψ is a Kähler potential on each fibre), then by a similar calculation to Equation (3.6) the function $h_{\mathcal{X},v}$ changes to

$$h_{\mathcal{X},v,\psi} = h_{\mathcal{X},v} + (J_{\mathcal{X}}v)\psi \in C^\infty(\mathcal{X});$$

we use this notation also when ψ is not K -invariant. We further denote

$$h_{b,v,\psi} = h_{\mathcal{X},v}|_{\mathcal{X}_b} \text{ and } \omega_{\mathcal{X}_b,\psi} = (\omega_{\mathcal{X}} + i\partial\bar{\partial}\psi)|_{\mathcal{X}_b},$$

which allows us to write for for each $\psi \in C^\infty(M)$ and $b \in B$ write

$$\mathfrak{k}_{b,\psi} = \{h_{b,v,\psi} - \hat{h}_{b,v,\psi} : v \in \mathfrak{k}\},$$

where the constant $\hat{h}_{b,v,\psi}$ is the average of $h_{b,v,\psi}$, defined in such a way that the function space consists of functions which integrate to zero:

$$\int_{\mathcal{X}_b} (h_{b,v,\psi} - \hat{h}_{b,v,\psi})\omega_{\mathcal{X}_b,\psi}^n = 0.$$

For $b = 0$, these are the holomorphy potentials with respect to $\omega_{0,\psi}$, hence $\mathfrak{k}_{0,\psi} \cong \mathfrak{k}$ as vector spaces; it follows that this isomorphism extends to a neighbourhood of $0 \in B$, and we shrink B so that this is the case. Our discussion extends to functions ψ of lower regularity without change.

For each $b \in B$ we are interested in the operator $L_k^2 \times \mathfrak{k} \rightarrow L_{k-6}^2$ defined by

$$(\psi, v) \rightarrow \text{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\mathcal{X}_b, \omega_{\mathcal{X}_b,\psi})) + h_{b,v,\psi}.$$

Here, we have included the space \mathcal{X}_b in the notation for clarity. Note that for $b = 0$, provided ψ is K -invariant the function $h_{0,v,\psi}$ is genuinely the holomorphy potential for the real holomorphic vector field v on \mathcal{X}_0 with respect to the Kähler metric $\omega_{\mathcal{X}_0}$. Importantly, it follows that for $b = 0$ this operator is

precisely the operator considered throughout the present section, and in particular Theorem 3.19 allows us to conclude for $b = 0$ that there exists a sequence

$$(\psi_\varepsilon, \nu_\varepsilon) \in L_k^2 \times \mathfrak{k}$$

such that

$$\text{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\mathcal{X}_0, \omega_b, \Psi)) = h_{0, \nu_\varepsilon, \psi_\varepsilon} \in \mathfrak{h}_{0, \psi_\varepsilon}.$$

We next explain how the same technique proves the following analogue for $b \neq 0$:

Proposition 3.19. *Perhaps after shrinking B , for all $0 < \varepsilon \ll 1$ there is a map*

$$\Psi_\varepsilon : B \rightarrow L_k^2(M, \mathbb{R})$$

such that for all $b \in B$

$$\text{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\mathcal{X}_b, \omega_b, \Psi)) \in \mathfrak{k}_{b, \Psi(b)}.$$

We mention two further properties of the Ψ_ε thus produced, before explaining how the same technique as Theorem 3.19 establishes Proposition 3.19. Firstly, as the proof of Proposition 3.19 ultimately uses the contraction mapping theorem, as a standard consequence of the contraction mapping theorem the b -dependent functions $\Psi_\varepsilon(b)$ are as regular as possible in $b \in B$ and ε (just as in [59, Proof of Theorem 1], for example). We will only need that they are actually, say, C^8 , to ensure that the Z -critical operator is twice differentiable, which is then guaranteed for sufficiently large k . Secondly, through the (smooth) identification $\mathcal{X} = B \times M$, we may identify Ψ_ε with a function on \mathcal{X} such that

$$\omega_{\mathcal{X}, \varepsilon} = \omega_{\mathcal{X}} + i\partial\bar{\partial}\Psi_\varepsilon \tag{3.24}$$

is relatively Kähler for all sufficiently small ε . The remaining property we will need is that the $\omega_{\mathcal{X}} + i\partial\bar{\partial}\Psi_\varepsilon$ produced in this manner is K -invariant, which is ultimately a consequence of K -invariance of all objects involved: the map $\mathcal{X} \rightarrow B$, the form $\omega_{\mathcal{X}}$ and the Z -critical operator itself.

Proposition 3.19 follows directly from the arguments on a fixed complex structure, so we only sketch the differences. On the central fibre $(\mathcal{X}_0, \mathcal{L}_0)$, the result is precisely Theorem 3.19. The three key ingredients in Theorem 3.19 were the construction of approximate solutions, the bound on the operator norm of the right inverse of the linearised operator and the control of the nonlinear operator. We mention how each aspect in turn adapts.

As a first step, we replace the initial form $\omega_{\mathcal{X}}$ with a K -invariant relatively Kähler metric (still denoted $\omega_{\mathcal{X}}$) that satisfies

$$S(\omega_b) \in \mathfrak{k}_{b,0};$$

this leaves the Kähler metric on X_0 unchanged but perturbs the metric on nearby fibres. The construction of such an $\omega_{\mathcal{X}}$ follows from an application of the implicit function theorem analogous to [61, Proposition 7], perturbing the Kähler metric on each fibre. The application of the implicit function theorem uses that the linearisation of the scalar curvature on the cscK manifold (X_0, ω) takes the form $-\mathcal{D}_0^*\mathcal{D}_0$ so that its kernel is isomorphic to $\mathfrak{k}_{0,0}$, that the linearisation for general $b \in B$ is a perturbation of this and that the function spaces $\mathfrak{h}_{b,0}$ are similarly perturbations of $\mathfrak{h}_{0,0}$.

The approximate solutions can then be constructed for all $b \in B$ since the property used to construct the approximate solutions was that the linearisation was to leading order the Lichnerowicz operator $-\mathcal{D}_0^*\mathcal{D}_0$ on the central fibre $(\mathcal{X}_0, \mathcal{L}_0)$. Since the linearisation for general $b \in B$ is a perturbation of $-\mathcal{D}_0^*\mathcal{D}_0$, it remains an isomorphism orthogonal to $\mathfrak{h}_{b,0}$ (as this is itself a perturbation of $\mathfrak{h}_{0,0}$), so the same argument applies to produce approximate solutions to any order. Similarly, calculated at the approximate solution, the linearised operator at b is a perturbation of the linearisation at $b = 0$, hence

the mapping properties are inherited from those on $(\mathcal{X}_0, \mathcal{L}_0)$, producing the appropriate bound on the operator norm of the right inverse. Here, we use in addition regularity of the operator $(b, \psi, \nu) \rightarrow h_{b, \nu, \psi}$ to ensure the linearised operators at the approximate solutions converge as $b \rightarrow 0$. As noted in Remark 3.22, the bounds on the nonlinear terms apply also with the complex structure allowed to vary. Thus, one can produce the desired $\Psi_\varepsilon(b)$ for each $b \in B$, and as above the contraction mapping theorem ensures regularity of $\Psi_\varepsilon(b)$ as one varies b or ε .

