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Extreme Version of Projectivity for Normed Modules Over Sequence Algebras

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Abstract. We define and study the so-called extreme version of the notion of a projective normed module. The relevant definition takes into account the exact value of the norm of the module in question, in contrast with the standard known definition that is formulated in terms of norm topology.

After the discussion of the case where our normed algebra A is just $\mathbb{C}$, we concentrate on the case of the next degree of complication, where A is a sequence algebra satisfying some natural conditions. The main results give a full characterization of extremely projective objects within the subcategory of the category of non-degenerate normed A-modules, consisting of the so-called homogeneous modules. We consider two cases, 'non-complete' and 'complete', and the respective answers turn out to be essentially different.

In particular, all Banach non-degenerate homogeneous modules consisting of sequences are extremely projective within the category of Banach non-degenerate homogeneous modules. However, neither of them, provided it is infinite-dimensional, is extremely projective within the category of all normed non-degenerate homogeneous modules. On the other hand, submodules of these modules consisting of finite sequences are extremely projective within the latter category.

1 Introduction: Formulation of the Main Results and Comments

The concept of a projective module is one of the most important ones in algebra. In particular, it plays the role of one of the three pillars of the whole building of homological algebra; the other two are the notions of injective module and a flat module. The first functional-analytic versions of the three mentioned notions appeared about forty years ago [5], [6], [7]. They were introduced in connection with the rise of interest, in functional analysis, to such topics as derivations of Banach algebras, radical extensions and amenability. The relevant definitions were given in the framework of a certain kind of relative homology, adapted to the context of functional analysis. They were formulated in terms of the norm topology of modules in question rather than the norm itself.

Quite recently, the birth of new areas of analysis, notably of quantum functional analysis (= operator space theory), has motivated the introduction and study of the so-called extreme versions of these notions (*cf.* [8], [14], [10]). The specific feature of the new versions is that they take into account the exact value of the norm.

Let A be a complex algebra, and X, Y, P left A-modules. When A is fixed, the term 'morphism' will always mean a morphism of A-modules. Let $\tau\colon Y \to X$ be a surjective morphism, and $\varphi\colon P \to X$ an arbitrary morphism. A morphism $\psi\colon P \to Y$

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is called a *lifting of* φ *across* τ, if the diagram

$$\begin{tikzcd} & Y \arrow[d, "\tau"] \\ P \arrow[ur, "\psi"] \arrow[r, "\varphi"] & X \end{tikzcd}$$

is commutative.

Now we suppose that A is a normed algebra, not necessary unital. Denote by **A-mod** the category of all left-normed A-modules and their bounded morphisms. Throughout this paper, all normed algebras and modules are supposed to be contractive; this means that for A and $X \in$ **A-mod** we have the multiplicative inequalities $\|ab\| \le \|a\|\,\|b\|$ and $\|a \cdot x\| \le \|a\|\,\|x\|$ for all $a, b \in A$, $x \in X$.

In what follows, a left A-module X is called *essential* (sometimes called 'non-degenerate') if the closure of the linear span of the set $\{a \cdot x : a \in A, x \in X\}$ is the whole X. A bounded morphism σ of A-modules is called a *near-retraction* if it is contractive (that is $\|\sigma\| \le 1$) and for every $\varepsilon > 0$ it has a right inverse morphism with norm $< 1 + \varepsilon$. A normed A-module X is called *near-retract* of an A-module Y if there exists a near-retraction from Y onto X.

As usual, the category of all normed spaces and bounded operators is denoted by **Nor**, and its full subcategory consisting of Banach spaces by **Ban**.

Recall that an operator, in particular, a module morphism, is called *coisometric* (also 'quotient map'), if it maps the open unit ball of the domain space onto the open unit ball of the range space. Equivalently, $\tau \colon Y \to X$ is coisometric if it is contractive and for every $x \in X$ and $\varepsilon > 0$ there exists $y \in Y$ such that $\tau(y) = x$ and $\|y\| < \|x\| + \varepsilon$.

Finally, our definition takes into account a certain full subcategory in **A-mod**, so far arbitrary chosen. We denote it by $\mathcal{K}$.

Definition 1 A module $P \in$ **A-mod** is called *extremely projective with respect to* $\mathcal{K}$, if for every coisometric morphism $\tau \colon Y \to X$ of modules in $\mathcal{K}$, every bounded A-module morphism $\varphi \colon P \to X$, and every $\varepsilon > 0$ there exists a lifting $\psi \colon P \to Y$ of φ across τ such that $\|\psi\| < \|\varphi\| + \varepsilon$.

In other words, P is extremely projective with respect to $\mathcal{K}$, if, for every τ as above, the operator $\mathbf{h}_A(P,\tau) \colon \mathbf{h}_A(P,Y) \to \mathbf{h}_A(P,X) \colon \psi \mapsto \tau\psi$ is also a coisometry.

Remark In the categorical language, P is extremely projective with respect to $\mathcal{K}$, if the covariant morphism functor $\mathbf{h}_A(P,\,\cdot\,) \colon \mathcal{K} \to$ **Nor** preserves extreme epimorphisms in relevant categories (*cf.* [2]). Hence the word 'extreme'.

Sometimes we shall come across another, somewhat weaker property than extreme projectivity. Namely, a module $P \in$ **A-mod** is called *topologically projective with respect to* $\mathcal{K}$ if for every $\tau \colon Y \to X$ and $\varphi \colon P \to X$ as before, there exists a (just) bounded lifting of φ across τ.

We would like to emphasize that we assume our modules to be Banach (= complete) only if it is explicitly stated. (Indeed, we shall see that some of the results and constructions in this paper deal with essentially non-complete modules).

First, let us look at the simplest case of (just) normed spaces, that is $A := \mathbb{C}$. In this situation, if a normed space P is extremely projective with respect to (the whole) **Nor**, we just say that P is *extremely projective in* **Nor**.

For an arbitrary non-empty set Λ, we denote by $l_1^0(\Lambda)$ the space of all finitely supported functions on Λ, equipped with the l_1-norm. (Thus $l_1^0(\Lambda)$ is a dense subspace in $l_1(\Lambda)$.) We also set $l_1^0(\varnothing) := l_1(\varnothing) := 0$.

The following proposition shows the importance of the spaces $l_1^0(\Lambda)$.

Proposition 0 *A normed space is extremely projective in* **Nor** *if and only if it is a near-retract in* **Nor** *(= $\mathbb{C}$-mod) of $l_1^0(\Lambda)$ for some index set Λ.*

If we are given a Banach space, we say that it is *extremely projective in* **Ban**, if it is extremely projective with respect to **Ban**. These spaces have a much more transparent description (*cf.* [4]):

The Grothendieck Theorem *A Banach space is extremely projective in* **Ban** *if and only if it is isometrically isomorphic to $l_1(\Lambda)$ for some index set Λ.*

(This theorem has a substantial operator space (= 'non-commutative') version; see [1, Thm. 3.14].)

With obvious modifications, one can define *topologically projective spaces in* **Nor** and *topologically projective spaces in* **Ban**.

The discussion of extremely projective spaces in **Nor** and in **Ban**, including the proof of the formulated proposition, will be presented in Section 3. In particular, we shall see that the same space can be extremely projective in **Ban** but not extremely, or even topologically, projective in **Nor** (Proposition 3.5).

From just spaces let us turn again to modules.

