ANZIAM J. 44(2002), 283–297

AN $M/G/1$ QUEUEING SYSTEM WITH FIXED FEEDBACK POLICY

BONG DAE CHOI[1] and BARA KIM[2]

(Received 14 September, 1998)

Abstract

We consider a single server queueing system where each customer visits the queue a fixed number of times before departure. A customer on his j th visit to the queue is defined to be a class-j -customer. We obtain the joint probability generating function for the number of class-j -customers and also obtain the Laplace-Stieltjes transform for the total response time of a customer.

1. Introduction

We consider a single server queueing system with fixed feedback policy where each customer visits the queue a fixed number of times m before departure. Customers in the queue, both those that are newly arrived and those that are fed back, are served in the order in which they joined the tail of the queue. The motivation for this work comes from the modelling of signalling system No. 7 [4, 12]. We assume that customer arrivals follow a Poisson process with intensity λ. Service times are independent and identically distributed regardless of the number of visits to the queue. Let X denote the generic random variable representing a service time. We denote the mean of X by $\bar{x}$, and its Laplace-Stieltjes Transform (LST) by $B^*(s)$.

It is easy to see that the system is ergodic if and only if the offered load $\rho \triangleq m\lambda\bar{x}$ is less than 1. To guarantee the stability of the system, we assume that $\rho < 1$. A customer on his j th visit to the queue is defined to be a class-j -customer ($1 \leq j \leq m$).

Queueing systems with various feedback policies have been investigated by many authors. Most feedback queueing systems have the Bernoulli feedback policy. In

[1]Department of Mathematics and Telecommunication Mathematics Research Center, Korea University, 1, Anam-dong, Sungbuk-ku, Seoul, 136-701, Korea; e-mail: bdchoi@semi.korea.ac.kr.
[2]School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta GA 30332-0205, USA; e-mail: bkim@isye.gatech.edu.

queueing systems with this policy, the memoryless property of the number of feedbacks of a customer makes it easy to analyze the system. Fewer results are known for feedback queueing systems in which the feedback policy is not Bernoulli. Baskett *et al.* [2] obtained the product form of the joint queue size distribution for the $M/M/1$ queueing system with several types of customers and general feedback policy. Simon [9] considered an $M/G/1$ priority queueing system with several types of customers and general bounded feedback policy and obtained a system of linear equations for the mean sojourn times for each class of customer type.

Adve and Nelson [1] obtained only the mean total response time for the $M/G/1$ queueing system with fixed feedback policy.

In this paper, we obtain the joint probability generating function (PGF) for the number of class-j-customers ($j = 1, \dots, m$) for the $M/G/1$ queueing system with fixed feedback policy. We also obtain the LST of the total response time of a customer. By differentiating the joint PGF and the LST, we obtain the first and second moments of the number of class-j-customers in the system and the total response time.

2. Stationary distribution of the system size

Let $Q_i(t)$, $1 \le i \le m$, be the number of class-i-customers in the queue at time t and let $\tau_{n,i}$ be the epoch of the beginning of the ith service of the nth arriving customer. Denote by $\pi_i(l_1, \dots, l_m)$ the probability that there are l_j class-j-customers in the queue just after the beginning of the ith service of an arbitrary customer, that is,

$$\pi_i(l_1, \dots, l_m) = \lim_{n\to\infty} P\{Q_1(\tau_{n,i}+) = l_1, \cdots, Q_m(\tau_{n,i}+) = l_m\}.$$

Let $\Pi_i(z_1, \dots, z_m)$ denote the joint PGF of $\pi_i(l_1, \dots, l_m)$, that is,

$$\begin{aligned}\Pi_i(z_1, \dots, z_m) &= \sum_{l_1=0}^{\infty} \cdots \sum_{l_m=0}^{\infty} \pi_i(l_1, \dots, l_m) z_1^{l_1}, \dots, z_m^{l_m} \\ &= \lim_{n\to\infty} E\left[z_1^{Q_1(\tau_{n,i}+)} \cdots z_m^{Q_m(\tau_{n,i}+)}\right].\end{aligned}$$

Let $\tau_{n,i}^{(e)}$ be the epoch of the end of the ith service completion of the nth arriving customer and let $\pi_i^{(e)}(l_1, \dots, l_m)$ be the probability that there are l_j class-j-customers in the queue just before the end of the ith service completion of an arbitrary customer, that is, $\pi_i^{(e)}(l_1, \dots, l_m) = \lim_{n\to\infty} P\{Q_1(\tau_{n,i}^{(e)}-) = l_1, \dots, Q_m(\tau_{n,i}^{(e)}-) = l_m\}$. We denote by $\Pi_i^{(e)}(z_1, \dots, z_m)$ be the joint PGF of $\pi_i^{(e)}(l_1, \dots, l_m)$, that is,

$$\begin{aligned}\Pi_i^{(e)}(z_1, \dots, z_m) &= \sum_{l_1=0}^{\infty} \cdots \sum_{l_m=0}^{\infty} \pi_i^{(e)}(l_1, \dots, l_m) z_1^{l_1}, \dots, z_m^{l_m} \\ &= \lim_{n\to\infty} E\left[z_1^{Q_1(\tau_{n,i}^{(e)}-)} \cdots z_m^{Q_m(\tau_{n,m}^{(e)}-)}\right].\end{aligned}$$

We observe that the PGF of the number of class-1-customers who arrive newly during a service time is $B^*(\lambda - \lambda z_1)$ [11]. Hence, for $i = 1, \dots, m$, $\Pi_i^{(e)}(z_1, \dots, z_m)$ is related to $\Pi_i(z_1, \dots, z_m)$ by

$$\Pi_i^{(e)}(z_1, \dots, z_m) = B^*(\lambda - \lambda z_1)\Pi_i(z_1, \dots, z_m). \tag{1}$$