As we see in Corollary 3.20, the important consequence of Proposition 3.19 is that a zero of a natural moment map on B is then actually a genuine Z -critical Kähler metric. Thus, we have reduced to a finite-dimensional moment map problem.

3.5. Solving the finite-dimensional problem

We now turn to the solution of the finite-dimensional moment map problem. Recall from Section 3.3.1 that to the Kuranishi family $(\mathcal{X}, \mathcal{L}) \rightarrow B$, the central charge Z_ε and a relatively Kähler metric $\omega_{\mathcal{X}}$, we have associated a closed complex $(n + 1, n + 1)$ -form on \mathcal{X} , which we denote here $\tilde{Z}_\varepsilon(\mathcal{X}, \omega_{\mathcal{X}, \varepsilon})$, hence producing a closed $(1, 1)$ -form on B through taking the associated fibre integral. Using the forms $\omega_{\mathcal{X}, \varepsilon} = \omega_{\mathcal{X}} + i\partial\bar{\partial}\Psi_\varepsilon$ produced by Proposition 3.19 (as in Equation (3.24)), we set

$$\Omega_\varepsilon = \text{Im} \left(e^{-i\varphi_\varepsilon} \int_{\mathcal{X}/B} \tilde{Z}_\varepsilon(\mathcal{X}, \omega_{\mathcal{X}, \varepsilon}) \right).$$

By K -invariance of Ψ_ε and $\mathcal{X} \rightarrow B$, the forms Ω_ε are K -invariant for all ε and are further Kähler for ε sufficiently small as they are perturbations of (the pullback under the Kuranishi map of) the Donaldson–Fujiki Kähler metric on $\mathcal{J}(M, \omega)$, just as in Proposition 3.8.

Lemma 3.25. *There exist a sequence of moment maps*

$$\mu_\varepsilon : B \rightarrow \mathfrak{k}^*$$

for the K -action on (B, Ω_ε) .

Proof. The K -action on B is induced by the linear K -action on the vector space $\tilde{H}^1(X_0, TX_0^{1,0})$. The linear K -action on B admits a canonical moment map with respect to the flat Kähler metric, and hence by the equivariant Darboux theorem also admits a moment map with respect to any other K -invariant symplectic form [26, Theorem 3.2]. One may alternatively more directly used that one can write $\Omega_\varepsilon = d\lambda_\varepsilon$ for λ_ε a K -invariant one-form to conclude the existence of a moment map [48, Exercise 5.2.2]. □

The moment maps μ_ε are only unique up to the addition of an element of $(\mathfrak{k}^*)^K$, where the latter denotes K -invariant elements of \mathfrak{k}^* under the coadjoint action. The next result ensures that we have chosen the geometrically appropriate sequence of moment maps. In what follows, for $\nu \in \mathfrak{k}$ we denote by $h_{\varepsilon, \nu}$ a function satisfying

$$i\partial\bar{\partial}h_{\varepsilon, \nu} = \mathcal{L}_{J_{\mathcal{X}}\nu}\omega_{\mathcal{X}, \varepsilon};$$

as in Section 3.4.6, as discussed there, such a choice is unique up to the addition of the pullback of a function from B and any choice suffices. We then denote $h_{\varepsilon, \nu, b}$ its restriction to a fibre \mathcal{X}_b and similarly denote $\omega_{b, \varepsilon} = \omega_{\mathcal{X}, \varepsilon}|_{\mathcal{X}_b}$, with the corresponding function spaces as in Section 3.4.6 denoted $\mathfrak{h}_{b, \varepsilon}$.

Lemma 3.26. *We may normalise μ_ε such that*

$$\langle \mu_\varepsilon, \nu \rangle(0) = \int_{\mathcal{X}_0} h_{\varepsilon, \nu, 0} \text{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(X_0, \omega_{b, \varepsilon})) \omega_{b, \varepsilon}^n.$$

Proof. Since adding an element of $(\mathfrak{k}^*)^K$ preserves the moment map condition, we need only check that the map $\mathfrak{k} \rightarrow \mathbb{C}$ defined by

$$v \rightarrow \int_{\mathcal{X}_0} h_{\varepsilon,v,0} \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(X_0, \omega_{b,\varepsilon})) \omega_{b,\varepsilon}^n$$

is K -invariant, where K acts on \mathfrak{k} by the adjoint action. But this follows since $\operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(X_0, \omega_{b,\varepsilon})) \omega_{b,\varepsilon}^n$ is a K -invariant (n, n) -form and

$$h_{\varepsilon,k \cdot v} = k^* h_{\varepsilon,v},$$

where $k \cdot v$ denotes the adjoint action. □

While we have little control over the moment maps μ_ε in general, on the orbit of interest we next show that solving the moment map problem produces Z -critical Kähler metrics. For this we denote by $b_0 \in B$ a point corresponding to the complex manifold (X, L) of interest so that $(X, L) \cong (\mathcal{X}_{b_0}, \mathcal{L}_{b_0})$.

We claim that $0 \in K^\mathbb{C}.b_0 \cap B$. Since there is a test configuration for our polarised manifold with central fibre X_0 , by versality of B there is a sequence of points b_l in B with $\mathcal{X}_{b_l} \cong X$ and $b_l \rightarrow 0$ as $l \rightarrow \infty$. To see this, note that the only point with fibre isomorphic to X_0 is 0 itself, and that by a result of Székelyhidi there is some $\mathbb{C}^* \hookrightarrow K^\mathbb{C}$ such that the specialisation of b_0 corresponds to a complex structure admitting a cscK metric [61, Theorem 2]. But cscK specialisations are actually unique by a result of Chen–Sun [9, Corollary 1.8], implying $0 \in K^\mathbb{C}.b_0 \cap B$. We note that we do not need to rely on the deep result of Chen–Sun to prove our main result: Without appealing to this, one could instead consider the cscK degeneration $(\mathcal{X}_{b_\infty}, \mathcal{L}_{b_\infty})$ of (X, L) in B produced by Székelyhidi and consider (X, L) as a deformation of $(\mathcal{X}_{b_\infty}, \mathcal{L}_{b_\infty})$ instead, arguing in the same way.