The interest in extreme versions of basic homological notions was stimulated by the fundamental Arveson–Wittstock Theorem in quantum functional analysis. First, extremely flat and extremely injective modules appeared (in [8] and [14]), however only for the special case of some highly non-commutative operator algebras and for the class of the so-called semi-Ruan modules in the capacity of the distinguished category $\mathcal{K}$. The results of the above cited papers led to some extension theorems of Hahn–Banach type for modules and to a conceptually new proof of the Arveson–Wittstock Extension Theorem in its non-coordinate presentation. This was done in [8] for the initial case of that theorem, dealing with operator spaces, and in [14] for the more sophisticated case of operator modules.

In the present paper we concentrate on projectivity and consider another class of "popular" algebras, which is opposite, in a sense, to those in [8], [14]. We mean commutative normed algebras, in fact algebras consisting of sequences (*cf.* [10]). These algebras apparently represent the next degree of complication after $\mathbb{C}$. Nevertheless we will show that even in this case, after the proper choice of $\mathcal{K}$, there is something to be said.

Denote by $\mathbf{p}^n$ the sequence $(0, \dots, 0, 1, 0, 0 \dots)$ with 1 as its n-th term, and by c_{00} the linear space of finite sequences, that is $\operatorname{span}\{\mathbf{p}^n; n = 1, 2, \dots\}$.

Definition 2 Let A be a normed algebra, consisting of some complex-valued sequences and equipped with the coordinate-wise operations. We say that A is a *sequence algebra*, if it contains, as a dense subalgebra, c_{00}, and $\|\mathbf{p}^n\| = 1; n \in \mathbb{N}$.

We see that the class of sequence algebras includes c_0, all $l_p; 1 \leq p < \infty$ (but not l_∞), Fourier algebras of discrete countable groups (after rearranging the respective domains as sequences), and many other algebras.

The main results of this paper are Theorems 1.1 and 1.2 below. They give, within a certain substantial class of normed modules over a sequence algebra, a full characterization of extremely projective modules with respect to that class. Theorem 1.1 essentially deals with, generally speaking, non-complete modules. (The interest in the 'non-Banach' case is derived from the papers [8], [14] on the extreme flatness.) However, from this theorem its Banach counterpart easily follows. This is Theorem 1.2, which in one important point sounds essentially different.

We proceed to define, after some preparatory observations, the distinguished class of modules playing the role of $\mathcal{K}$ (see above).

Let A be a sequence algebra and X a normed A-module. Often, when there is no danger of confusion, for $x \in X$ we shall write x_n instead of $\mathbf{p}^n \cdot x$ and call the latter the *n-th coordinate* of x. Of course, we have $\mathbf{p}^n \cdot x_n = x_n$. Further, we set $X_n := \{\mathbf{p}^n \cdot x; x \in X\}$ for every $n \in \mathbb{N}$. We see that X_n is a subspace of X and, moreover, a submodule with the outer multiplication

$$a \cdot x = a_n x; \quad a = (a_1, \dots, a_n, \dots) \in A, x \in X_n. \tag{1.1}$$

It will be called the *n-th coordinate subspace (or submodule)* of X.

Definition 3 An A-module X is called *homogeneous* if, for every $x, y \in X$, the inequalities $\|x_n\| \leq \|y_n\|; n \in \mathbb{N}$ imply that $\|x\| \leq \|y\|$.

We see that for elements x, y in a homogeneous module the equalities $\|x_n\| = \|y_n\|; n = 1, 2, \dots$ imply that $\|x\| = \|y\|$. Thus in a homogeneous module the norm of an element is completely determined by the norms of its coordinates.

For many typical sequence algebras the class of homogeneous modules is fairly wide. In particular, it is easy to show that all essential normed modules over c_0, consisting of complex-valued sequences, are homogeneous. In addition, l_p-sums, $1 \leq p \leq \infty$, of arbitrary families of normed spaces are obviously homogeneous A-modules. (In both examples we mean the coordinate-wise outer multiplication.)

On the other hand, a homogeneous normed A-module X is obviously faithful, that is for every $x \in X$ the equality $a \cdot x = 0$ for all $a \in A$ implies $x = 0$.

In the following two theorems A is an arbitrary sequence algebra. We denote by $\mathcal{H}$ the full subcategory in **A-mod**, consisting of all essential homogeneous modules, and by $\overline{\mathcal{H}}$ the full subcategory in $\mathcal{H}$, consisting of Banach modules.

We call an element x of a given normed A-module X *finite* if $x_n = 0$ for all sufficiently large n. We denote by X_{00} the submodule of X, consisting of finite elements. If X is faithful, in particular, homogeneous, then for every $x \in X_{00}$ we obviously have $x = \sum_n x_n$, and X_{00} is exactly the *algebraic* direct sum of coordinate submodules $X_n; n \in \mathbb{N}$.

We say that X *is of finite type*, (or *has finite type*), if $X = X_{00}$.

Theorem 1.1 *A module $X \in \mathcal{H}$ is extremely projective with respect to $\mathcal{H}$ if and only if it satisfies the following two conditions:*

(i) *For every $n \in \mathbb{N}$, the n-th coordinate subspace X_n is extremely projective in* **Nor**, *or equivalently (see Proposition 0 above), X_n is a near-retract of $l_1^0(\Lambda_n)$ for some index set Λ_n.*

(ii) *X is of finite type.*

Moreover, if X is at least topologically projective, (ii) *is again valid.*

The Banach counterpart of the formulated theorem is the following.

Theorem 1.2 *A module $X \in \overline{\mathcal{H}}$ is extremely projective with respect to $\overline{\mathcal{H}}$ if and only if for every $n \in \mathbb{N}$ the n-th coordinate subspace X_n is extremely projective as a Banach space, or equivalently (see above), X_n is isometrically isomorphic to $l_1(\Lambda_n)$ for some index set Λ_n.*

Thus, speaking informally, in both theorems the answer depends not on the norm on the whole module but only on the norms of its coordinate subspaces. In particular, all Banach homogeneous modules consisting of sequences are extremely projective with respect to $\overline{\mathcal{H}}$, but neither of them is extremely projective with respect to $\mathcal{H}$, when it is infinite-dimensional. On the other hand, submodules of these modules consisting of finite sequences are extremely projective with respect to $\mathcal{H}$.

2 Preparatory Observations

First, we consider the general case of an arbitrary normed algebra A and an arbitrary distinguished full subcategory $\mathcal{K}$ in **A-mod**.

Proposition 2.1 *Let Q be a normed A-module, and P its near-retract. Assume that Q is extremely projective with respect to $\mathcal{K}$. Then the same is true for P.*

Proof Suppose we are given τ, φ and ε as in Definition 1. Fix any $\delta > 0$ such that $\|\varphi\|\delta + \delta + \delta^2 < \varepsilon$. Then for $\varphi_0 := \varphi\sigma$, there exists its lifting $\psi_0 \colon Q \to Y$ with $\|\psi_0\| < \|\varphi_0\| + \delta$, and also bounded morphisms $\sigma \colon Q \to P$, $\rho \colon P \to Q$ with $\sigma\rho = \mathbf{1}_P$ and $\|\rho\| < 1 + \delta$. We see that $\psi := \psi_0\rho$ is a lifting we need. ∎

Denote by $\overline{\mathcal{K}}$ the full subcategory of **A-mod** whose objects are Banach A-modules that are completions of modules from $\mathcal{K}$.