Next we will find a relation between $\Pi_{i+1}(z_1, \dots, z_m)$ and $\Pi_i(z_1, \dots, z_m)$. For $i = 1, 2, \dots, m-1$, suppose that there are l_j class-j-customers ($j = 1, \dots, m$) in the queue at $\tau_{n,i}^{(e)}-$. The nth arriving customer will begin his $(i+1)$th service after all of the customers in the queue at $\tau_{n,i}^{(e)}-$ receive service. A class-j-customer ($j = 1, 2, \dots, m-1$) becomes a class-$(j+1)$-customer after his service completion and a class-m-customer departs the system permanently after his service completion. Hence the number of class-1-customers in the queue at $\tau_{n,i+1}+$ is the number of new arrivals during the total service times of $l_1 + \cdots + l_m$ customers, and the number of class-j-customers in the queue at $\tau_{n,i+1}+$ is l_{j-1} ($j = 2, 3, \dots, m$). From this observation, given that there are l_j class-j-customers at $\tau_{n,i}^{(e)}-$, ($j = 1, \dots, m$), the joint PGF of the number of class-j-customers in the queue at $\tau_{n,i+1}+$ ($j = 1, 2, \dots, m$) is given by

$$\begin{aligned}
&E\left[z_1^{Q_1(\tau_{n,i+1}+)} \cdots z_m^{Q_m(\tau_{n,i+1}+)} \,\middle|\, Q_1(\tau_{n,i}^{(e)}-) = l_1, \dots, Q_m(\tau_{n,i}^{(e)}-) = l_m\right] \\
&\quad = [B^*(\lambda - \lambda z_1)]^{l_1 + \cdots + l_m} z_2^{l_1} \cdots z_m^{l_{m-1}} \\
&\quad = [z_2 B^*(\lambda - \lambda z_1)]^{l_1} [z_3 B^*(\lambda - \lambda z_1)]^{l_2} \cdots [z_m B^*(\lambda - \lambda z_1)]^{l_{m-1}} [B^*(\lambda - \lambda z_1)]^{l_m}.
\end{aligned}$$

For $i = 1, \dots, m-1$, we have

$$\begin{aligned}
&\Pi_{i+1}(z_1, \cdots, z_m) \\
&\quad = \lim_{n\to\infty} E\left[z_1^{Q_1(\tau_{n,i+1}+)} \cdots z_m^{Q_m(\tau_{n,i+1}+)}\right] \\
&\quad = \lim_{n\to\infty} \sum_{l_1=0}^{\infty} \cdots \sum_{l_m=0}^{\infty} P\left\{Q_1(\tau_{n,i}^{(e)}-) = l_1, \dots, Q_m(\tau_{n,i}^{(e)}-) = l_m\right\} \\
&\qquad \times E\left[z_1^{Q_1(\tau_{n,i+1}+)} \cdots z_m^{Q_m(\tau_{n,i+1}+)} \,\middle|\, Q_1(\tau_{n,i}^{(e)}-) = l_1, \dots, Q_m(\tau_{n,i}^{(e)}-) = l_m\right] \\
&\quad = \sum_{l_1=0}^{\infty} \cdots \sum_{l_m=0}^{\infty} \pi_i^{(e)}(l_1, \dots, l_m) \\
&\qquad \times [z_2 B^*(\lambda - \lambda z_1)]^{l_1} \cdots [z_m B^*(\lambda - \lambda z_1)]^{l_{m-1}} [B^*(\lambda - \lambda z_1)]^{l_m} \\
&\quad = \Pi_i^{(e)}\left(z_2 B^*(\lambda - \lambda z_1), z_3 B^*(\lambda - \lambda z_1), \dots, z_m B^*(\lambda - \lambda z_1), B^*(\lambda - \lambda z_1)\right).
\end{aligned}$$

Substituting (1) into the above equation, we have the following relation between

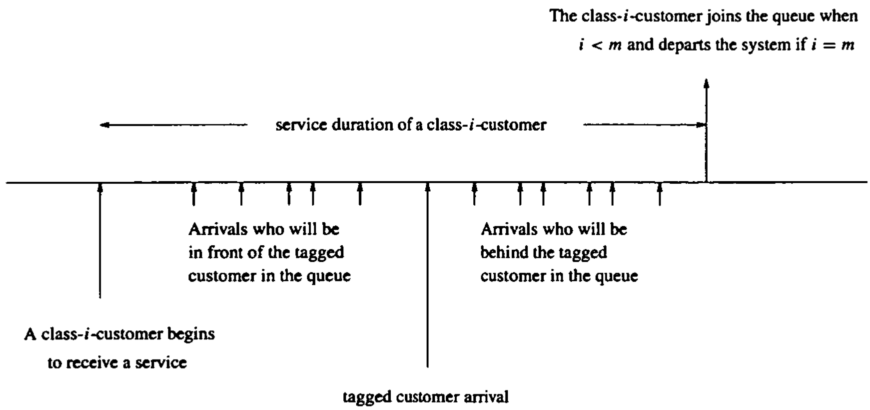

FIGURE 1. Tagged customer who arrives while a class-i-customer is being served

$\Pi_{i+1}(z_1, \dots, z_m)$ and $\Pi_i(z_1, \dots, z_m)$:

$$\begin{aligned}
&\Pi_{i+1}(z_1, \cdots, z_m) \\
&\quad = B^*(\lambda - \lambda z_2 B^*(\lambda - \lambda z_1)) \\
&\qquad \times \Pi_i(z_2 B^*(\lambda - \lambda z_1), z_3 B^*(\lambda - \lambda z_1), \dots, z_m B^*(\lambda - \lambda z_1), B^*(\lambda - \lambda z_1)). \quad (2)
\end{aligned}$$

Next, we will find an equation (see (5) below) relating $\Pi_1(z_1, \dots, z_n)$ to

$$\Pi_i(z_2 B^*(\lambda - \lambda z_1), z_3 B^*(\lambda - \lambda z_1), \dots, z_m B^*(\lambda - \lambda z_1), B^*(\lambda - \lambda z_1)),$$

$i = 1, \dots, m$. Given that a class-i-customer is being served when a tagged customer arrives from outside, let $\psi_i(l_1, \dots, l_m; l'_1, \dots, l'_m)$ be the probability that there are l_j class-j-customers in front of the tagged customer and l'_j class-j-customers behind the tagged customer in the system immediately after the end of the remaining service of the customer who was being served at the arrival epoch of the tagged customer. Immediately after the end of the remaining service, the number of class-j-customers in front of the tagged customer is the number of class-j-customers who were in the queue when the class-i-customer in service at the arrival epoch of the tagged customer started service, for $j = 2, \dots, m$. The number of class-1-customers in front of the tagged customer is the number of class-1-customers who were in the queue when the class-i-customer in service at the arrival epoch of the tagged customer started service plus the number of new arrivals during the elapsed service time. Behind the tagged customer, for $i = 1, \dots, m-1$, there is only one class-$(i+1)$-customer who just finished service and returned to the queue and there are class-1-customers who arrived during the remaining service time. When $i = m$, there are only class-1-customers behind the tagged customer, because the class-m-customer in service departs the system after the end of the remaining service. At an arbitrary time, given that a

https://doi.org/10.1017/S1446181100013948 Published online by Cambridge University Press

customer is being served, the joint PGF of the number of customer arrivals during the elapsed service time and during the remaining service time is [11]

$$\frac{B^*(\lambda - \lambda z_1) - B^*(\lambda - \lambda z_1')}{\lambda \bar{x}(z_1 - z_1')}.$$

Hence, by the above observation and Figure 1, the joint PGF $\Psi_i(z_1, \ldots, z_m; z_1', \ldots, z_m')$ of $\psi_i(l_1, \ldots, l_m; l_1', \ldots, l_m')$ for $i = 1, \ldots, m$ is given by

$$\Psi_i(z_1, \ldots, z_m; z_1', \ldots, z_m') = z_{i+1}' \frac{B^*(\lambda - \lambda z_1) - B^*(\lambda - \lambda z_1')}{\lambda \bar{x}(z_1 - z_1')} \Pi_i(z_1, \ldots, z_m), \quad (3)$$

where $z_{m+1}' \triangleq 1$.