Corollary 3.20. *For any $b \in \overline{K^\mathbb{C}.b_0}$, the moment map $\mu_\varepsilon(b)$ is given by*

$$\langle \mu_\varepsilon(b), v \rangle = \int_{\mathcal{X}_b} h_{\varepsilon,v,b} \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}(\mathcal{X}_b, \omega_\varepsilon, b)) \omega_{\varepsilon,b}^n.$$

Proof. By Theorem 3.6, the operator $B \rightarrow \mathfrak{k}^*$ assigning

$$b \rightarrow \left[v \rightarrow \int_{\mathcal{X}_b} h_{\varepsilon,v,b} \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\mathcal{X}_b, \omega_\varepsilon, b)) \omega_{\varepsilon,b}^n \right]$$

is a moment map for the K -action on $K^\mathbb{C}.b_0$. It follows that this operator agrees with μ_ε up to the addition of an element of $(\mathfrak{k}^*)^K$, namely that for all $b \in K^\mathbb{C}.b_0$

$$\langle \mu_\varepsilon, v \rangle(b) = \int_{\mathcal{X}_b} h_{\varepsilon,v,b} \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\mathcal{X}_b, \omega_\varepsilon, b)) \omega_{\varepsilon,b}^n + \langle \xi_\varepsilon, v \rangle$$

for some $\xi_\varepsilon \in (\mathfrak{k}^*)^K$ independent of $b \in K^\mathbb{C}.b_0$. We claim that $\xi_\varepsilon = 0$, which will imply the result for all $b \in K^\mathbb{C}.b_0$. Fixing $v \in \mathfrak{k}$ and taking any sequence $b_l \in K^\mathbb{C}.b_0$ converging to $0 \in K^\mathbb{C}.b_0$, the fact that $\langle \mu_\varepsilon, v \rangle(b)$ and $\int_{\mathcal{X}} h_{\varepsilon,v} \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(X, \omega_\varepsilon)) \omega_\varepsilon^n$ both converge to $\int_{\mathcal{X}_0} h_{\varepsilon,v} \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(X_0, \omega_\varepsilon)) \omega_\varepsilon^n$ implies that $\langle \xi_\varepsilon, v \rangle = 0$, as claimed. The result for points $b \in \overline{K^\mathbb{C}.b_0}$ also allowed to lie in the closure of the orbit of b_0 follow by continuity. □

We appeal to a version of the Kempf–Ness theorem to construct zeroes of the moment maps μ_ε .

Proposition 3.21 [15, Corollary 4.6, Propositions 4.8, 4.9]. *Suppose b_0 satisfies the condition the following stability condition: For all $0 < \varepsilon \ll 1$ and for all one-parameter subgroups λ_v of $K^\mathbb{C}$*

associated to $v \in \mathfrak{k}$ such that

$$\lim_{t \rightarrow 0} \lambda_v(t).b = \lim_{t \rightarrow \infty} \exp(-itv).b_0 = b' \in B$$

exists and lies in B , we have

$$\langle \mu_\varepsilon, v \rangle(b') < 0.$$

Then there exists a sequence of points $b_\varepsilon \in K^C.b_0$ such that $\mu_\varepsilon(b_\varepsilon) = 0$.

Remark 3.27. The proof of this result uses that B is an open ball in a vector space where the action as linear, as it passes to an associated projective problem. It is also important that ε is taken to be sufficiently small, as the proof proceeds by considering the gradient flow associated to the moment map problem, and the condition that ε be taken to be small is used to ensure that the flow converges in B .

We are now in a position to prove our main result, for which are assumptions are the same as throughout.

Corollary 3.22. *Suppose (X, L) is asymptotically Z -stable. Then (X, L) admits Z_ε -critical Kähler metrics for all $0 < \varepsilon \ll 1$.*

Proof. A one-parameter subgroup λ_v of G associated to $v \in \mathfrak{k}$ such that

$$\lim_{t \rightarrow 0} \lambda_v(t).b = \lim_{t \rightarrow \infty} \exp(-itv).b_0 = b' \in B$$

exists and lies in B induces a test configuration $(\mathcal{Y}, \mathcal{L}_\mathcal{Y})$ for (X, L) , with central fibre $(\mathcal{X}_{b'}, \mathcal{L}_{b'})$ by restricting the Kuranishi family $(\mathcal{X}, \mathcal{L}) \rightarrow B$ (as in [61, Proof of Theorem 2]). By the form of the moment map μ_ε given in Corollary 3.20 and Proposition 3.6,

$$\langle \mu_\varepsilon, v \rangle(b') = -\operatorname{Im} \left(\frac{Z_\varepsilon(\mathcal{Y}, \mathcal{L}_\mathcal{Y})}{Z_\varepsilon(X, L)} \right). \tag{3.25}$$

Thus, asymptotic Z -stability of (X, L) forces the condition $\langle \mu_\varepsilon, v \rangle(b') < 0$, meaning that Proposition 3.21 implies the existence of zeroes b_ε of the moment maps μ_ε . In terms of the function space $\mathfrak{h}_{b_\varepsilon, \varepsilon}$, this means that for any $h \in \mathfrak{h}_{b_\varepsilon, \varepsilon}$

$$\int_{\mathcal{X}_{b_\varepsilon}} h \operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\mathcal{X}_{b_\varepsilon}, \omega_\varepsilon, b_\varepsilon)) \omega_{\varepsilon, b_\varepsilon}^n = 0,$$

again by the form of the moment map μ_ε given in Corollary 3.20. But since by Proposition 3.19

$$\operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\mathcal{X}_{b_\varepsilon}, \omega_\varepsilon, b_\varepsilon)) \in \mathfrak{h}_{b_\varepsilon, \varepsilon},$$

it follows from nondegeneracy of the L^2 -inner product that

$$\operatorname{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\mathcal{X}_{b_\varepsilon}, \omega_\varepsilon, b_\varepsilon)) = 0.$$

Elliptic regularity implies these solutions are actually smooth, concluding the result. □

3.6. Existence implies stability

We return to the base of the Kuranishi space B and along with its universal family $(\mathcal{X}, \mathcal{L}) \rightarrow B$. Our hypothesis is that (X, L) admits Z_ε -critical Kähler metrics for all ε sufficiently small, in a way that is compatible with our proof of that ‘stability implies existence’. That is, we assume that there is a

sequence of relative Kähler metrics $\omega_{\mathcal{X},\varepsilon} \in c_1(\mathcal{L})$ such that for each ε there is a $b_\varepsilon \in K^{\mathbb{C}} \cdot b_0$, where $(\mathcal{X}_{b_0}, \mathcal{L}_{b_0}) \cong (X, L)$ such that

$$\text{Im}(e^{-i\varphi_\varepsilon} \tilde{Z}_\varepsilon(\mathcal{X}_b, \omega_\varepsilon|_{\mathcal{X}_b})) = 0.$$

Recall from the proof of Corollary 3.22 that each $\mathbb{C}^* \hookrightarrow K^{\mathbb{C}}$ produces a test configuration for (X, L) .

