Bull. Aust. Math. Soc. **89** (2014), 5–7
doi:10.1017/S0004972713000452

ADDITION TO 'AN UPPER BOUND FOR THE NUMBER OF ODD MULTIPERFECT NUMBERS'

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(Received 16 January 2013; accepted 7 February 2013; first published online 7 June 2013)

Abstract

The main result in the earlier paper (by the first author) is improved as follows. The number of odd multiperfect numbers with at most r distinct prime factors is bounded by $4^{r^2}/2^{r+2}(r-1)!$.

2010 *Mathematics subject classification*: primary 11A25.
Keywords and phrases: odd perfect numbers, k-perfect numbers.

1. Introduction

The terminology and notation in [4] are continued in this note.

In [4], the first author proved that for each positive integer r, the number of odd multiperfect numbers N with $\omega(N) \le r$ is bounded by 4^{r^2} when r is large enough. The purpose here is to use a similar method to that in [4] to obtain the following improved estimate, valid for all r.

THEOREM 1.1. *Let r be a positive integer. The number of odd multiperfect numbers with at most r distinct prime factors is bounded by* $4^{r^2}/2^{r+2}(r-1)!$.

Theorem 1.1 is a corollary of the following result.

THEOREM 1.2. *Let x and r be positive integers. The number of odd multiperfect numbers* $N \le x$ *with at most r distinct prime factors is bounded by* $\binom{\lfloor \log_3 x\rfloor + r-1}{r-1}2^{r-2}$.

2. Proofs

PROOF OF THEOREM 1.2. The proof is essentially a modification of the proof in [4]. Suppose that $N \le x$ is odd k-perfect, $k \ge 2$ and $\omega(N) \le r$. By a result in [1], we have $r \ge k^2 - 1 \ge 3$ and $k < r$. Write $N = AB$, where $A := \prod_{p^e \| N, p>2r} p^e$ and $B := \prod_{p^e \| N, p \le 2r} p^e$. We have

$$\frac{\sigma(A)}{A} = \prod_{p^e \| A}\Big(1 + \frac{1}{p} + \cdots + \frac{1}{p^e}\Big) < \prod_{p|A}\Big(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\Big),$$

Supported by NSF of China (No. 11271142) and the Guangdong Provincial Natural Science Foundation (No. S2012010009942).

and so

$$\frac{A}{\sigma(A)} > \prod_{p|A}\Big(1-\frac{1}{p}\Big) \ge 1 - \sum_{p|A}\frac{1}{p} \ge 1 - \frac{r}{2r+1} > \frac{1}{2}. \tag{2.1}$$

Thus $\sigma(A) < 2A$, which implies that $B > 1$. Since N is k-perfect, $\sigma(AB) = kAB$, and hence

$$\frac{k}{2}B < \frac{A}{\sigma(A)}kB = \sigma(B) \le kB, \tag{2.2}$$

with equality on the right precisely when $A = 1$. Suppose that $A \ne 1$. Then, by (2.2),

$$\sigma(B) > \frac{kB}{2} \quad \text{and} \quad \sigma(B) \mid kAB. \tag{2.3}$$

If $\gcd(A, \sigma(B)) = 1$, then by the second formula of (2.3), $\sigma(B) \mid kB$, and so $\sigma(B) \le kB/2$, which contradicts (2.3). Therefore there is a prime p dividing $\gcd(A, \sigma(B))$, which means that $\sigma(B)$ has a prime factor p with $p > 2r$ and $\gcd(p, B) = 1$ by the definition of A. Let p_1 be the smallest prime divisor of $\sigma(B)$ with $p_1 > 2r$. Then $p_1 \mid A$ since $k < r$. Suppose that $p_1^{e_1} \parallel A$, where $e_1 \ge 1$. Then if we put

$$A' := A/p_1^{e_1} \quad \text{and} \quad B' := Bp_1^{e_1},$$

it is clear that (2.1)–(2.3) hold with A' and B' replacing A and B. By the same argument as in [3], continuing the above procedure, we eventually obtain a factorisation

$$A = p_1^{e_1} p_2^{e_2} \cdot \cdots \cdot p_t^{e_t},$$

where $t = \omega(A) = \omega(N) - \omega(B) \le r - 1$.

We note that the prime p_1 depends only on B, while, for $i > 1$, the prime p_i depends only on B and the exponents $e_1, \ldots, e_{i-1}$, and, moreover, p_t and e_t depend only on B and the exponents $e_1, \ldots, e_{t-1}$. It follows that for a given B the cofactor A (if $A > 1$) is entirely determined by $e_1, \ldots, e_{t-1}$, and we have $e_i \le \log_5 x$ for $i = 1, \ldots, t$.

Let $B = q_1^{f_1} q_2^{f_2} \cdots q_s^{f_s}$. Then $f_j \le \log_3 x$ for $j = 1, \ldots, s$, and $s + t = r$. Let m be the number of odd primes not exceeding $2r$, so $m \le r - 2$. To estimate the number of possibilities for B and $e_1, \ldots, e_{t-1}$, we first choose s odd primes $(1 \le s \le r-2)$ from the first $m \le r - 2$ odd primes. Then choose $f_j \le \log_3 x$ for $j = 1, \ldots, s$ and $e_i \le \log_5 x$ for $i = 1, \ldots, t-1$ with $s + t = r$ and obviously $e_1 + \cdots + e_{t-1} + f_1 + \cdots + f_s \le \log_3 x$. The number of possibilities for $e_1 + \cdots + e_{t-1} + f_1 + \cdots + f_s \le \log_3 x$ is equal to the number of nonnegative integer solutions of the equation

$$e_1 + \cdots + e_{t-1} + f_1 + \cdots + f_s + y = \lfloor \log_3 x \rfloor, \tag{2.4}$$

which is $\binom{\lfloor \log_3 x\rfloor + r - 1}{r-1}$. Therefore the number of possibilities for B and $e_1, \ldots, e_t$ is bounded by

$$\binom{\lfloor \log_3 x\rfloor + r - 1}{r-1} \sum_{i=0}^{r-2} \binom{r-2}{i} \le \binom{\lfloor \log_3 x\rfloor + r - 1}{r-1} 2^{r-2}.$$

This completes the proof of Theorem 1.2. □

https://doi.org/10.1017/S0004972713000452 Published online by Cambridge University Press

Proof of Theorem 1.1. By a result of Nielsen [2], we have $N < 2^{4^r}$. By Theorem 1.2, we may take $x = 2^{4^r}$. Then, since $4^r \log_3 2 + r - 1 \le 4^r$, the number of odd k-perfect numbers N with $\omega(N) \le r$ is bounded by

$$\binom{\lfloor 4^r \log_3 2 \rfloor + r - 1}{r - 1} 2^{r-2} \le \frac{4^{r(r-1)} 2^{r-2}}{(r-1)!} = \frac{4^{r^2}}{2^{r+2}(r-1)!}.$$

This proves the theorem. □

Acknowledgement

The authors are grateful to the editor for his valuable suggestions.

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