A tagged customer finds the server idle with probability $1 - \rho$ and a class-i-customer ($i = 1, \ldots, m$) being served with probability ρ/m at the arrival epoch from the outside. Suppose that there are l_j class-j-customers in front of the tagged customer and l_j' class-j-customers behind the tagged customer in the system immediately after the end of the remaining service of the customer who was being served at the arrival epoch of the tagged customer. Then the tagged customer begins his first service after $l_1 + \cdots + l_m$ customers are served. At the first visit of the tagged customer to the server, the number of class-j-customers in the queue is $l_{j-1} + l_j'$ for $j = 2, \ldots, m$, and the number of class-1-customers is l_1' plus the number of new arrivals during the $l_1 + \cdots + l_m$ service times. Therefore the joint PGF of the number of each class of customers in the queue at the first visit of the tagged customer to the server is given by

$$\begin{aligned}
&\Pi_1(z_1, \ldots, z_m) \\
&\quad = 1 - \rho + \frac{\rho}{m} \sum_{i=1}^{m} \sum_{l_1=0}^{\infty} \cdots \sum_{l_m=0}^{\infty} \sum_{l_1'=0}^{\infty} \cdots \sum_{l_m'=0}^{\infty} \psi_i(l_1, \ldots, l_m; l_1', \ldots, l_m') \\
&\qquad \times \left([B^*(\lambda - \lambda z_1)]^{l_1 + \cdots + l_m} z_1^{l_1'} \right) z_2^{l_1 + l_2'} \cdots z_m^{l_{m-1} + l_m'} \\
&\quad = 1 - \rho + \frac{\rho}{m} \sum_{i=1}^{m} \sum_{l_1=0}^{\infty} \cdots \sum_{l_m=0}^{\infty} \sum_{l_1'=0}^{\infty} \cdots \sum_{l_m'=0}^{\infty} \psi_i(l_1, \ldots, l_m; l_1', \ldots, l_m') \\
&\qquad \times [z_2 B^*(\lambda - \lambda z_1)]^{l_1} \cdots [z_m B^*(\lambda - \lambda z_1)]^{l_{m-1}} [B^*(\lambda - \lambda z_1)]^{l_m} z_1^{l_1'} \cdots z_m^{l_m'} \\
&\quad = 1 - \rho \\
&\qquad + \frac{\rho}{m} \sum_{i=1}^{m} \Psi_i\big(z_2 B^*(\lambda - \lambda z_1), \ldots, z_m B^*(\lambda - \lambda z_1), B^*(\lambda - \lambda z_1), z_1, \ldots, z_m\big). \quad (4)
\end{aligned}$$

https://doi.org/10.1017/S1446181100013948 Published online by Cambridge University Press

Substituting (3) into the above equation, we get

$$\begin{aligned}
&\Pi_1(z_1, \dots, z_m) \\
&\quad = 1 - \rho + \frac{\rho}{m} \sum_{i=1}^{m} z_{i+1} \frac{B^*(\lambda - \lambda z_2 B^*(\lambda - \lambda z_1)) - B^*(\lambda - \lambda z_1)}{\lambda \bar{x}(z_2 B^*(\lambda - \lambda z_1) - z_1)} \\
&\qquad \times \Pi_i\big(z_2 B^*(\lambda - \lambda z_1), z_3 B^*(\lambda - \lambda z_1), \dots, z_m B^*(\lambda - \lambda z_1), B^*(\lambda - \lambda z_1)\big), \qquad (5)
\end{aligned}$$

where $z_{m+1} \triangleq 1$.

So far we have obtained (2) and (5) for $\Pi_i(z_1, \dots, z_m)$, $i = 1, \dots, m$. We are going to solve these equations explicitly. For the sake of simplicity, let $z = (z_1, \dots, z_m)$,

$$\begin{aligned}
f(z) &= (f_1(z), \dots, f_m(z)) \\
&= (z_2 B^*(\lambda - \lambda z_1), \dots, z_m B^*(\lambda - \lambda z_1), B^*(\lambda - \lambda z_1)), \qquad (6) \\
g(z) &= B^*(\lambda - \lambda z_2 B^*(\lambda - \lambda z_1)), \\
h(z) &= \frac{B^*(\lambda - \lambda z_2 B^*(\lambda - \lambda z_1)) - B^*(\lambda - \lambda z_1)}{\lambda \bar{x}(z_2 B^*(\lambda - \lambda z_1) - z_1)}.
\end{aligned}$$

Then (5) and (2) become

$$\Pi_1(z) = 1 - \rho + \frac{\rho}{m} \sum_{i=1}^{m} z_{i+1} h(z) \Pi_i(f(z)), \qquad (7)$$

$$\Pi_{i+1}(z) = g(z) \Pi_i(f(z)), \quad \text{for } i = 1, 2, \dots, m-1. \qquad (8)$$

In matrix form, (7) and (8) become

$$\Pi(z) = A(z) \Pi(f(z)) + (1 - \rho) e_1, \qquad (9)$$

where $\Pi(z) = (\Pi_1(z), \dots, \Pi_m(z))^T$, the superscript T denoting transposition,

$$A(z) = \begin{bmatrix}
\frac{\rho}{m} z_2 h(z) & \frac{\rho}{m} z_3 h(z) & \cdots & \frac{\rho}{m} z_{m-1} h(z) & \frac{\rho}{m} z_m h(z) & \frac{\rho}{m} h(z) \\
g(z) & 0 & \cdots & 0 & 0 & 0 \\
0 & g(z) & & 0 & 0 & 0 \\
\vdots & & \ddots & & \vdots & \vdots \\
0 & 0 & & g(z) & 0 & 0 \\
0 & 0 & \cdots & 0 & g(z) & 0
\end{bmatrix} \qquad (10)$$

and e_j is the m-dimensional column vector with all zeros except for the jth element 1. Iterating (9) u times gives

$$\Pi(z) = \left[\prod_{n=0}^{u} A\left(f^{(n)}(z)\right)\right] \Pi\left(f^{(u+1)}(z)\right) + (1 - \rho) \sum_{n=0}^{u} \left[\prod_{k=0}^{n-1} A\left(f^{(k)}(z)\right)\right] e_1,$$