Theorem 3.28. *In the above situation, for each test configuration $(\mathcal{Y}, \mathcal{L}_\mathcal{Y})$ arising from the action of $K^{\mathbb{C}}$, we have*

$$\text{Im}\left(\frac{Z_\varepsilon(\mathcal{Y}, \mathcal{L}_\mathcal{Y})}{Z_\varepsilon(X, L)}\right) > 0$$

for all $0 < \varepsilon \ll 1$.

This of course is equivalent to our definition of asymptotic Z -stability with respect to these test configurations, which used $k = \varepsilon^{-1}$ rather than ε . We note that, in principle, (X, L) could admit Z_ε -critical Kähler metrics which are ‘far’ from the cscK degeneration (X_0, L_0) and hence do not arise from this construction. Thus, this is a truly local result.

Proof. This is a formal consequence of standard finite-dimensional moment map theory. By Corollary 3.20, each b_ε is actually a zero of a genuine finite-dimensional moment map with respect to the Kähler metrics Ω_ε on B . It then follows by convexity of the log norm functional associated to the moment map that for any \mathbb{C}^* -action induced by $J_B v$, with J_B the almost complex structure on B and $b_{\varepsilon,0}$ the specialisation of b_ε , the value $\langle \hat{\mu}_\varepsilon, v \rangle(b_{\varepsilon,0})$ is negative. But by Equation (3.25), we have

$$\langle \hat{\mu}_\varepsilon, v \rangle(b_0) = -\text{Im}\left(\frac{Z_\varepsilon(\mathcal{Y}, \mathcal{L}_\mathcal{Y})}{Z_\varepsilon(X, L)}\right),$$

proving the result. □

Remark 3.29. This is truly a local result, and it would be very interesting to obtain a global analogue. In principle, (X, L) could admit Z_ε -critical Kähler metrics that are ‘far’ from the cscK metric on $(\mathcal{X}_0, \mathcal{L}_0)$ to which our result would not apply, although this seems unlikely in practice. Furthermore, there are many other test configurations for (X, L) not arising from the Kuranishi space of $(\mathcal{X}_0, \mathcal{L}_0)$ for which we do not obtain stability with respect to.

4. The higher rank case

We now extend our results to central charges involving higher Chern classes. Our exposition is brief, as the details are broadly similar to the ‘rank one’ case, with a small number of exceptions. The first main difficulty is to extend the slope formula for the Z -energy to the setting where higher Chern classes are involved. The idea to overcome this is to reduce to the ‘rank one’ case by projectivising, so we use of the Segre classes $s_k(X)$ of X . The second difficulty is that it is not clear that taking the variation of the Z -energy in this context actually produces a partial differential equation, so we simply include this as a hypothesis.

We thus consider a central charge of the form

$$Z_k(X, L) = \sum_{l=0}^n \rho_l k^l \int_X L^l \cdot f(s(X)) \cdot \Theta,$$

for some ρ, Θ and $f(s(X))$ now an arbitrary polynomial in the Segre classes $s_1(X), \dots, s_n(X)$ of X . The substantial difference is in the equation itself: The Euler–Lagrange equation of the Z -energy no longer produces a partial differential equation.

4.1. Stability

It is straightforward to extend the notion of stability, provided the central fibre of the test configuration is smooth, which we hence assume. Given such a test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) , we associate to a term of the central charge of the form

$$\int_X L^l \cdot s_{m_1}(X) \cdot \dots \cdot s_{m_j}(X) \cdot \Theta$$

an intersection number

$$\int_{\mathcal{X}} \mathcal{L}^{l+1} \cdot s_{m_1}(T_{\mathcal{X}/B}) \cdot \dots \cdot s_{m_j}(T_{\mathcal{X}/B}) \cdot \Theta,$$

where $s_m(T_{\mathcal{X}/\mathbb{P}^1})$ is the m^{th} Segre class of the relative holomorphic tangent bundle $T_{\mathcal{X}/B}$, which is a holomorphic vector bundle as $\mathcal{X} \rightarrow \mathbb{P}^1$ is a holomorphic submersion. The notion of stability is then just as before: We require

$$\text{Im} \left(\frac{Z_{\varepsilon}(\mathcal{X}, \mathcal{L})}{Z_{\varepsilon}(X, L)} \right) > 0 \text{ for } 0 < \varepsilon \ll 1$$

Remark 4.1. One approach to defining the numerical invariant of interest more generally, when \mathcal{X} is smooth but \mathcal{X}_0 is singular, is as follows. Recall that the Segre classes are multiplicative in short exact sequences. Thus, when $\mathcal{X} \rightarrow \mathbb{P}^1$ is a smooth morphism, we have

$$s(T_{\mathcal{X}}) = s(T_{\mathcal{X}/\mathbb{P}^1})s(T_{\mathbb{P}^1}),$$

where each of these denotes the holomorphic tangent bundle. When \mathcal{X} has smooth total space but \mathcal{X}_0 is singular so that $s(T_{\mathcal{X}/\mathbb{P}^1})$ and $s(T_{\mathbb{P}^1})$ are both defined, one can use this to define analogues of $s(T_{\mathcal{X}/\mathbb{P}^1})$ and as \mathcal{X} is smooth, one can still make sense of the intersection of cycles on \mathcal{X} itself. It seems challenging to give a reasonable definition when \mathcal{X} is singular, meaning intersection theory of cycles is not defined.

4.2. Z-energy

We now fix a Kähler metric $\omega \in c_1(L)$ and recall some general theory of Bott–Chern forms. Good expositions are given by Donaldson [22, Section 1.2] and Tian [64, Section 1]. The Kähler metric ω induces a Hermitian metric on the holomorphic tangent bundle and hence induces a Chern–Weil representative $s_j(\omega)$ of the Segre classes $s_j(X)$ for all j through the general theory of Bott–Chern forms. Suppose now that $\omega_{\psi} = \omega + i\partial\bar{\partial}\psi$ is another Kähler metric in the same class, producing another representative of $s_j(X)$. Then the theory of Bott–Chern forms implies that there is a $(j - 1, j - 1)$ -form $\text{BC}_j(\psi)$ such that

$$s_j(\omega + i\partial\bar{\partial}\psi) - s_j(\omega) = i\partial\bar{\partial} \text{BC}_j(\psi).$$

To draw the parallel with the theory we have developed in the rank one case, note that $s_1(\omega) = -\text{Ric}(\omega)$, so

$$\text{BC}_1(\psi) = \log \left(\frac{\omega_{\psi}^n}{\omega^n} \right),$$

which is a function that appeared many times in Section 3.2.