Proposition 2.2 *Suppose that for every module in $\mathcal{K}$ its completion also belongs to $\mathcal{K}$. Then, for every $P \in$* **A-mod**, *which is extremely projective with respect to $\mathcal{K}$, its completion $\overline{P}$ is extremely projective with respect to $\overline{\mathcal{K}}$.*

Proof Suppose we are given a coisometric morphism $\tau \colon Y \to X; X, Y \in \overline{\mathcal{K}}$, a bounded morphism $\varphi \colon \overline{P} \to X$ and $\varepsilon > 0$. Consider the restriction φ_0 of φ to P. Then there exists a lifting $\psi_0 \colon P \to Y$ of φ_0 with $\|\psi_0\| < \|\varphi_0\| + \varepsilon$. Since Y is complete, ψ_0 has the continuous extension $\psi \colon \overline{P} \to Y$, which is obviously a lifting of φ. Therefore the previous estimate of $\|\psi_0\|$ gives the desired estimate of $\|\psi\|$. ∎

In the remaining part of this section we concentrate on the case when A is a sequence algebra. This material will be used in Section 3.

Note that an arbitrary morphism $\varphi\colon Y \to X$ in **A-mod** gives rise, in an obvious way, to the sequence of its birestrictions between the respective coordinate submodules (*cf.* Introduction). These morphisms will be denoted by $\varphi_n\colon Y_n \to X_n$, $n = 1, 2, \dots$, and called *coordinate submorphisms of* φ.

Proposition 2.3 *If* $\tau\colon Y \to X$ *is a coisometric morphism in* **A-mod**, *then for every* n, $\tau_n\colon Y_n \to X_n$ *is also a coisometry.*

Proof Take $x \in X_n$ and $\varepsilon > 0$. Then there exists $\widetilde{y} \in Y$ with $\tau(\widetilde{y}) = x$ and $\|\widetilde{y}\| < \|x\| + \varepsilon$. Hence for $y := \mathbf{p}^n \cdot \widetilde{y}$ we have $\tau(y) = x$ and $\|y\| \le \|\widetilde{y}\|$. ∎

Fix, for a moment, $n \in \mathbb{N}$ and denote by $\mathcal{K}_n$ the full subcategory in **Nor** whose objects are n-th coordinate subspaces of modules from $\mathcal{K}$.

Proposition 2.4 *Suppose* $\mathcal{K}$ *contains, for every module in* $\mathcal{K}$, *the* n*-th coordinate submodule of the latter. Assume that a normed* A*-module* P *is extremely projective with respect to* $\mathcal{K}$. *Then* P_n *is extremely projective in* **Nor** *with respect to* $\mathcal{K}_n$.

Proof Take $F, E \in \mathcal{K}_n$, a coisometric operator $\tau\colon F \to E$, a bounded operator $\varphi\colon P_n \to E$ and $\varepsilon > 0$. Our E and F are underlying spaces of n-th coordinate submodules, say X and Y, of some modules in $\mathcal{K}$. Hence $X, Y \in \mathcal{K}$, and (1.1) implies that τ, as a map between these modules, is a module morphism.

Consider the map $\widetilde{\varphi}\colon P \to X\colon x \mapsto \varphi(x_n)$. It follows again from (1.1) that it is a morphism of A-modules. Also we obviously have $\|\widetilde{\varphi}\| = \|\varphi\|$. Therefore, the assumption on P provides a lifting $\widetilde{\psi}\colon P \to Y$ of $\widetilde{\varphi}$ such that $\|\widetilde{\psi}\| < \|\varphi\| + \varepsilon$.

Denote by ψ the restriction of $\widetilde{\psi}$ to P_n. Then we easily see that it is a lifting of φ, and $\|\psi\| \le \|\widetilde{\psi}\|$. The rest is clear. ∎

Let us turn to the special properties of homogeneous modules.

For every $N = 1, 2, \dots$ we set $\mathbf{P}^N := \sum_{n=1}^{N} \mathbf{p}^n \in A$.

Proposition 2.5 *If an* A*-module* X *is essential and homogeneous, then for every* $x \in X$ *we have*

$$x = \lim_{N\to\infty} \mathbf{P}^N \cdot x.$$

Proof Take x and $\varepsilon > 0$. By Definition 2, there is $y \in X$ of the form $\sum_{k=1}^{n} a^k \cdot z^k$; $a^k \in c_{00}, z^k \in X$ with $\|x - y\| < \varepsilon/2$. For all $N \in \mathbb{N}$ we have

$$\|x - \mathbf{P}^N \cdot x\| \le \|x - y\| + \|y - \mathbf{P}^N \cdot y\| + \|\mathbf{P}^N \cdot y - \mathbf{P}^N \cdot x\|.$$

But, because of the choice of y, for some $M \in \mathbb{N}$ we have $y = \mathbf{P}^N \cdot y$ for all $N > M$. Besides, the homogeneity of X implies that $\|\mathbf{P}^N \cdot (y - x)\| \le \|y - x\|$. Therefore for all $N > M$ we have $\|x - \mathbf{P}^N \cdot x\| < \varepsilon$. ∎

Proposition 2.6 *If an* A*-module* X *is essential and homogeneous, the same is true for its completion* $\overline{X}$. *(In other words,* $X \in \mathcal{H}$ *implies* $\overline{X} \in \overline{\mathcal{H}}$.*)*

Proof Clearly, $\overline{X}$ is essential. Let us prove that it is homogeneous. Take $x, y \in \overline{X}$ with $\|x_n\| \leq \|y_n\|$ for all n. We must show that $\|x\| \leq \|y\|$. By Proposition 2.5, we can assume that for some N we have $x_n = y_n = 0$ for all $n > N$.

Choose sequences $x^k, y^k \in X$, $k = 1, 2, \ldots$, converging to x and y, respectively. Using Proposition 2.5, we can assume that $x^k_n = 0$ whenever $x_n = 0$.

For all n we have $\lim_{k\to\infty} \|x^k_n\| = \|x_n\|$ and $\lim_{k\to\infty} \|y^k_n\| = \|y_n\|$. We see that for every $\varepsilon > 0$ there exists a natural M such that for all $k > M$ and for all n we have $\|x^k_n\| \leq (1+\varepsilon)\|y^k_n\|$. But X is homogeneous, and therefore for the same k we have $\|y^k\| \leq \|(1+\varepsilon)x^k\|$. It remains to pass to limits. ∎

Now we recall a property of Banach (but not just normed) spaces, which is very well known and which is easy to prove. Namely, *if $\sum_{n=1}^{\infty} x_n$ is a converging series in a Banach space E, then there exists another series in E, say $\sum_{n=1}^{\infty} \bar{x}_n$, such that, for some sequence $\lambda_n \in \mathbb{R}$, $\lambda_n \geq 1$, and $\lambda_n \to \infty$, we have $\bar{x}_n = \lambda_n x_n$, and this new series is still convergent.* From this one can easily deduce the following.

Proposition 2.7 *Let X be essential and homogeneous. Then for every x in its completion $\overline{X}$ (in particular, in X) there exists $\bar{x} \in \overline{X}$ such that $\bar{x}_n = \lambda^x_n x_n$, where $\lambda^x_n \in \mathbb{R}$, $\lambda^x_n \geq 1$, and $\lambda^x_n \to \infty$.*

Speaking informally, the coordinates of $\bar{x}$, being proportional to those of x, tend to 0 essentially more slowly.