https://doi.org/10.1017/S1446181100013948 Published online by Cambridge University Press

where $f^{(n)}(\cdot)$ is the n-fold composition function of $f(\cdot)$, $\prod_{n=0}^{u} A\left(f^{(n)}(z)\right)$ denotes $A(z)A(f(z))\cdots A\left(f^{(u)}(z)\right)$ and the empty product means the $m\times m$ identity matrix I. The absolute value of $\left(\prod_{n=0}^{u} A(f^{(n)}(z))\right)_{ij}$ is bounded above by $(A^u)_{ij}$ for all $1 \le i,j \le m$ and z with $|z_1| \le 1, \ldots, |z_m| \le 1$, where $A \triangleq A(\mathbf{1}^T)$, and $\mathbf{1}$ is the m-dimensional column vector with all 1's. Since $A^u \to 0$ as $u \to \infty$, we have $\prod_{n=0}^{u} A(f^{(n)}(z)) \to 0$ as $u \to \infty$, and thus we have obtained $\Pi_i(z)$ explicitly as follows:

$$\Pi(z) = (\Pi_1(z), \ldots, \Pi_m(z))^T = (1-\rho)\sum_{n=0}^{\infty}\left[\prod_{k=0}^{n-1} A\left(f^{(k)}(z)\right)\right] e_1. \tag{11}$$

Now we are ready to find the joint PGF of the number of class-j-customers in the system. Let N_j, $1 \le j \le m$, be the number of class-j-customers in the system at steady state, including the customer in service if a class-j-customer is being served. Let $P(z)$ be the joint PGF of the number of class-j-customers in the system at steady state, that is, $P(z) = E\left[z_1^{N_1}\cdots z_m^{N_m}\right]$. At an arbitrary time, given that a customer is being served, the PGF of the number of customer arrivals during the elapsed service time is $(1 - B^*(\lambda - \lambda z_1))/\lambda\bar{x}(1 - z_1)$ [11]. Hence we have

$$\begin{aligned} P(z) &= 1 - \rho + \frac{\rho}{m}\frac{1 - B^*(\lambda - \lambda z_1)}{\lambda\bar{x}(1 - z_1)}\left(z_1\Pi_1(z) + \cdots + z_m\Pi_m(z)\right) \\ &= 1 - \rho + \frac{1 - B^*(\lambda - \lambda z_1)}{1 - z_1}\left(z_1\Pi_1(z) + \cdots + z_m\Pi_m(z)\right). \end{aligned}$$

In summary, we have the following theorem.

THEOREM 1. *The joint PGF* $P(z) = E\left[z_1^{N_1}\cdots z_m^{N_m}\right]$ *of the numbers* $N_1, \ldots, N_m$ *of class-1-customers, ..., class-m-customers in the system at steady state is given by*

$$P(z) = 1 - \rho + \frac{1 - B^*(\lambda - \lambda z_1)}{1 - z_1}\left(z_1\Pi_1(z) + \cdots + z_m\Pi_m(z)\right), \tag{12}$$

where $\Pi(z)$ *is given by* (11).

3. Stationary distribution of the total response time

The total response time of a customer is defined as the duration of the time from a customer's arrival until his departure from the system after his mth service completion.

Given that a class-i-customer is being served when a tagged customer arrives from outside, let $\phi_i(t; l_1, \ldots, l_m; l'_1, \ldots, l'_m)$ be the probability that there are l_j class-j-customers in front of the tagged customer and l'_j customers behind the tagged customer in the system after the end of the remaining service of the customer who was being served at the arrival epoch of the tagged customer, and the remaining service time is less than or equal to t. Define $\Phi_i^*(s; z_1, \ldots, z_m; z_1', \ldots, z_m')$ as

$$\Phi_i^*(s; z_1, \ldots, z_m; z_1', \ldots, z_m') = \int_0^\infty \sum_{l_1=0}^\infty \cdots \sum_{l_m=0}^\infty \sum_{l_1'=0}^\infty \cdots \sum_{l_m'=0}^\infty \phi_i(dt; l_1, \ldots, l_m; l_1', \ldots, l_m') e^{-st} z_1^{l_1} \cdots z_m^{l_m} {z_1'}^{l_1'} \cdots {z_m'}^{l_m'}.$$

At an arbitrary time, given that a customer is being served, the joint transform of the remaining service time, the number of customer arrivals during the elapsed service time and during the remaining service time is needed. To do this, let X^+ be the remaining service time and A^- (A^+) be the number of customer arrivals during the elapsed (remaining) service time, respectively. Then we have from [11] that

$$E\left[e^{-sX^+} z_1^{A^-} {z_1'}^{A^+} \,\middle|\, \text{busy} \right] = \frac{B^*(\lambda - \lambda z_1) - B^*(s + \lambda - \lambda z_1')}{\bar{x}(\lambda z_1 + s - \lambda z_1')}.$$

Therefore, by a derivation similar to that of (3), $\Phi_i^*(s; z_1, \ldots, z_m; z_1', \ldots, z_m')$ is given by

$$\Phi_i^*(s; z_1, \ldots, z_m; z_1', \ldots, z_m') = \frac{B^*(\lambda - \lambda z_1) - B^*(s + \lambda - \lambda z_1')}{\bar{x}(\lambda z_1 + s - \lambda z_1')} z_{i+1}' \Pi_i(z_1, \ldots, z_m), \quad i = 1, \ldots, m. \tag{13}$$

Let $S_{n,i}$ be the sojourn time of the nth arriving customer from his arrival until the beginning of his ith service. For $i = 1, \ldots, m$, let $W_i^*(s; z_1, \ldots, z_m)$ be the joint transform of the sojourn time of a tagged customer until the beginning of his ith service and the number of class-j-customers ($j = 1, \ldots, m$) in the queue just after the beginning of the ith service of the tagged customer, that is,

$$W_i^*(s; z_1, \ldots, z_m) = \lim_{n\to\infty} E\left[e^{-sS_{n,i}} z_1^{Q_1(\tau_{n,i}+)}, \ldots, z_m^{Q_m(\tau_{n,i}+)} \right].$$

A customer finds the server idle with probability $1 - \rho$ and a class-i-customer being served with probability ρ/m at the arrival epoch from the outside. The joint transform of a service time and the number of customer arrivals during that service time is given by $B^*(s + \lambda - \lambda z_1)$. A class-$j$-customer becomes a class-$(j+1)$-customer after his service completion for $j = 1, \ldots, m-1$, and a class-m-customer departs the system permanently. Therefore, by a derivation similar to that of (4), we have, using the notation $\hat{s} = s + \lambda - \lambda z_1$,

$$W_1^*(s; z_1, \ldots, z_m) = (1 - \rho) + \frac{\rho}{m} \sum_{i=1}^m \Phi_i^*(s; z_2 B^*(\hat{s}), \ldots, z_m B^*(\hat{s}), B^*(\hat{s}); z_1, \ldots, z_m).$$

https://doi.org/10.1017/S1446181100013948 Published online by Cambridge University Press