With this in hand, we define Deligne functionals in a similar manner to Section 3.1 and analogously to work of Elkik [27]. A Kähler metric $\omega \in c_1(L)$ induces a metric on the holomorphic tangent bundle T_X . This produces representatives of the Segre classes $s_j(T_X)$, and changing ω to ω_{ψ} changes

the representatives of the Segre classes through the Bott–Chern forms. We also fix a representative $\theta \in \Theta$.

We associate to the intersection number $\int_X L^l \cdot s_{m_1}(X) \cdot \dots \cdot s_{m_j}(X) \cdot \Theta$ the value

$$\frac{1}{l+1} \langle \psi, \dots, \psi; \text{BC}_{m_1}(\psi), \dots, \text{BC}_{m_j}(\psi); \theta \rangle \in \mathbb{R}$$

given by

$$\begin{aligned} & \langle \psi, \dots, \psi; \text{BC}_{m_1}(\psi), \dots, \text{BC}_{m_j}(\psi); \theta \rangle \\ &= \int_X \psi \omega_\psi^l \wedge s_{m_1}(\omega_\psi) \wedge \dots \wedge s_{m_j}(\omega_\psi) \wedge \theta \\ &+ \dots + \int_X \text{BC}_{m_j}(\psi) \omega^l \wedge s_{m_1}(\omega) \wedge \dots \wedge s_{m_{j-1}}(\omega) \dots \wedge \theta \end{aligned}$$

by analogy with the usual theory of Deligne functionals. The basic properties of this functional extend directly: There is a natural analogue of the ‘change of metric’ formula, which follows by definition, and the curvature property of Proposition 3.3. The curvature property is proven by Tian when $\theta = 0$ [64, Proposition 1.4] for general functionals of this kind, but the proof applies to the general case.

By linearity we have produced a functional $E_Z : \mathcal{H}_\omega \rightarrow \mathbb{R}$ on the space of Kähler metrics, which we call the Z -energy as before. In the case that $\theta = 0$, a variational formula for the Deligne functional can be found in the work of Donaldson [22, Proposition 6 (ii)], and a similar result holds in general. We will not make use of the precise variational formula, beyond the fact that the Euler–Lagrange equation is independent of initial Kähler metric ω chosen. Thus, the Euler–Lagrange equation is only a condition on ω_ψ and not ω itself. We note, however, that to phrase the Euler–Lagrange equation as a partial differential equation requires a further understanding of the linearisation of the Bott–Chern classes.

Definition 4.1. We say that ω_ψ is a Z -critical Kähler metric if it is a critical point of the Z -energy.

To clarify this condition, let

$$F_{Z,\psi} : f \rightarrow \frac{d}{dt} E_Z(\omega_\psi + t i \partial \bar{\partial} f) \tag{4.1}$$

be the derivative of the Z -energy. Then a Z -critical Kähler metric is a zero of the map

$$\begin{aligned} C^\infty(X, \mathbb{R}) &\rightarrow C^\infty(X, \mathbb{R})^*, \\ \psi &\rightarrow F_{Z,\psi}. \end{aligned}$$

In the ‘rank one’ case, from Proposition 3.5 the map $F_{Z,\psi}$ is given by

$$F_{Z,\psi}(f) = \int f \operatorname{Im}(e^{-i\varphi} \tilde{Z}(\omega_\psi)) \omega_\psi^n,$$

resulting in the Euler–Lagrange equation being equivalent to the partial differential equation

$$\operatorname{Im}(e^{-i\varphi} \tilde{Z}(\omega_\psi)) = 0.$$

Note in general that the operator $F_{Z,\psi}$ is linear in ψ and so takes the form

$$F_{Z,\psi} = \int_X L(\psi) \operatorname{Im}(e^{-i\varphi} \hat{Z}(\omega)) \omega^n,$$

for some linear differential operator L and some $\hat{Z}(\omega)$ which we do not explicitly derive. Let L^* denote the formal adjoint of L .

Definition 4.2. We say that Z is *analytic* if the condition

$$\text{Im}(e^{-i\varphi_\varepsilon} L^* \hat{Z}_\varepsilon(\omega)) = 0$$

is a partial differential equation for ω for all $0 < \varepsilon \ll 1$.

We remark that Pingali has, in a special case, linearised $c_2(\omega)$ and has even established an ellipticity result under hypotheses on the geometry of the manifold in question [52, Lemma 3.1].

Example 4.2. Set

$$Z_k(X, L) = \sum_{l=0}^n \int_X k^l i^{n-l+1} L^l .c_{n-l}(X).$$

The variation of the Deligne functional associated to each term $\int_X L^l .c_{n-l}(X)$ has been calculated by Weinkove [67, Lemma 5.1] (who does not use the Deligne functional terminology) to be

$$\int_X \psi c_{n-l}(\omega) \wedge \omega^l,$$

so the induced equation is a fourth-order partial differential equation only involving the Chern forms of ω . In fact, for $k \gg 0$ small variants of the resulting Z -critical equation have been studied by Leung (under the name ‘almost Kähler-Einstein metrics’ [44]) and Futaki (under the name ‘constant perturbed scalar curvature Kähler metrics’ [32]). Note that, as the equation is fourth order, it is automatically elliptic for $k \gg 0$ as the leading order term of the linearisation is Δ^2 , with this term coming from the linearisation of the scalar curvature. Thus, this is an analytic central charge. Leung and Futaki both use the inverse function theorem to produce solutions to their equations for $k \gg 0$; as these equations are fourth order, their applications of the inverse function theorem do not require the techniques we developed in Theorem 3.19, where the main difficulties were caused by the jump from a fourth order to a sixth-order partial differential equation.

We must produce an analogue of the slope formula of Proposition 3.6, which is the reason we make use of Segre classes rather than Chern classes. As in that situation, a test configuration smooth over \mathbb{C} gives rise to a path ψ_t of Kähler potentials, which in addition induces representatives of the Segre classes. Denote, as was done in the earlier situation of Section 3.1, h the function on \mathcal{X} induced by the \mathbb{C}^* -action and the relatively Kähler metric $\omega_{\mathcal{X}}$. In addition, denote ω_0 the restriction of $\omega_{\mathcal{X}}$ to \mathcal{X}_0 and set $\tau = -\log |t|^2$.