At the end of the section we describe a certain way to construct some new homogeneous modules, starting with a given homogeneous module of finite type.

Suppose we have an algebraic A-module X of finite type. Set $M_X := \{n \in \mathbb{N} : X_n \neq 0\}$ and denote by $c^+_{00}(X)$ the set (actually, cone) of all finite non-negative sequences $\xi = (\ldots, \xi_n, \ldots)$ such that $\xi_n = 0$ whenever $n \notin M_X$.

Now assume that our X is a normed homogeneous A-module. Introduce the function $f_X \colon c^+_{00}(X) \to \mathbb{R} \colon \xi \mapsto \|x\|$, where x is an (obviously existing) element of X with $\|x_n\| = \xi_n$. By homogeneity of X, this function is well defined.

Proposition 2.8 *The function f_X has the following properties:*

(i) *if $\xi \in c^+_{00}(X)$ is not zero, then $f_X(\xi) > 0$;*

(ii) *if $\xi \in c^+_{00}(X)$ and $\lambda > 0$, then $f_X(\lambda\xi) = \lambda f_X(\xi)$;*

(iii) *if $\xi, \eta \in c^+_{00}(X)$, and $\xi \leq \eta$, then $f_X(\xi) \leq f_X(\eta)$;*

(iv) *if $\xi \in c^+_{00}(X)$ and $a \in A$, then $f_X(|a|\xi) \leq \|a\| f_X(\xi)$;*

(v) *if $\xi, \eta \in c^+_{00}(X)$, then $f_X(\xi + \eta) \leq f_X(\xi) + f_X(\eta)$;*

(vi) *if $n \in M_X$, then $f_X(\mathbf{p}^n) = 1$.*

Proof The properties (i)–(iv) are immediate.

(v) Take $x \in X$ such that $\|x_n\| = (\xi + \eta)_n$ for all n and thus $f_X(\xi + \eta) = \|x\|$. If $\xi_n + \eta_n > 0$ for a given n, then we set $\lambda_n := \xi_n/(\xi_n + \eta_n)$ and $\mu_n := \eta_n/(\xi_n + \eta_n)$; otherwise we set $\lambda_n = \mu_n = 0$. After this, we set $y := \sum_n \lambda_n x_n$ and $z := \sum_n \mu_n x_n$. We have $f_X(\xi) = \|y\|$ and $f_X(\eta) = \|z\|$. But $x = y + z$ and hence $\|x\| \leq \|y\| + \|z\|$.

(vi) Take $x \in X_n$ of norm 1. Then the sequence $(\|x_1\|, \|x_2\|, \ldots)$ is exactly $\mathbf{p}^n$. ∎

Sometimes we shall refer to f_X as to the *function associated with the module X*.

Thus we got a function from a module. Proceed in the opposite direction.

Let X be an algebraic A-module X of finite type such that, for $x \in X$, the equalities $\mathbf{p}^n \cdot x = 0; n \in \mathbb{N}$ imply that $x = 0$. Suppose that every coordinate subspace X_n is equipped with a norm, say $\|\cdot\|_n$. Fix an arbitrary function $f\colon c_{00}^+(X) \to \mathbb{R}$ possessing the properties (i)–(v) of the previous proposition. Now for $x \in X$ we set $\|x\| := f(\xi)$, where $\xi_n := \|x_n\|_n$.

Proposition 2.9 *The assignment $x \mapsto \|x\|$ is a norm, making X a homogeneous A-module. If, in addition, f has the property* (vi), *then, for every n, the restriction of $\|\cdot\|$ to X_n coincides with the initial norm $\|\cdot\|_n$.*

Proof Of the properties of a norm, only the triangle inequality is not immediate. Let $x = y + z$ in X. Take the sequences $\xi, \eta, \zeta \in c_{00}^+(X)$ such that $\xi_n := \|x_n\|_n$, $\eta_n := \|y_n\|_n$ and $\zeta_n := \|z_n\|_n$. We have $\xi \le \eta + \zeta$, and the properties (iii) and (v) of f imply $f(\xi) \le f(\eta + \zeta) \le f(\eta) + f(\zeta)$. Hence $\|x\| \le \|y\| + \|z\|$.

Now take $a \in A$ and $x \in X$. Then, for sequences ξ and η with $\xi_n := \|x_n\|_n$ and $\eta_n := \|(a \cdot x)_n\|_n$, we have $\eta = |a|\xi$. Therefore the property (iv) of f implies that $\|a \cdot x\| \le \|a\|\|x\|$.

Thus X became a normed module, which is obviously homogeneous.

Finally, if $x \in X_n$, then the sequence $(\|x_1\|_1, \|x_2\|_2, \dots)$ coincides with $\|x_n\|_n \mathbf{p}^n$, and therefore $\|x\| = \|x_n\|_n f(\mathbf{p}^n)$. The last assertion follows. ■

We shall denote the constructed homogeneous module by X^f.

Proposition 2.10

(i) *If X is a homogeneous module of finite type, then $X^{f_X} = X$.*

(ii) *If we have the data of Proposition 2.9, and f satisfies the properties* (i)–(vi) *of Proposition 2.8, then $f_{X^f} = f$.*

Proof (i) is immediate, and (ii) follows from Proposition 2.9. ■

Finally, suppose that we have two A-modules in the pure algebraic sense, say X and Y, and, for every n, a linear operator $\varphi_n\colon X_n \to Y_n$ is given. Suppose, further, that X is of finite type. Then there exists, for every $x \in X$, the well-defined element $\varphi(x) := \sum_n \varphi_n(x_n) \in Y$. In this way we obtain a map $\varphi\colon X \to Y$, which is, of course, a morphism. We shall call it the *morphism generated by the operators φ_n*. Note that for every x and n we obviously have

$$\varphi(x)_n = \varphi_n(x_n). \tag{2.1}$$

Proposition 2.11 *Let X and Y be normed homogeneous modules of finite type with $c_{00}^+(X) = c_{00}^+(Y)$ and with the same associated function. Suppose that for every n we are given a bounded operator $\varphi_n\colon X_n \to Y_n$, and $C := \sup\{\|\varphi_n\|; n \in \mathbb{N}\} < \infty$. Then the morphism φ generated by operators φ_n is bounded, and $\|\varphi\| = C$.*

Proof Take $x \in X$. Let ξ and η be the sequences with $\xi_n := \|x_n\|$ and $\eta_n := \|\varphi(x)_n\|$. It follows from (2.1) and the assumption on φ_n that $\eta_n \le C\xi_n$. Since $f_X = f_Y$, Proposition 2.8 (iii) and (ii) imply $f_Y(\eta) \le C f_X(\xi)$, that is, $\|\varphi(x)\| \le C\|x\|$. Thus $\|\varphi\| \le C$. Further, we see from (2.1) that, for every n, the restriction of φ to X_n is φ_n. Hence $C \le \|\varphi\|$. ■

3 On Extremely Projective Non-complete and Complete Normed Spaces

Looking for extremely projective spaces in the 'non-complete context', one inevitably pays attention to the spaces $l_1^0(\Lambda)$, mentioned in the introduction. For every $\nu \in \Lambda$, we denote by e_ν the function in $l_1^0(\Lambda)$, taking ν to 1 and other points of Λ to 0. Clearly, the set $\{e_\nu; \nu \in \Lambda\}$ is a linear basis in $l_1^0(\Lambda)$. In what follows, such a basis will be called *natural*.