By substituting (13) into the above equation, we have

$$W_1^*(s; z_1, \dots, z_m) = (1-\rho) + \frac{\rho}{m} \frac{B^*(\lambda - \lambda z_2 B^*(\lambda - \lambda z_1)) - B^*(\hat{s})}{\bar{x}(\lambda z_2 B^*(\lambda - \lambda z_1) + s - \lambda z_1)} \times \sum_{i=1}^{m} z_{i+1} \Pi_i\big(z_2 B^*(\hat{s}), \dots, z_m B^*(\hat{s}), B^*(\hat{s})\big). \qquad (14)$$

For $i = 1, \dots, m-1$, by a derivation similar to that of (2), $W_{i+1}^*(s; z_1, \dots, z_m)$ is obtained from $W_i^*(s; z_1, \dots, z_m)$ by

$$W_{i+1}^*(s; z_1, \dots, z_m) = B^*\big(s + \lambda - \lambda z_2 B^*(\hat{s})\big) W_i^*\big(s; z_2 B^*(\hat{s}), \dots, z_m B^*(\hat{s}), B^*(\hat{s})\big). \qquad (15)$$

Let T be the generic total response time of a customer. The total response time of a customer is the sojourn time from his arrival until the beginning of his mth service plus his last service time. Therefore the LST $T^*(s) \triangleq E\big[e^{-sT}\big]$ of a total response time T is given by $T^*(s) = B^*(s)\, W_m^*(s; 1, \dots, 1)$.

In summary, we have the following theorem.

THEOREM 2. *The LST $T^*(s)$ of the total response time T is given by*

$$T^*(s) = B^*(s)\, W_m^*(s; 1, \dots, 1), \qquad (16)$$

where $W_m^*(s; z_1, \dots, z_m)$ is obtained by applying (14) *and* (15) *iteratively.*

4. Moments of the system size and total response time

4.1. Mean system size and mean total response time The mean number of class-j-customers at steady state is obtained by differentiating the joint PGF $P(z)$ and evaluating at $z = \mathbf{1}^T$, that is,

$$E[N_j] = \frac{\partial}{\partial z_j} P(z)\bigg|_{z=\mathbf{1}^T}. \qquad (17)$$

By differentiating (12) with respect to z_j and evaluating at $z = \mathbf{1}^T$, we have

$$\frac{\partial}{\partial z_j} P(z)\bigg|_{z=\mathbf{1}^T} = \rho \frac{\lambda \bar{x^2}}{2\bar{x}} \delta_{j1} + \frac{\rho}{m} + \frac{\rho}{m} \mathbf{1}^T \frac{\partial}{\partial z_j} \Pi(z)\bigg|_{z=\mathbf{1}^T}, \qquad (18)$$

where $\bar{x^2} \triangleq E[X^2]$. By differentiating (11) with respect to z_j, evaluating at $z = \mathbf{1}^T$ and premultiplying by $\mathbf{1}^T$ on both sides, we have

$$\mathbf{1}^T \frac{\partial}{\partial z_j} \Pi(z)\bigg|_{z=\mathbf{1}^T} = (1-\rho) \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} \mathbf{1}^T A^k \left[\nabla_{F^k e_j} A(z)\big|_{z=\mathbf{1}^T}\right] A^{n-k-1} e_1,$$

where F is the $m \times m$ matrix whose (i,j) entry F_{ij} is $\frac{\partial}{\partial z_j} f_i(z)\big|_{z=\mathbf{1}^T}$, $f_i(z)$ is given by (6), and for any m-dimensional column vector q, $\nabla_q A(z)$ is the $m \times m$ matrix whose (i,j) entry is $\sum_{k=1}^m \frac{\partial}{\partial z_k} A_{ij}(z) q_k$. By interchanging summations, the above equation becomes

$$\mathbf{1}^T \frac{\partial}{\partial z_j} \Pi(z)\bigg|_{z=\mathbf{1}^T} = (1-\rho)\sum_{k=0}^{\infty} \mathbf{1}^T A^k \left[\nabla_{F^k e_j} A(z)\big|_{z=\mathbf{1}^T}\right](I-A)^{-1} e_1. \tag{19}$$

On the other hand, let J be the $m \times m$ matrix whose (i,j) entry is 1 if $i = j+1$ and 0 otherwise. Then $A(z)$ can be expressed by

$$A(z) = \frac{\rho}{m} h(z) e_1 (z_2, \dots, z_m, 1) + g(z) J.$$

Therefore $\nabla_q A(z)|_{z=\mathbf{1}^T}$ is given by

$$\begin{aligned}\nabla_q A(z)\big|_{z=\mathbf{1}^T} = {} & \frac{\rho}{m}\frac{\lambda \bar{x^2}}{2\bar{x}}\left(\left(1+\frac{\rho}{m}\right) q^T e_1 + q^T e_2\right) e_1 \mathbf{1}^T \\ & + \frac{\rho}{m} e_1 q^T J + \left(\left(\frac{\rho}{m}\right)^2 q^T e_1 + \left(\frac{\rho}{m}\right) q^T e_2\right) J.\end{aligned}$$

Noting that $F = A^T$ and substituting the above equation into (19), we have

$$\begin{aligned}& \mathbf{1}^T \frac{\partial}{\partial z_j} \Pi(z)\bigg|_{z=\mathbf{1}^T} \\ & = (1-\rho)\sum_{k=0}^{\infty} \mathbf{1}^T A^k \left\{ \frac{\rho}{m}\frac{\lambda \bar{x^2}}{2\bar{x}} \left(\left(1+\frac{\rho}{m}\right)(e_1^T F^k e_j) + (e_2^T F^k e_j)\right) e_1 \mathbf{1}^T \right. \\ & \left. \quad + \frac{\rho}{m} e_1 e_j^T A^k J + \left(\frac{\rho}{m}\right)^2 (e_1^T F^k e_j) J + \frac{\rho}{m}(e_2^T F^k e_j) J \right\} (I-A)^{-1} e_1. \end{aligned} \tag{20}$$