Proposition 4.3. *We have equalities*

$$F_{Z, \mathcal{X}_0, \omega_0}(h) = \lim_{\tau \rightarrow \infty} \frac{d}{d\tau} E_Z(\psi_\tau) = \text{Im} \left(\frac{Z(\mathcal{X}, \mathcal{L})}{Z(X, L)} \right).$$

Proof. The Segre classes are defined in such a way that

$$s_j(X) = \sigma_*(\mathcal{O}(1)^{n-1+j}),$$

where σ_* denotes the push-forward of a cycle through the map $\sigma : \mathbb{P}(T_X) \rightarrow X$ and $\mathcal{O}(1)$ is the relative hyperplane class. On the analytic side, the Hermitian metric on T_X induces a Hermitian metric on $\mathcal{O}(1)$, with curvature ω_{FS} which restricts to a Fubini–Study metric on each fibre. We then have, for example, from [21, Proposition 1.1] or [38], an equality of forms

$$\int_{\mathbb{P}(T_X)/X} \omega_{FS}^{n-1+j} = s_j(\omega),$$

which is simply the metric counterpart of the usual defining property of the Segre classes.

Now, suppose ω_ψ is another Kähler metric on X , giving representatives $s_j(\omega_\psi)$ of the Segre classes. Then

$$s_j(\omega_\psi) - s_j(\omega) = \int_{\mathbb{P}(T_X)/X} (\omega_{\psi,FS}^{n-1+j} - \omega_{FS}^{n-1+j}).$$

Writing

$$\omega_{\psi,FS} - \omega_{FS} = i\partial\bar{\partial}\psi_{FS},$$

this means that

$$\int_{\mathbb{P}(T_X)/X} \psi_{FS} \wedge \left(\sum_{q=0}^{n-2+j} \omega_{\psi,FS}^q \wedge \omega_{FS}^{n-2+j-q} \right) = \text{BC}_j(\psi) \tag{4.2}$$

since taking $i\partial\bar{\partial}$ commutes with the fibre integral and

$$\int_{\mathbb{P}(T_X)/X} (\omega_{\psi,FS}^{n-1+j} - \omega_{FS}^{n-1+j}) = \int_{\mathbb{P}(T_X)/X} i\partial\bar{\partial}\psi_{FS} \wedge \left(\sum_{q=0}^{n-2+j} \omega_{\psi,FS}^q \wedge \omega_{FS}^{n-2+j-q} \right).$$

We note here that Bott–Chern classes are only defined modulo closed forms of one degree lower, and so strictly speaking this is merely a representative of the Bott–Chern class.

We return to our integral $E_Z(\psi_\tau)$ of interest, and as usual we focus on one term of the form

$$\langle \psi, \dots, \psi; \text{BC}_{m_1}(\psi), \dots, \text{BC}_{m_j}(\psi); \theta \rangle.$$

The Segre class construction allows us to reduce to the line bundle case, where the result has already been established.

Suppose first that $j = 1$, meaning we only have one Segre class involved in the intersection number. Then the equality

$$\int_{\mathbb{P}(TX)} \psi_{FS} \left(\sum_{l=0}^{n-2+j} \omega_{\psi,FS}^l \wedge \omega_{FS}^{n-2+j-l} \right) \wedge \sigma^*\beta = \int_X \text{BC}_j(\psi) \wedge \beta$$

that we have established in Equation (4.2) allows us to conclude that the Deligne functional

$$\langle \psi, \dots, \psi, \text{BC}_m(\psi); \theta \rangle$$

can be computed on $\mathbb{P}(T_X)$ as

$$\langle \psi, \dots, \psi, \psi_{FS}, \dots, \psi_{FS}; \theta \rangle_{\mathbb{P}(TX)},$$

where we pull back ω_ψ to $\mathbb{P}(TX)$ to consider it as a form on $\mathbb{P}(TX)$.

In the case that multiple Bott–Chern forms are involved, we simply iterate this construction as follows. After following this procedure once, we have only $j - 1$ Segre classes remaining on $\mathbb{P}(TX)$. But we can pull back T_X through $\sigma : \mathbb{P}(TX) \rightarrow X$, and in this way by functoriality the Segre forms computed with respect to the metric induced by $\sigma^*\omega$ are the pullback of the Segre form computed on X . Thus, applying the same procedure, we reduce to only $j - 2$ higher Segre classes, and repeating we eventually reduce to the line bundle case. What remains is to compute the asymptotic slope of the Deligne functional along the path of metrics induced by the test configuration.

Projectivising $T_{\mathcal{X}/\mathbb{C}}$, we obtain a family $\mathbb{P}(T_{\mathcal{X}/\mathbb{C}}) \rightarrow \mathbb{C}$ which admits a \mathbb{C}^* -action, and is essentially a smooth test configuration for $\mathbb{P}(T_X)$ without a choice of line bundle. The relatively Kähler metric $\omega_{\mathcal{X}}$ produces a Hermitian metric on $T_{\mathcal{X}/\mathbb{C}}$ and, assuming there is only one Segre class $s_m(X)$ involved in the intersection number, we obtain that the limit derivative of the Deligne functional is

$$\int_{\mathbb{P}(T_{\mathcal{X}/\mathbb{P}^1})} \mathcal{L}^{l+1} \cdot \mathcal{O}(1)^{m+n-1} \cdot \Theta = \int_{\mathcal{X}} \mathcal{L}^{l+1} \cdot s_m(T_{\mathcal{X}/\mathbb{P}^1}) \cdot \Theta.$$

Iterating this procedure by pulling back the relative tangent bundle to $\mathbb{P}(T_{\mathcal{X}/\mathbb{C}})$ produces the slope formula in general. The computation of the slope as an integral over \mathcal{X}_0 is completely analogous. \square

4.3. Final steps

We now assume that Z is an admissible central charge, in the sense of Section 3, which means that $\text{Re}(\rho_{n-1}) < 0, \text{Re}(\rho_{n-2}) > 0, \text{Re}(\rho_{n-3}) = 0$ and $\theta_1 = \theta_2 = \theta_3 = 0$. These mean that the new terms in the Segre class enter at order ε^4 , meaning the structure of the equation at lower order is the same as in the ‘rank one’ case.

We finally explain how to prove our main result in the higher rank case:

Theorem 4.3. *Let Z be an analytic admissible central charge. Suppose (X, L) has discrete automorphism group and is analytically K -semistable. If it is in addition asymptotically Z -stable, then it admits Z_ε -critical Kähler metrics for all ε sufficiently small.*

The proof is, from here, very similar to the ‘rank one’ case. The moment map interpretation is exactly as in the ‘rank one’ case. Indeed, the construction of the sequence of Kähler metrics Ω_ε on B is identical to Equation (3.11), as it does not use anything concerning the structure of the equation. Then the moment map property proven there does not actually use that the Euler–Lagrange equation is actually a partial differential equation, but rather just uses formal properties. Thus, we see obtain an analogous moment map interpretation.