The following two statements must be well known, at least as folklore.

Proposition 3.1 *Let E be a normed space, $\alpha\colon \Lambda \to E$ a map with a bounded image. Then there exists a bounded operator $\varphi\colon l_1^0(\Lambda) \to E$, uniquely defined by $e_\nu \mapsto \alpha(\nu)$, $\nu \in \Lambda$. Moreover, $\|\varphi\| = \sup\{\|\alpha(\nu)\| \,;\, \nu \in \Lambda\}$.*

Proposition 3.2 *The space $l_1^0(\Lambda)$, where Λ is an arbitrary index set, is extremely projective in* **Nor**.

Proof Take a coisometry $\tau\colon F \to E$ in **Nor**, a bounded operator $\varphi\colon l_1^0(\Lambda) \to E$ and $\varepsilon > 0$. Fix, for every $\nu \in \Lambda$, an arbitrary $y_\nu \in F$ with $\tau(y_\nu) = \varphi(e_\nu)$ and $\|\tau(y_\nu)\| < \|\varphi(e_\nu)\| + \varepsilon$. Then the operator $\psi\colon l_1^0(\Lambda) \to F$, well defined by $e_\nu \mapsto y_\nu$, is a lifting of φ across τ. Finally, Proposition 3.1 implies that $\|\psi\| < \|\varphi\| + \varepsilon$. ■

Let E be a normed space. Denote its unit sphere by S_E. Consider the normed space $l_1^0(S_E)$ (*i.e.*, $l_1^0(\Lambda)$ with S_E as Λ) and its natural linear basis $e_x; x \in S_E$. Then Proposition 3.1 provides a bounded operator $\tau_E\colon l_1^0(S_E) \to E\colon e_x \mapsto x$, $x \in S_E$; obviously, it is a coisometry. We shall call it the *canonical coisometry for E*.

Now we can prove Proposition 0, formulated in the introduction.

Proof The 'if' part follows from Propositions 3.2 and 2.1. In the latter we set **A-mod** $:= \mathcal{K} =:$ **Nor** and $Q := l_1^0(\Lambda)$. Conversely, suppose that a certain P is extremely projective in **Nor**; we can assume that $P \neq 0$. Consider the canonical coisometry $\tau_P\colon l_1^0(S_P) \to P$. Then for every $\varepsilon > 0$ there exists a lifting $\rho\colon P \to l_1^0(S_P)$ of the identity operator $\mathbf{1}_P$ on P such that $\|\rho\| < \|\mathbf{1}_P\| + \varepsilon$. The rest is clear. ■

Of course, a much more sound and transparent statement would be: a normed space is extremely projective as a normed space if and only if it is isometrically isomorphic to $l_1^0(\Lambda)$ for some index set Λ. But is it true? We do not know. The question seems reasonable, especially because it has, as a background, the Grothendieck Theorem that was formulated in the introduction.

Remark In the literature, the latter theorem is usually cited as [4] (see, *e.g.*, [11, p. 182]). Basically, the attribution of the result to Grothendieck is correct. At the same time, despite the fact that the paper [4] contains all needed ingredients for the proof, the theorem itself is not explicitly formulated (perhaps because the author just thought it unnecessary). By 'ingredients' we mean the following two statements, formulated (needless to say, in equivalent terms) and completely proved:

(i) a Banach space, which is topologically projective in **Ban** and isometrically isomorphic to some $L_1(\Omega, \mu)$, is isometrically isomorphic to $l_1(\Lambda)$ [4, Prop. 2];

(ii) A Banach space F is isometrically isomorphic to some $L_1(\Omega,\mu)$ if and only if it is extremely flat, that is, for every isometry $i\colon F\to G$ of Banach spaces the operator $\mathbf{1}_E \widehat{\otimes} i\colon E\widehat{\otimes} F\to E\widehat{\otimes} G$ is also an isometry (here $\widehat{\otimes}$ is the symbol of the projective tensor product of Banach spaces and bounded operators).

Thus, to complete the proof, one must show that every extremely projective Banach space is extremely flat. Apparently, the traditional way to do it is to use a nontrivial criterion of extreme flatness, namely Proposition 1 in [4], and then to verify the relevant condition. For this aim one takes a certain family of operators (indexed by ε from Definition 1, and then, applying the Banach–Alaoglu Theorem, proceeds to the cluster point of this family with respect to a suitable weak* topology. See, *e.g.*, [13, 27.4.2].

Here we suggest what seems to be a shorter way.

As is well known, the property of a bounded operator, in particular $\mathbf{1}_E\widehat{\otimes} i$, to be isometric is equivalent to the property of its adjoint to be coisometric. But $(\mathbf{1}_E\widehat{\otimes} i)^*$, by the standard Banach version of the law of adjoint associativity (see, *e.g.*, [2, p. ix], [15, III.B.26] or [9, p. 180]) is isometrically equivalent to the operator

$$\mathcal{B}(E,i^*)\colon \mathcal{B}(E,G^*)\to\mathcal{B}(E,F^*)\colon \varphi\mapsto i^*\varphi,$$

where $\mathcal{B}(\,\cdot\,,\,\cdot\,)$ denotes the space of all bounded operators between relevant Banach spaces. But E is extremely projective, and also, since i is isometric, i^* is coisometric. Therefore, by Definition 1, $\mathcal{B}(E,i^*)$ is coisometric, and we are done.

Returning to general normed spaces, combining the Grothendieck Theorem with the suitable particular case of Proposition 2.2, we immediately get the following proposition.

Proposition 3.3 *Suppose P is extremely projective in* **Nor**. *Then it is, up to an isometric isomorphism, a dense subspace in $l_1(\Lambda)$ for some index set Λ.* ■

Thus the question, formulated above, can be posed in the following somewhat more detailed form. For a given Λ, which dense subspaces of $l_1(\Lambda)$ are extremely projective in **Nor** (like $l_1^0(\Lambda)$), and which are not?

Concluding this section, we point out another necessary condition of the property under discussion.

Proposition 3.4 *Let P be a separable normed space. Suppose that it is extremely, or at least topologically, projective in* **Nor**. *Then it has at most countable linear dimension.*

Proof Consider the canonical coisometry $\tau_P\colon l_1^0(S_P)\to P$ for P. Then the identity operator $\mathbf{1}_P$ has a bounded lifting $\psi\colon P\to l_1^0(S_P)$ across τ.

Choose an arbitrary dense subset, say $\{x_n; n\in\mathbb{N}\}$, in P. Since $l_1^0(S_P)$ has the *linear* basis e_x, $x\in S_P$, every $\psi(x_n)$, $n\in\mathbb{N}$, has a form $\sum_{k=1}^{m_n}\lambda_k^{(n)}e^n_{x_k}$ for some $m_n\in\mathbb{N}$, $x_k^n\in S_P$ and $\lambda_k^{(n)}\in\mathbb{C}$, $k=1,\dots,m_n$. Therefore all $\psi(x_n)$, $n\in\mathbb{N}$, belong to the linear span of all $e_{x_k^n}$, $n\in\mathbb{N}$, $k=1,\dots,m_n$, denoted for brevity by F. The latter is obviously closed in $l_1^0(S_P)$, the set $\{x_n; n\in\mathbb{N}\}$ is dense in P, and ψ is continuous. Consequently, ψ maps P into F and, being a right inverse to τ, it is injective. Therefore, ψ implements a linear isomorphism of P onto its image in F, and this image, of course, has at most countable linear dimension. ■

Proposition 3.5 *Let Λ be an arbitrary infinite index set. Then the space $l_1(\Lambda)$ (being, thanks to Grothendieck, extremely projective in* **Ban***) is not topologically, and hence extremely, projective in* **Nor**.