Since $(I-A)\mathbf{1} = (1-\rho)e_1$, we have $(I-A)^{-1}e_1 = (1-\rho)^{-1}\mathbf{1}$. Substituting this into (20), we have

$$\begin{aligned}& \mathbf{1}^T \frac{\partial}{\partial z_j} \Pi(z)\bigg|_{z=\mathbf{1}^T} \\ & = \sum_{k=0}^{\infty} \mathbf{1}^T A^k \left\{ \left(\rho \frac{\lambda \bar{x^2}}{2\bar{x}}\left(1+\frac{\rho}{m}\right) + (1-\rho)\frac{\rho}{m}\left(1-\frac{\rho}{m}\right)\right) e_1 + \left(\frac{\rho}{m}\right)^2 \mathbf{1} \right\} e_1^T F^k e_j \\ & \quad + \sum_{k=0}^{\infty} \mathbf{1}^T A^k \left\{ \left(\rho\frac{\lambda \bar{x^2}}{2\bar{x}} - (1-\rho)\frac{\rho}{m}\right) e_1 + \frac{\rho}{m}\mathbf{1} \right\} e_2^T F^k e_j \\ & \quad + \frac{\rho}{m}\sum_{k=0}^{\infty} \mathbf{1}^T A^k e_1 \mathbf{1}^T F^k e_j\end{aligned}$$

$$= \left\{ \left(\rho \frac{\lambda \overline{x^2}}{2\bar{x}} \left(1 + \frac{\rho}{m}\right) + (1-\rho)\frac{\rho}{m}\left(1 - \frac{\rho}{m}\right) \right) e_1^T + \left(\frac{\rho}{m}\right)^2 \mathbf{1}^T \right\} \sum_{k=0}^{\infty} F^k \mathbf{1} e_1^T F^k e_j$$
$$+ \left\{ \left(\rho \frac{\lambda \overline{x^2}}{2\bar{x}} - (1-\rho)\frac{\rho}{m} \right) e_1 + \frac{\rho}{m}\mathbf{1} \right\} \sum_{k=0}^{\infty} F^k \mathbf{1} e_2^T F^k e_j$$
$$+ \frac{\rho}{m} e_1^T \sum_{k=0}^{\infty} F^k \mathbf{1}\mathbf{1}^T F^k e_j. \tag{21}$$

Since $e_2^T = e_1^T(F - (\rho/m)I)$ and $\mathbf{1}^T = (1-\rho)e_1^T(I-F)^T$, we have

$$\sum_{k=0}^{\infty} F^k \mathbf{1} e_2^T F^k = \left(\sum_{k=0}^{\infty} F^k \mathbf{1} e_1^T F^k \right) \left(F - \frac{\rho}{m} I \right), \tag{22}$$
$$\sum_{k=0}^{\infty} F^k \mathbf{1}\mathbf{1}^T F^k = (1-\rho) \left(\sum_{k=0}^{\infty} F^k \mathbf{1} e_1^T F^k \right) (I-F)^{-1}. \tag{23}$$

Let $M \triangleq \sum_{k=0}^{\infty} F^k \mathbf{1} e_1^T F^k$ (given explicitly in the appendix). Substituting (22) and (23) into (21), we have

$$\mathbf{1}^T \frac{\partial}{\partial z_j} \Pi(z) \bigg|_{z=\mathbf{1}^T} = \rho \frac{\lambda \overline{x^2}}{2\bar{x}} e_1^T M (I+F) e_j + (1-\rho)\frac{\rho}{m} e_1^T M (I-F) e_j$$
$$+ \frac{\rho}{m} \mathbf{1}^T M F e_j + (1-\rho)\frac{\rho}{m} e_1^T M (I-F)^{-1} e_j. \tag{24}$$

By direct calculation, we have

$$\left((I-F)^{-1}\right)_{ij} = \begin{cases} \dfrac{1}{1-\rho}\left(1 - \dfrac{i-1}{m}\rho\right), & \text{if } i \le j; \\[2ex] \dfrac{1}{1-\rho}\left(\rho - \dfrac{i-1}{m}\rho\right), & \text{if } i > j. \end{cases}$$

By substituting the above equation and (33) into (24), we have

$$\mathbf{1}^T \frac{\partial}{\partial z_1} \Pi(z) \bigg|_{z=\mathbf{1}^T} = \frac{\rho}{(1-\rho)(1+\rho/m)} \left(\frac{1+\rho}{2} \frac{\lambda \overline{x^2}}{\bar{x}} + \frac{m-1}{m}\rho \right), \tag{25}$$
$$\mathbf{1}^T \frac{\partial}{\partial z_j} \Pi(z) \bigg|_{z=\mathbf{1}^T} = \frac{\rho}{(1-\rho)(1+\rho/m)} \left(\frac{\lambda \overline{x^2}}{\bar{x}} + 1 - \frac{\rho}{m} \right). \tag{26}$$

By substituting (25), (26) and (18) into (17) with $c_x^2 \triangleq (\mathrm{Var}[X])/(\bar{x})^2$, we have

$$E[N_1] = \frac{\rho}{m} + \frac{\rho}{m} \frac{\rho}{2(1-\rho)(1+\rho/m)} \left\{ \left(1 - \frac{m-2}{m}\rho\right) c_x^2 + 1 + \rho \right\},$$

$$E[N_j] = \frac{\rho}{m} + \frac{\rho}{m}\frac{\rho}{(1-\rho)(1+\rho/m)}\left(\frac{\rho}{m}c_x^2 + 1\right), \quad j = 2, \ldots, m.$$

The system size $N \triangleq N_1 + \cdots + N_m$ has the mean

$$E[N] = \rho + \rho\frac{\rho((1+\rho)c_x^2/m + 2 - (1-\rho)/m)}{2(1-\rho)(1+\rho/m)}. \tag{27}$$

By Little's formula, the mean total response time $E[T]$ is given by

$$E[T] = m\bar{x} + m\bar{x}\frac{\rho((1+\rho)c_x^2/m + 2 - (1-\rho)/m)}{2(1-\rho)(1+\rho/m)}. \tag{28}$$

Note that (28) coincides with the result given by Adve and Nelson [1, (5)], except for a slight difference which is due to a misprint in [1].