The application of the implicit function theorem is much the same. By analyticity of the central charge, the same reasoning as Section 3.4 demonstrates that the linearisation is an isomorphism, and the quantitative inverse function theorem allows us to construct a potential Ψ such that the Z_ε -critical operator lies in $\mathfrak{k}_{k,B}^2$, where we use the same notation as Section 3.4. Here, we use that analyticity of the central charge implies the equation is an *elliptic* partial differential equation.

The solution to the finite-dimensional problem applies, as it is a general result in Kähler geometry, and the local converse is, again, identical in the higher rank case.

Acknowledgements. I thank Frances Kirwan, John McCarthy, Jacopo Stoppa, Gábor Székelyhidi and especially Lars Sektnan for several interesting discussions on this circle of ideas, Michael Hallam, Yoshi Hashimoto and Eiji Inoue for technical advice.

Funding statement. I was funded by a Royal Society University Research Fellowship (URF\R1\201041) for the duration of this work.

Competing interests. The authors have no competing interest to declare.

Data availability statement. Not applicable.

Ethical standards. The research meets all ethical guidelines, including adherence to the legal requirements of the study country.

References

- [1] D. Arcara and A. Bertram, ‘Bridgeland-stable moduli spaces for K -trivial surfaces. With an appendix by Max Lieblich’, *J. Eur. Math. Soc. (JEMS)* **15**(1) (2013), 1–38.
- [2] V. Alexeev, ‘Moduli spaces $M_{g,n}(W)$ for surfaces’, in *Higher-Dimensional Complex Varieties (Trento, 1994)* (de Gruyter, Berlin, 1996), 1–22.
- [3] A. Bayer, ‘Polynomial Bridgeland stability conditions and the large volume limit’, *Geom. Topol.* **13**(4) (2009), 2389–2425.

- [4] A. Bayer, ‘Wall-crossing implies Brill–Noether: Applications of stability conditions on surfaces’, in *Algebraic Geometry: Salt Lake City 2015*, Proc. Sympos. Pure Math., 97.1 (Amer. Math. Soc., Providence, RI, 2018), 3–27.
- [5] T. Bridgeland, ‘Stability conditions on triangulated categories’, *Ann. of Math. (2)* **166**(2) (2007), 317–345.
- [6] T. Brönnle, ‘Deformation constructions of extremal metrics’, Ph.D. thesis (2009).
- [7] T. Brönnle, ‘Extremal Kähler metrics on projectivized vector bundles’, *Duke Math. J.* **164**(2) (2015), 195–233.
- [8] G. Chen, ‘The J-equation and the supercritical deformed Hermitian–Yang–Mills equation’. To appear in *Invent. Math.*
- [9] X. Chen and S. Sun, ‘Calabi flow, geodesic rays, and uniqueness of constant scalar curvature Kähler metrics’, *Ann. of Math. (2)* **180**(2) (2014), 407–454.
- [10] X. X. Chen and G. Tian, ‘Ricci flow on Kähler–Einstein surfaces’, *Invent. Math.* **147**(3) (2002), 487–544.
- [11] T. C. Collins, A. Jacob and S. T. Yau, ‘(1,1) forms with specified Lagrangian phase: A priori estimates and algebraic obstructions’, *Camb. J. Math.* **8**(2) (2020), 407–452.
- [12] T. C. Collins and S. T. Yau, ‘Moment maps, nonlinear PDE, and stability in mirror symmetry’, Preprint, 2018, [arXiv:1811.04824](https://arxiv.org/abs/1811.04824).
- [13] R. Dervan, ‘Uniform stability of twisted constant scalar curvature Kähler metrics’, *Int. Math. Res. Not. IMRN* (15) (2016), 4728–4783.
- [14] R. Dervan, ‘Stability conditions in geometric invariant theory’, Preprint, 2022, [arXiv:2207.04766](https://arxiv.org/abs/2207.04766).
- [15] R. Dervan, J. B. McCarthy and L. M. Sektnan, ‘Z-critical connections and Bridgeland stability conditions’, Preprint, 2020, [arXiv:2012.10426](https://arxiv.org/abs/2012.10426).
- [16] R. Dervan and P. Naumann, ‘Moduli of polarised manifolds via canonical Kähler metrics’, Preprint, 2018, [arXiv:1810.02576](https://arxiv.org/abs/1810.02576).
- [17] R. Dervan and J. Ross, ‘K-stability for Kähler manifolds’, *Math. Res. Lett.* **24**(3) (2017), 689–739.
- [18] R. Dervan and J. Ross, ‘Stable maps in higher dimensions’, *Math. Ann.* **374**(3–4) (2019), 1033–1073.
- [19] R. Dervan and L. M. Sektnan, ‘Extremal metrics on fibrations’, *Proc. Lond. Math. Soc. (3)* **120**(4) (2020), 587–616.
- [20] R. Dervan and L. M. Sektnan, ‘Uniqueness of optimal symplectic connections’. To appear in *Forum Math. Sigma*.
- [21] S. Diverio, ‘Segre forms and Kobayashi–Lübke inequality’, *Math. Z.* **283**(3–4) (2016), 1033–1047.
- [22] S. K. Donaldson, ‘Anti self-dual Yang–Mills connections over complex algebraic surfaces and stable vector bundles’, *Proc. London Math. Soc. (3)* **50**(1) (1985), 1–26.
- [23] S. K. Donaldson, ‘Remarks on Gauge theory, complex geometry and 4-manifold topology’, in *Fields Medallists’ Lectures, World Sci. Ser. 20th Century Math. 5* (World Sci. Publ., River Edge, NJ, 1997), 384–403.
- [24] S. K. Donaldson, ‘Scalar curvature and stability of toric varieties’, *J. Differential Geom.* **62**(2) (2002), 289–349.
- [25] S. K. Donaldson, ‘Lower bounds on the Calabi functional’, *J. Differential Geom.* **70**(3) (2005), 453–472.
- [26] S. Dwivedi, J. Herman, L. Jeffrey and T. van den Hurk, *Hamiltonian Group Actions and Equivariant Cohomology*, Springer Briefs in Mathematics (Springer, Cham, 2019), xi+132 pp.
- [27] R. Elkik, ‘Métriques sur les fibrés d’intersection’, *Duke Math. J.* **61**(1) (1990), 303–328.
- [28] J. Fine, ‘Constant scalar curvature Kähler metrics on fibred complex surfaces’, *J. Differential Geom.* **68**(3) (2004), 397–432.
- [29] W. Fischer and H. Grauert, ‘Lokal-triviale Familien kompakter komplexer Mannigfaltigkeiten’, *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II* 1965 (1965), 89–94.
- [30] A. Fujiki, ‘Moduli space of polarized algebraic manifolds and Kähler metrics’, *Sugaku Expositions* **5**(2) (1992) 173–191.
- [31] A. Fujiki and G. Schumacher, ‘The moduli space of extremal compact Kähler manifolds and generalized Weil–Peterson metrics’, *Publ. Res. Inst. Math. Sci.* **26**(1) (1990), 101–183.
- [32] A. Futaki, ‘Harmonic total Chern forms and stability’, *Kodai Math. J.* **29**(3) (2006), 346–369.
- [33] P. Gauduchon, *Calabi’s Extremal Kähler Metrics: An elementary Introduction*. Book available online.
- [34] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order, Classics in Mathematics* (Springer-Verlag, Berlin, 2001), xiv+517 pp.
- [35] M. Hallam, ‘Geodesics in the space of relatively Kähler metrics’, *J. Lond. Math. Soc.* **108**(3) (2023), 1036–1081.
- [36] Y. Hashimoto, ‘Existence of twisted constant scalar curvature Kähler metrics with a large twist’, *Math. Z.* **292**(3–4) (2019), 791–803.
- [37] B. Hassett and D. Hyeon, ‘Log minimal model program for the moduli space of stable curves: The first flip’, *Ann. of Math. (2)* **177**(3) (2013), 911–968.
- [38] D. Guler, ‘On Segre forms of positive vector bundles’, *Canad. Math. Bull.* **55**(1) (2012), 108–113.
- [39] E. Inoue, ‘The moduli space of Fano manifolds with Kähler–Ricci solitons’, *Adv. Math.* (357) (2019), 1–65.
- [40] A. Jacob and S. T. Yau, ‘A special Lagrangian type equation for holomorphic line bundles’, *Math. Ann.* **369**(1–2) (2017), 869–898.
- [41] C. LeBrun and S. Simanca, ‘Extremal Kähler metrics and complex deformation theory’, *Geom. Funct. Anal.* **4**(3) (1994), 298–336.
- [42] M. Lejmi and G. Székelyhidi, ‘The J-flow and stability’, *Adv. Math.* **274** (2015), 404–431.
- [43] N. C. Leung, ‘Einstein type metrics and stability on vector bundles’, *J. Differential Geom.* **45**(3) (1997), 514–546.
- [44] N. C. Leung, ‘Bando Futaki invariants and Kähler–Einstein metric’, *Comm. Anal. Geom.* **6**(4) (1998), 799–808.
- [45] N. C. Leung, S. T. Yau and E. Zaslow, ‘From special Lagrangian to Hermitian–Yang–Mills via Fourier–Mukai transform’, *Adv. Theor. Math. Phys.* **4**(6) (2000), 1319–1341.
- [46] C. Li, ‘Geodesic rays and stability in the cscK problem’, To appear in *Ann. Sci. Éc. Norm. Supér.*