Proof Clearly, $l_1(\Lambda)$ has l_1 as its near-retract. Therefore, by virtue of Proposition 2.1, it is sufficient to prove our assertion for the latter space. But l_1 is separable, and its linear dimension, by an old theorem of Lövig [12], is continuum. ■

4 The Proof of Theorems 1.1 and 1.2

Up to the end of the paper, A is a fixed sequence algebra, the word "module" means "A-module", and "morphism" means "morphism of A-modules".

Theorem 4.1 *Let P be a homogeneous normed A-module of finite type such that for every n, $P_n = l_1^0(\Lambda_n)$ for an index set Λ_n. Then P is extremely projective with respect to $\mathcal{H}$.*

Proof For every n we denote by $e^n_\nu; \nu \in \Lambda_n$ the natural basis in $l_1^0(\Lambda_n)$, by S the unit sphere in P and by $E(S)$ the set of such z in S that, for every n, z_n is a multiple of e^n_ν for some $\nu \in \Lambda_n$.

We need two preparatory assertions. In both of them X is an arbitrary normed module, and $\varphi\colon P \to X$ is a bounded morphism.

Lemma 1 *For every $x \in P$ and $N \in \mathbb{N}$ there exists $x^{(N)} \in P$ such that $x^{(N)}_N$ is a multiple of some e^N_ν, $\nu \in \Lambda_N$, $x^{(N)}_n = x_n$ for all $n \neq N$, $\|x^{(N)}\| = \|x\|$, and $\|\varphi(x^{(N)})\| \geq \|\varphi(x)\|$.*

Proof We can assume that $x_N \neq 0$, and thus it has a form $\sum_{k=1}^m \lambda_k e^N_{\nu_k}$, where $\lambda_k \neq 0$, $\nu_k \in \Lambda_N$. Set $\mu_k := |\lambda_k|/\|x_N\|$ and consider, for every $k = 1, \dots, m$, the element $x^k \in P$ with the coordinates $x^k_N := \mu_k^{-1}\lambda_k e^N_{\nu_k}$ and $x^k_n := x_n$ for $n \neq N$. Obviously, for all k we have $\|x^k_N\| = \|x_N\|$. Since P is homogeneous, this implies $\|x^k\| = \|x\|$. Further, since $\sum_{k=1}^m |\lambda_k| = \|x_N\|$, we have $\sum_{k=1}^m \mu_k x^k = x$. Therefore $\sum_{k=1}^m \mu_k \varphi(x^k) = \varphi(x)$, and we see that $\varphi(x)$ is a convex combination of elements $\varphi(x^k)$ in X. Consequently, for at least one of k we have $\|\varphi(x^k)\| \geq \|\varphi(x)\|$. Thus the appropriate x^k has all properties of the desired $x^{(N)}$.

Lemma 2 *We have $\|\varphi\| = \sup\{\|\varphi(z)\|; z \in E(S)\}$.*

Proof Take an arbitrary $x \in S$. Apply to this element the previous lemma for the case $N := 1$. Then apply to the resulting element x^1 (also belonging, by the construction, to S) the same lemma, this time for the case $N := 2$, and so on. Since x is finite, eventually we come to an element, say z, belonging to $E(S)$ and such that $\|\varphi(z)\| \geq \|\varphi(x)\|$. The rest is clear.

The End of the Proof of Theorem 4.1

Suppose we are given modules $X, Y \in \mathcal{H}$, a coisometric morphism $\tau\colon Y \to X$, a bounded morphism $\varphi\colon P \to X$, and $\varepsilon > 0$. Consider the coordinate submorphisms $\varphi_n\colon P_n \to X_n, \tau_n\colon Y_n \to X_n$, and choose $\delta > 0$ with $(1+\delta)^2\|\varphi\| < \|\varphi\| + \varepsilon/2$. By

Proposition 2.3, all τ_n are coisometries. Therefore for every $\nu \in \Lambda_n$ there exists an element $y_\nu^n \in Y_n$ such that $\tau_n(y_\nu^n) = \varphi(e_\nu^n)$ and $\|y_\nu^n\| \le \|\varphi(e_\nu^n)\| + \delta\|\varphi(e_\nu^n)\|$. Observe that Proposition 3.1 provides, for every n, an operator $\psi_n\colon P_n \to Y_n$, well defined by $e_\nu^n \mapsto y_{n_\nu}$, $\nu \in \Lambda_n$. Consider the morphism $\psi\colon P \to Y$, generated (see above) by the operators ψ_n. Clearly, it is a lifting of φ across τ.

Now take $x \in E(S)$. For every n we have $x_n = \lambda_\nu^n e_\nu^n$ for some $\nu \in \Lambda_n$ and $\lambda_\nu^n \in \mathbb{C}$. Hence $\varphi_n(x_n) = \lambda_\nu^n \varphi(e_\nu^n)$ and $\psi_n(x_n) = \lambda_\nu^n y_\nu^n$. Therefore we have

$$\|\psi_n(x_n)\| \le (1+\delta)\|\varphi_n(x_n)\|. \tag{4.1}$$

Since τ is a coisometry, there exists $z \in Y$ with $\tau(z) = \varphi(x)$ and $\|z\| \le \|\varphi(x)\| + \delta\|\varphi(x)\|$. Note that $\tau(z_n) = \tau(z)_n = \varphi(x)_n = \varphi_n(x_n)$ and hence $\|\varphi_n(x_n)\| \le \|z_n\|$. Together with (4.1), this gives $\|\psi(x)_n\| = \|\psi_n(x_n)\| \le (1+\delta)\|z_n\|$. From this, by homogeneity, we obtain that

$$\|\psi(x)\| \le (1+\delta)\|z\| \le (1+\delta)^2\|\varphi(x)\| \le (1+\delta)^2\|\varphi\| < \|\varphi\| + \varepsilon/2.$$

Therefore the previous lemma gives $\|\psi\| \le \|\varphi\| + \varepsilon/2$. The rest is clear. ■

The Proof of the 'If' Part of Theorem 1.1

Now the coordinate subspaces P_n of our homogeneous module P of finite type are arbitrary spaces that are extremely projective in **Nor**. By virtue of Proposition 0, formulated in the Introduction and proved in Section 2, for every $n \in \mathbb{N}$ there exists an index set Λ_n and an operator $\sigma_n\colon l_1^0(\Lambda_n) \to P_n$ which is a near-retraction.

Set, for brevity, $Q_n := l_1^0(\Lambda_n)$ (thus, in particular, $Q_n = 0$ exactly when $P_n = 0$). Denote by Q the algebraic sum $\bigoplus_n Q_n$. Clearly, Q is an A-module of finite type with respect to the outer multiplication, well defined, for $a \in A$ and $x \in Q$ of the form $x = \sum_n x_n; x_n \in Q_n$, by $a \cdot x := \sum_n a_n x_n$.