4.2. Second moments of the system size and the total response time In this subsection, we restrict ourselves to the case of $m = 2$ to avoid complicated notation. To find the second-order partial derivatives of $P(z_1, z_2)$ at $z = \mathbf{1}^T$, we need to calculate the second-order partial derivatives of $\Pi_1(z_1, z_2)$ and $\Pi_2(z_1, z_2)$. To do this, we use (9) (or equivalently, (7) and (8)). By differentiating (7) and (8) with respect to z_j ($j = 1, 2$) and evaluating at $z_1 = z_2 = 1$, we obtain a system of four equations with four unknowns $\frac{\partial}{\partial z_j}\Pi_i(z)\big|_{z=\mathbf{1}^T}$ $(i, j = 1, 2)$. By solving the system of equations, we have

$$\frac{\partial}{\partial z_1}\Pi_1(z)\bigg|_{z=\mathbf{1}^T} = \frac{\lambda^2\overline{x^2} + \rho^3/4}{(1-\rho)(1+\rho/2)}, \quad \frac{\partial}{\partial z_1}\Pi_2(z)\bigg|_{z=\mathbf{1}^T} = \frac{\rho^2/2 - \rho^3/4 + \rho\lambda^2\overline{x^2}}{(1-\rho)(1+\rho/2)},$$
$$\frac{\partial}{\partial z_2}\Pi_1(z)\bigg|_{z=\mathbf{1}^T} = \frac{\rho/2 - \rho^2/4 + \lambda^2\overline{x^2}}{(1-\rho)(1+\rho/2)}, \quad \frac{\partial}{\partial z_2}\Pi_2(z)\bigg|_{z=\mathbf{1}^T} = \frac{\rho/2 - \rho^2/4 + \lambda^2\overline{x^2}}{(1-\rho)(1+\rho/2)}.$$

By differentiating (7) and (8) twice with respect to z_i and z_j $(i, j = 1, 2)$ and evaluating at $z_1 = z_2 = 1$, we obtain a system of six equations with six unknowns $\frac{\partial^2}{\partial z_1^2}\Pi_i(z)\big|_{z=\mathbf{1}^T}$, $\frac{\partial^2}{\partial z_1 \partial z_2}\Pi_i(z)\big|_{z=\mathbf{1}^T}$, $\frac{\partial^2}{\partial z_2^2}\Pi_i(z)\big|_{z=\mathbf{1}^T}$ $(i = 1, 2)$. By solving the system of equations, we obtain $\frac{\partial^2}{\partial z_1^2}\Pi_i(z)\big|_{z=\mathbf{1}^T}$, $\frac{\partial^2}{\partial z_1 \partial z_2}\Pi_i(z)\big|_{z=\mathbf{1}^T}$, $\frac{\partial^2}{\partial z_2^2}\Pi_i(z)\big|_{z=\mathbf{1}^T}$ $(i = 1, 2)$. By differentiating (12) twice with respect to z_i and z_j $(i, j = 1, 2)$ and evaluating at $z_1 = z_2 = 1$, we obtain $\frac{\partial^2}{\partial z_i \partial z_j}P(z_1, z_2)\big|_{z=\mathbf{1}^T}$ $(i, j = 1, 2)$. From these, $\mathrm{Var}[N]$ is given by

$$\begin{aligned}
\mathrm{Var}[N] = \big\{&\lambda^3\bar{x}^3(2 + 2\rho - \rho^2 - \tfrac{5}{2}\rho^3 - \tfrac{7}{8}\rho^4 + \tfrac{1}{4}\rho^5 + \tfrac{1}{8}\rho^6) \\
&+ \lambda^4(\overline{x^2})^2(6 + \tfrac{3}{2}\rho + 6\rho^2 + \tfrac{3}{2}\rho^3 - \tfrac{3}{2}\rho^4) \\
&+ \lambda^2\overline{x^2}(9 + 6\rho - \tfrac{33}{4}\rho^2 - \tfrac{9}{8}\rho^3 - \tfrac{9}{16}\rho^4 + \tfrac{9}{8}\rho^5 + \tfrac{15}{16}\rho^6 - \tfrac{3}{8}\rho^7) \\
&+ 3\rho - \tfrac{3}{2}\rho^2 - 3\rho^3 + \tfrac{3}{2}\rho^4 + \tfrac{3}{4}\rho^5 + \tfrac{9}{16}\rho^6 - \tfrac{9}{32}\rho^7 - \tfrac{9}{32}\rho^8 + \tfrac{3}{32}\rho^9\big\} \\
&/\big\{3(1+\rho/2)^2(1-\rho)^2(1 + \rho^2/4 - \rho^3/8)\big\},
\end{aligned}$$

where $\bar{x}^3 \triangleq E[X^3]$.

Next we will calculate the variance of the total response time Var $[T]$ of a customer. By substituting $m = 2$ into (14)–(16), the LST $T^*(s)$ of the total response time T is given by

$$\begin{aligned} T^*(s) = {} & (1-\rho)B^*(s)B^*(s+\lambda-\lambda B^*(s)) + \rho/2B^*(s)B^*(s+\lambda-\lambda B^*(s)) \\ & \times \frac{B^*(\lambda-\lambda B^*(s)B^*(\lambda-\lambda B^*(s))) - B^*(s+\lambda-\lambda B^*(s))}{\bar{x}(\lambda B^*(s)B^*(\lambda-\lambda B^*(s)) + s - \lambda B^*(s))} \\ & \times \{B^*(s)\Pi_1(B^*(s)B^*(s+\lambda-\lambda B^*(s)), B^*(s+\lambda-\lambda B^*(s))) \\ & + \Pi_2(B^*(s)B^*(s+\lambda-\lambda B^*(s)), B^*(s+\lambda-\lambda B^*(s)))\}. \end{aligned}$$

By differentiating the above equation twice with respect to s and evaluating at $s = 0$, Var $[T]$ is given by

$$\begin{aligned} \mathrm{Var}[T] = {} & \left\{\lambda\bar{x}^3\left(2+\rho-\tfrac{3}{2}\rho^2-\tfrac{7}{4}\rho^3+\tfrac{1}{4}\rho^5\right)+\lambda^2(\bar{x}^2)^2\left(9+3\rho+\tfrac{33}{4}\rho^2-\tfrac{3}{2}\rho^3-\tfrac{3}{4}\rho^4\right)\right. \\ & \left.+\bar{x}^2\left(6+\tfrac{9}{2}\rho-\tfrac{3}{4}\rho^2-\tfrac{9}{8}\rho^4+\tfrac{3}{8}\rho^5\right)+(\bar{x})^2\left(6-3\rho^2+3\rho^3-\tfrac{15}{8}\rho^4+\tfrac{3}{8}\rho^5\right)\right\} \\ & /\left\{3(1-\rho)^2\left(1+\tfrac{1}{2}\rho+\tfrac{1}{4}\rho^2-\tfrac{1}{16}\rho^4\right)\right\} \\ & -\left\{\bar{x}(2-\rho^2)+\lambda\bar{x}^2(1+\rho)\right\}^2/\left(1-\tfrac{1}{2}\rho-\tfrac{1}{2}\rho^2\right)^2. \end{aligned}$$

Appendix. The calculation of $M \triangleq \sum_{k=0}^{\infty} F^k \mathbf{1} e_1^T F^k$

We have that M satisfies $M = \mathbf{1}e_1^T + FMF$. Postmultiplying by e_j on both sides, we obtain

$$Me_1 = \mathbf{1} + \frac{\rho}{m} F \sum_{k=1}^{m} Me_k, \tag{29}$$

$$Me_{j+1} = FMe_j, \quad , j = 1, \ldots, m-1. \tag{30}$$

Applying (30) recursively, we have

$$Me_j = F^{j-1}Me_1, \quad j = 1, 2, \ldots, m. \tag{31}$$

Substituting (31) into (29), we have

$$Me_1 = \left(I - \frac{\rho}{m}\sum_{k=1}^{m} F^k\right)^{-1} \mathbf{1} = \left(I - F^{m+1}\right)^{-1}\mathbf{1},$$

where we used the Cayley-Hamilton theorem in the last equality.