- [47] C. Li, X. Wang and C. Xu, ‘On the proper moduli spaces of smoothable Kähler–Einstein Fano varieties’, *Duke Math. J.* **168**(8) (2019), 1387–1459.
- [48] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, Oxf. Grad. *Texts Math.* (Oxford University Press, Oxford, 2017), xi+623 pp.
- [49] P. E. Newstead, *Introduction to Moduli Problems and Orbit Spaces*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, 51 (Narosa Publishing House, New Delhi, 1978), vi+183 pp.
- [50] Y. Odaka, ‘A generalization of the Ross–Thomas slope theory’, *Osaka J. Math.* **50**(1) (2013), 171–185.
- [51] Y. Odaka, ‘Compact moduli spaces of Kähler–Einstein Fano varieties’, *Publ. Res. Inst. Math. Sci.* **51**(3) (2015), 549–565.
- [52] V. P. Pingali, ‘A note on higher extremal metrics’, *Trans. Amer. Math. Soc.* **370**(10) (2018), 6995–7010.
- [53] D. Piyaratne and Y. Toda, ‘Moduli of Bridgeland semistable objects on 3-folds and Donaldson–Thomas invariants’, *J. Reine Angew. Math.* **747** (2019), 175–219.
- [54] C. Scarpa, ‘The Hitchin–cscK system’, Ph.D. thesis (2020).
- [55] E. Schlitzer and J. Stoppa, ‘Deformed Hermitian Yang–Mills connections, extended gauge group and scalar curvature’. *To appear in J. Lond. Math. Soc.*
- [56] L. M. Sektnan and C. Tipler, ‘Hermitian Yang–Mills connections on pullback bundles’, Preprint, 2020, [arXiv:2006.06453](https://arxiv.org/abs/2006.06453).
- [57] Z. Sjöström Dyrefelt, ‘K-semistability of cscK manifolds with transcendental cohomology class’, *J. Geom. Anal.* **28**(4) (2018), 2927–2960.
- [58] J. Song and B. Weinkove, ‘Energy functionals and canonical Kähler metrics’, *Duke Math. J. (1)* **137**, 159–184.
- [59] G. Székelyhidi, ‘On blowing up extremal Kähler manifolds’, *Duke Math. J.* **161**(8) (2012), 1411–1453.
- [60] G. Székelyhidi, *An Introduction to Extremal Kähler Metrics, Graduate Studies in Mathematics, 152* (American Mathematical Society, Providence, RI, 2014), xvi+192 pp.
- [61] G. Székelyhidi, ‘The Kähler–Ricci flow and K -polystability’, *Amer. J. Math. (4)* **132** (2014), 1077–1090.
- [62] M. Taylor, *Partial Differential Equations I. Basic Theory*, second edn., Applied Mathematical Sciences, 115 (Springer, New York, 2011), xxii+ 654 pp.
- [63] G. Tian, ‘Kähler–Einstein metrics with positive scalar curvature’, *Invent. Math.* **130**(1) (1997), 1–37.
- [64] G. Tian, ‘Bott–Chern forms and geometric stability’, *Discrete Contin. Dynam. Systems* **6**(1) (2000), 211–220.
- [65] Y. Toda, ‘Moduli stacks and invariants of semistable objects on K3 surfaces’, *Adv. Math.* **217**(6) (2008), 2736–2781.
- [66] X. Wang, ‘Height and GIT weight’, *Math. Res. Lett.* **19**(4) (2012), 909–926.
- [67] B. Weinkove, ‘Higher K-energy functionals and higher Futaki invariants’, Preprint, 2002, [arXiv:math/0204271](https://arxiv.org/abs/math/0204271).
- [68] C. Xu, ‘K-stability of Fano varieties: an algebro-geometric approach’, *EMS Surv. Math. Sci.* **8** (1/2) (2021), 265–354.
- [69] S. T. Yau, ‘Open problems in geometry’, in *Differential Geometry: Partial Differential Equations on Manifolds (Los Angeles, CA, 1990)*, Proc. Sympos. Pure Math., vol. 54, 1–28.
- [70] S. Zhang, ‘Heights and reductions of semi-stable varieties’, *Compositio Math.* **104**(1) (1996), 77–105.