Of course, $c_{00}^+(Q) = c_{00}^+(P)$, and hence we can endow Q with a norm by the recipe of Proposition 2.9, taking, in the capacity of f, the function f_P.

We obtain a homogeneous A-module of finite type. By virtue of Theorem 4.1, it is extremely projective with respect to the category $\mathcal{H}$.

Take $\varepsilon > 0$. As we know, for every n the operator σ_n has a right inverse operator, say ρ_n, with the norm $< 1 + \varepsilon/2$. Consider the morphisms $\sigma\colon Q \to P$ and $\rho\colon P \to Q$, generated by the sequences σ_n and ρ_n, respectively. Obviously, ρ is a right inverse to σ. By virtue of Proposition 2.10 (ii), $f_Q = f_P$, and we can apply Proposition 2.11 to both morphisms. Therefore, since all σ_n are contractive, the same is true for σ, and the estimate for $\|\rho_n\|$ gives $\|\rho\| \le \varepsilon/2 < \varepsilon$. Thus σ is a near-retraction of normed modules, and it remains to apply Proposition 2.1.

The End of the Proof of Theorem 1.1

Now we suppose that a module $P \in \mathcal{H}$ is extremely projective with respect to the category $\mathcal{H}$. Since every coordinate submodule of a homogeneous module is itself homogeneous, we can use Proposition 2.4. This implies that P has the property (i).

To complete the proof of Theorem 1.1, we proceed to show that every module in $\mathcal{H}$ that is topologically projective with respect to $\mathcal{H}$ has finite type. For brevity in what follows, for an arbitrary A-module X we denote the set $X \setminus X_{00}$ by X^∞.

Suppose that for $X \in \mathcal{H}$ we have $X^\infty \neq \varnothing$. Our nearest aim is to construct, starting with X, another module $\mathbf{Y} \in \mathcal{H}$ that will serve as the domain of a future coisometry onto X.

For every $x \in X^\infty$, choose and fix an arbitrary $\bar{x} \in \overline{X}$ and the sequence λ_n^x with the properties indicated in Proposition 2.7. Further, denote by Y^x the submodule $\{\mu\bar{x} + b \cdot \bar{x}; \mu \in \mathbb{C}, b \in A\}$ in $\overline{X}$, that is the submodule, *algebraically* generated by $\bar{x}$. After this, we introduce the A-module

$$\mathbf{Y} := \Big(\bigoplus_{x \in X^\infty} Y^x\Big) \oplus X_{00},$$

the *algebraic* direct sum of the indicated modules. In what follows, for a given $\mathbf{y} \in \mathbf{Y}$ the notation $\mathbf{y} = \{y^{x_k}; k = 1, \dots, m, y^*\}$ means that $y^{x_k} \in Y^{x_k}$ and $y^* \in X_{00}$ are direct summands of this element and all other summands are zeroes. Note that $\mathbf{y}_n$, the n-th coordinate of our $\mathbf{y}$ can be presented as $\{y_n^{x_k}; k = 1, \dots, m, y_n^*\}$.

We want to equip $\mathbf{Y}$ with a norm. To avoid a misunderstanding, we shall denote this future norm by $\|\cdot\|_{\mathbf{Y}}$, whereas the already given norm on X, as well as on its completion $\overline{X}$, will be denoted just by $\|\cdot\|$.

We begin with the coordinate submodules of $\mathbf{Y}$. Fix, for a moment, $n \in \mathbb{N}$ and for an arbitrary $\mathbf{y} = \{y^{x_k}; k = 1, \dots, m, y^*\} \in \mathbf{Y}_n$ (where, of course, $y^{x_k} \in Y_n^{x_k}$ and $y^* \in X_n$), set

$$\|\mathbf{y}\|_n := \sum_{k=1}^m \|y_n^{x_k}\| + \|y_n^*\|.$$

Evidently, $\|\cdot\|_n$ is a norm on $\mathbf{Y}_n$. Further, for every $n \in \mathbb{N}$, we have $X_n = 0$ if and only if $\mathbf{Y}_n = 0$. It follows that, in the notation of Section 1, $c_{00}^+(\mathbf{Y}_{00}) = c_{00}^+(X_{00})$. Denote by f the function associated with the module X_{00}. Using the recipe of Proposition 2.9, we introduce, with the help of that f and the norms $\|\cdot\|_n$, the norm on $\mathbf{Y}_{00}$. Denote this norm by $\|\cdot\|_{\mathbf{Y}}^0$.

Thus $\mathbf{Y}_{00}$ becomes a homogeneous A-module. Now take an arbitrary $\mathbf{y} \in \mathbf{Y}$ and consider numbers $\|\mathbf{P}^N \cdot \mathbf{y}\|_{\mathbf{Y}}^0$ for all $N \in \mathbb{N}$. By homogeneity, they form an increasing sequence. Further, note that for every $\mathbf{y} \in \mathbf{Y}_{00}$ with the only non-zero direct summand, say y, we have $\|\mathbf{y}\|_{\mathbf{Y}}^0 = \|y\|$. It follows that for every $\mathbf{y} = \{y^{x_k}; k = 1, \dots, m, y^*\} \in \mathbf{Y}$ and N we have

$$\|\mathbf{P}^N \cdot \mathbf{y}\|_{\mathbf{Y}}^0 \leq \sum_{k=1}^m \|\mathbf{P}^N \cdot y^{x_k}\| + \|\mathbf{P}^N \cdot y^*\| \leq \sum_{k=1}^m \|y^{x_k}\| + \|y^*\|.$$

Thus the sequence $\|\mathbf{P}^N \cdot \mathbf{y}\|_{\mathbf{Y}}^0$, $N = 1, 2, \dots$, converges; denote its limit by $\|\mathbf{y}\|_{\mathbf{Y}}$. It is easy to check that $\|\cdot\|_{\mathbf{Y}}$ is also a norm, this time on all $\mathbf{Y}$, and it makes the latter an essential homogeneous A-module.

Of course, if a certain $\mathbf{y} \in \mathbf{Y}$ belongs either to Y^x for some $x \in X^\infty$, or to X_{00}, then $\|\mathbf{y}\|_{\mathbf{Y}}$ is the norm of our element in the respective submodule of $\overline{X}$. On the other hand, if $\mathbf{y}$ belongs to $\mathbf{Y}_n$ for some n, that is $\mathbf{y}$ can be written as $\{y^{x_k}; k = 1, \dots, m, y^*\}$ with all $y^{x_k} \in (\overline{X})_n$ and $y^* \in X_n$, then we obviously have

$$\|\mathbf{y}\|_{\mathbf{Y}} = \|\mathbf{y}\|_n = \sum_{k=1}^m \|y^{x_k}\| + \|y^*\|. \tag{4.2}$$

Our next aim is to introduce a morphism of A-modules $\tau\colon \mathbf{Y} \to X$.

It is sufficient to define τ on direct summands of $\mathbf{Y}$. First, take a summand of the form $Y^x; x \in X^\infty$ and its element, say y. The latter, as we remember, has the form $\mu\bar{x} + b \cdot \bar{x}$ for some $\mu \in \mathbb{C}$ and $b \in A$. We set $\tau(y) := \mu x + b \cdot x$. Since we have, of course, $\tau(y)_n = (\lambda_n^x)^{-1} y_n$ (*cf.* Proposition 2.7), and X is homogeneous, our $\tau(y)$ is uniquely defined by y. This gives rise to the well defined map from Y^x into X, which is obviously a morphism of A-modules. In the case of the remaining direct summand, X_{00}, we define τ as just the natural embedding into X.