Since

$$\sum_{k=0}^{m} e^{(2lk\pi/(m+1))\sqrt{-1}} = \begin{cases} m+1, & \text{if } m+1 \text{ divides } l; \\ 0, & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned} Me_1 = \sum_{l=0}^{\infty} F^{(m+1)l}\mathbf{1} &= \frac{1}{m+1}\sum_{l=0}^{\infty}\sum_{k=0}^{m} e^{(2lk\pi/(m+1))\sqrt{-1}} F^l \mathbf{1} \\ &= \frac{1}{m+1}\sum_{k=0}^{m}\left(I - e^{(2k\pi/(m+1))\sqrt{-1}} F\right)^{-1}\mathbf{1}. \end{aligned} \tag{32}$$

By a tedious calculation of cofactors, for $|z| \le 1$, we obtain

$$\left((I - zF)^{-1}\right)_{ij} = \begin{cases} \dfrac{\frac{\rho}{m}(1 + z + \cdots + z^{m-i})z^j}{1 - \frac{\rho}{m}(z + z^2 + \cdots + z^m)}, & \text{if } i > j; \\ \dfrac{\{1 - \frac{\rho}{m}(z + z^2 + \cdots + z^{i-1})\}z^{j-i}}{1 - \frac{\rho}{m}(z + z^2 + \cdots + z^m)}, & \text{if } i \le j. \end{cases}$$

Summing the above equation over j yields

$$\left((I - zF)^{-1}\mathbf{1}\right)_i = \frac{1 + z + \cdots + z^{m-i}}{1 - \frac{\rho}{m}(z + z^2 + \cdots + z^m)}.$$

Substituting the above equation into (32) with $z = e^{(2k\pi/(m+1))\sqrt{-1}}$, we get

$$\begin{aligned} Me_1 &= \frac{1}{(m+1)(1-\rho)}(m, m-1, \dots, 1)^T \\ &\quad + \frac{1}{m+1}\sum_{k=1}^{m}\frac{1}{(1+\rho/m)(1 - e^{\frac{2k\pi}{m+1}\sqrt{-1}})} \\ &\quad \times \left(1 - e^{\frac{2km\pi}{m+1}\sqrt{-1}}, 1 - e^{\frac{2k(m-1)\pi}{m+1}\sqrt{-1}}, \dots, 1 - e^{\frac{2k\pi}{m+1}\sqrt{-1}}\right)^T \\ &= \frac{1}{(m+1)(1-\rho)}(m, m-1, \dots, 1)^T \\ &\quad + \frac{1}{(m+1)(1+\rho/m)}\sum_{k=1}^{m}\left(\sum_{j=0}^{m-1} e^{\frac{2kj\pi}{m+1}\sqrt{-1}}, \sum_{j=0}^{m-2} e^{\frac{2kj\pi}{m+1}\sqrt{-1}}, \dots, 1\right)^T \\ &= \frac{1}{(1-\rho)(1+\rho/m)}\left(1, 1 - \frac{1}{m}\rho, 1 - \frac{2}{m}\rho, \dots, 1 - \frac{m-1}{m}\rho\right)^T. \end{aligned}$$

https://doi.org/10.1017/S1446181100013948 Published online by Cambridge University Press

Substituting the above equation into (31), we obtain $Me_j = (M_{1j}, \ldots, M_{mj})^T$:

$$M_{ij} = \begin{cases} \dfrac{1-(i-1)\rho/m}{(1-\rho)(1+\rho/m)}, & \text{if } i+j \le m+1; \\ \dfrac{(m-i+1)\rho/m}{(1-\rho)(1+\rho/m)}, & \text{if } i+j > m+1. \end{cases} \tag{33}$$

Acknowledgement

This work was supported in part by a research program from KOSEF (grant No. 98-0101-02-01-3).

References

[1] V. S. Adve and R. Nelson, "The relationship between Bernoulli and fixed feedback policies for the $M/G/1$ queue", *Oper. Res.* **42** (1994) 380–385.

[2] F. Baskett, K. M. Chandy, R. R. Muntz and F. G. Palacios, "Open, closed and mixed networks of queues with different classes of customers", *J. ACM* **22** (1975) 248–260.

[3] O. J. Boxma and U. Yechiali, "An $M/G/1$ queue with multiple types of feedback and gated vacations", *J. Appl. Prob.* **34** (1997) 773–784.

[4] B. D. Choi, S. H. Choi, C. G. Park and D. K. Sung, "Analysis of a leaky bucket control scheme in the signaling system no. 7 network", *IEE Proceeding Communications* **145** (1998) 25–32.

[5] B. D. Choi, B. Kim and S. H. Choi, "An $M/G/1$ queue with multiple types of feedback, gated vacations and FCFS policy", *Comput. Oper. Res.* **44** (22) (2002) to appear.

[6] B. D. Choi, B. Kim and S. H. Choi, "On the $M/G/1$ Bernoulli feedback queue with multi-class customers", *Comput. Oper. Res.* **27** (3) (2000) 269–286.

[7] B. D. Choi and V. G. Kulkarni, "Feedback retrial queueing system", in *Queueing and related models*, (Oxford Univ. Press, New York, 1992) 93–105.

[8] K. Rege, "On the $M/G/1$ queue with Bernoulli feedback", *Oper. Res. Lett.* **14** (1993) 163–170.

[9] B. Simon, "Priority queues with feedback", *J. ACM* **31** (1984) 134–149.

[10] L. Takacs, "A single-server queue with feedback", *Bell System Technical J.* **42** (1963) 509–519.

[11] H. Takagi (ed.), *Queueing Analysis: A Foundation of Performance Evaluation, Vol. 1, Vacation and Priority Systems, Part 1* (Elsevier Science Publishers, North-Holland, Amsterdam, 1991).

[12] G. Willmann and P. J. Kühn, "Performance modeling of signaling no. 7", *IEEE Communications Magazine* **28** (1990) 44–56.

[13] M. A. Wortman and R. Disney, "The $M/GI/1$ Bernoulli feedback queue with vacations", *Queueing Systems Theory Appl.* **9** (1991) 353–364.