Thus τ is defined. Note that for $\mathbf{y} \in \mathbf{Y}$, say $\mathbf{y} = \{y^{x_k}; k = 1, \dots, m, y^*\}$, we obviously have

$$\tau(\mathbf{y}_n) = \sum_{k=1}^{m} (\lambda_n^{x_k})^{-1} y_n^{x_k} + y_n^*. \tag{4.3}$$

Proposition 4.2 *The morphism τ is coisometric.*

Proof Clearly, τ has a well defined birestriction $\tau_{00}\colon \mathbf{Y}_{00} \to X_{00}$, and τ_{00} is generated by its coordinate submorphisms $(\tau_{00})_n\colon (\mathbf{Y}_{00})_n \to (X_{00})_n$; here, of course, $(\mathbf{Y}_{00})_n = \mathbf{Y}_n$ and $(X_{00})_n = X_n$.

Fix, for a moment, $n \in \mathbb{N}$ and take $\mathbf{y} = \{y_n^{x_k}; k = 1, \dots, m, y_n^*\} \in \mathbf{Y}_n$. We have $\tau(y_n^{x_k}) = (\lambda_n^{x_k})^{-1} y_n^{x_k}$, where, by Proposition 2.7, $\lambda_n^{x_k} \geq 1$. Combining this with (4.2), we see that the norm of $(\tau_{00})_n(\mathbf{y})$, that is of $\tau(\mathbf{y})$, does not exceed $\|\mathbf{y}\|$.

From this, by Proposition 2.11, we have $\|\tau_{00}(\mathbf{y})\| \leq \|\mathbf{y}\|$ for all $\mathbf{y} \in \mathbf{Y}_{00}$ and hence $\|\tau(\mathbf{P}^N \cdot \mathbf{y})\| \leq \|\mathbf{P}^N \cdot \mathbf{y}\|$ for all $\mathbf{y} \in \mathbf{Y}$ and $N \in \mathbb{N}$. But, by Proposition 2.5, $\tau(\mathbf{y}) = \lim_{N\to\infty} \mathbf{P}^N \cdot \tau(\mathbf{y})$. In addition, τ is a module morphism. Hence τ is contractive.

It remains to display, for a given $x \in X$ and $\varepsilon > 0$, a certain $\mathbf{y} \in \mathbf{Y}$ with $\tau(\mathbf{y}) = x$ and $\|\mathbf{y}\|_{\mathbf{Y}} < \|x\| + \varepsilon$. If $x \in X_{00}$, then the copy of x in the direct summand X_{00} of $\mathbf{Y}$ already fits. If $x \in X^\infty$, we recall that $\overline{X} \in \mathcal{H}$ (Proposition 2.6), and therefore, by Proposition 2.5, there exists $N \in \mathbb{N}$ such that $\|\bar{x} - \mathbf{P}^N \cdot \bar{x}\| < \varepsilon$.

Now consider $\mathbf{y} \in \mathbf{Y}$ with at most two non-zero direct summands, namely $\bar{x} - \mathbf{P}^N \cdot \bar{x} \in Y^x$ and $\mathbf{P}^N \cdot x \in X_{00}$. We see that $\tau(\mathbf{y}) = x$, and

$$\|\mathbf{y}\|_{\mathbf{Y}} \leq \|\bar{x} - \mathbf{P}^N \cdot \bar{x}\| + \|\mathbf{P}^N \cdot x\| < \varepsilon + \|x\|. \qquad \blacksquare$$

Proposition 4.3 *Let X, non-empty X^∞, $\mathbf{Y}$ and τ be as before. Then the identity morphism $\mathbf{1}_X$ has no bounded lifting across τ. In other words, there is no bounded morphism of A-modules $\psi\colon X \to \mathbf{Y}$ making the diagram*

$$\begin{tikzcd} & \mathbf{Y} \arrow[d, "\tau"] \\ X \arrow[ur, "\psi"] \arrow[r, "\mathbf{1}"'] & X \end{tikzcd}$$

commutative.

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Proof Let ψ be an (algebraic) morphism of A-modules, being a lifting of $\mathbf{1}_X$ across τ. Fix an arbitrary $x \in X^\infty$. Denote, for brevity, $\psi(x)$ by $\mathbf{y}$ and write it as $\mathbf{y} = \{y^{x_k}; k = 1, \dots, m, y^*\}$ (see above). Since ψ is a morphism, $\psi(x_n)$ coincides with $\mathbf{y}_n$ and thus can be written as $\{y_n^{x_k}; k = 1, \dots, m, y_n^*\}$, where $y_n^{x_k} \in (Y^{x_k})_n$ and $y_n^* \in X_n$. Since y^* is a finite sequence, the equality (4.2) transforms to $\|\mathbf{y}_n\| = \sum_{k=1}^m \|y_n^{x_k}\|$ for sufficiently large n. But by (4.3) for these n we have also

$$x_n = \tau\psi(x_n) = \tau(\mathbf{y}_n) = \sum_{k=1}^{m} (\lambda_n^{x_k})^{-1} y_n^{x_k}.$$

Therefore for all sufficiently big n we have

$$\|x_n\| \leq \zeta_n \sum_{k=1}^{m} \|y_n^{x_k}\| = \zeta_n \|\mathbf{y}_n\|_{\mathbf{Y}},$$

where we set $\zeta_n := \max\{(\lambda_n^{x_k})^{-1}; k = 1, \dots, m\}$. Since ζ_n tends to 0, the set of numbers $\|\psi(x_n)\|_{\mathbf{Y}} / \|x_n\|$, taken over all n with $x_n \neq 0$, is not bounded. This shows, of course, that the morphism ψ is not bounded. ■

Recall that the module $\mathbf{Y}$ above belongs, together with X, to the category $\mathcal{H}$. Therefore the last assertion of Theorem 1.1 is valid. Since every extremely projective module is topologically projective, we obtain the 'only if' part of the theorem. This completes the proof of Theorem 1.1.

The Proof of Theorem 1.2

'If' Part Suppose we are given a module $X \in \overline{\mathcal{H}}$ with $X_n = l_1(\Lambda_n); n = 1, 2, \dots$. Consider its submodule P of finite type with $P_n := l_1^0(\Lambda_n)$. Of course, P belongs to $\mathcal{H}$, and hence, by Theorem 4.1, it is extremely projective with respect to that category. But the completion of P is obviously our initial X. Therefore, by Propositions 2.6 and 2.2 combined, X is extremely projective with respect to $\overline{\mathcal{H}}$.

'Only If' Part Suppose that $X \in \overline{\mathcal{H}}$ is extremely projective with respect to $\overline{\mathcal{H}}$. Clearly, $\overline{\mathcal{H}}$ satisfies the condition on $\mathcal{K}$, formulated in Proposition 2.5, and $\mathcal{K}_n$, for $\mathcal{K} := \overline{\mathcal{H}}$, is **Ban**. Consequently, Proposition 2.5 implies that for every n, X_n is extremely projective in **Ban**. It remains to apply the Grothendieck Theorem, formulated in the introduction.

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