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On Classification of Certain C*-Algebras

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Abstract. We consider C^* -algebras which are inductive limits of finite direct sums of copies of $C([0, 1]) \otimes O_2$. For such algebras, the lattice of closed two-sided ideals is proved to be a complete invariant.

1 The Problem and the Result

We consider the following class of C^* -algebras: inductive limits of finite direct sums of copies of $C([0, 1]) \otimes \mathcal{O}_2$, where \mathcal{O}_2 denotes the Cuntz algebra with two generators. Therefore, an algebra *A* from this class can be represented as the limit

$$A_1 \to A_2 \to \cdots \to A$$

with each A_i being isomorphic to

$$\bigoplus_{j=1}^{n_i} C([0,1]) \otimes \mathbb{O}_2$$

We prove that a complete invariant for this class of C^* -algebras is the lattice of closed two-sided ideals of the algebra. More precisely, we have proved the following theorem.

Theorem 1 Let A and B be two C^* -algebras as above. If their lattices of closed two-sided ideals J(A) and J(B) are isomorphic as lattices, then the C^* -algebras A and B are isomorphic.

2 Partial Case Considered by J. Mortensen

Jacob Mortensen [3] has solved the above problem in a particular case. The invariant is the same, namely the lattice of closed two-sided ideals $\mathcal{I}(A)$ of the algebra A. The problem was solved only for the algebras with totally ordered ideals.

Theorem 2 (Mortensen's Classification Theorem) (See [3, Theorem 5.1.1]) Let A and B be two C^* -algebras as above, and assume that J(A) and J(B) are totally ordered. If the lattices J(B) and J(A) are isomorphic, then the algebras A and B are isomorphic.

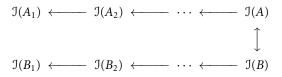
Sketch of Mortensen's Proof Suppose *A* and *B* are two algebras as above, and assume that $\mathcal{I}(A) \cong \mathcal{I}(B)$.

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Main "tool": for a homomorphism of C^* -algebras $\varphi \colon C \to D$ one can construct a map $\hat{\varphi} \colon \mathfrak{I}(D) \to \mathfrak{I}(C)$, namely: for each $I \in \mathfrak{I}(D)$, one puts $\hat{\varphi}(I) \equiv \varphi^{-1}(I)$.

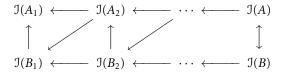
(Remark: the map $\hat{\varphi}$ is not in general a homomorphism of lattices, it is only infimum-preserving.)

Then, one obtains the following diagram:



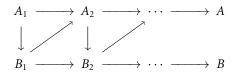
where the horizontal arrows come from the construction of *A* and *B*, and the vertical arrow represents the given isomorphism of lattices.

Mortensen completes this diagram as follows:

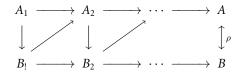


to get an approximately commuting diagram (after passing to a subsequence and renumbering). In this process, he strongly relays on the condition of total ordering of $\mathcal{I}(A)$ and $\mathcal{I}(B)$.

Then, he uses his existence and uniqueness theorems to "lift" each map of the type $\mathcal{I}(D) \to \mathcal{I}(C)$ to the corresponding C^* -algebra homomorphism $C \to D$. The whole diagram above can be lifted to the corresponding diagram for C^* -algebras:



Mortensen manages to do this in such a way that the resulting diagram is the approximate intertwining in the sense of Elliott [2]. Therefore, there exists an isomorphism ρ between the limit C^* -algebras A and B completing the above diagram:



Finally, one can prove that the map $\hat{\rho} \colon \mathfrak{I}(B) \to \mathfrak{I}(A)$ coming from ρ as above coincides with the given map.

3 General Case

Our goal in this section is to prove Theorem 1 which generalizes Mortensen's Theorem 2.

Remark We don't assume anymore that $\mathcal{J}(A)$ and $\mathcal{J}(B)$ are totally ordered.

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3.1 New "Tool"

To eliminate the condition of total ordering in the Mortensen's setting, we add another "tool". For a C^* -algebra homomorphism $\varphi \colon C \to D$ we consider the map between the lattices of ideals $\check{\varphi} \colon \mathfrak{I}(C) \to \mathfrak{I}(D)$ acting in forward direction (while $\hat{\varphi}$ is acting in backward direction). The map $\check{\varphi}$ is defined naturally: for each $I \in \mathfrak{I}(C), \check{\varphi}(I)$ is the ideal in $\mathfrak{I}(D)$ generated by the image $\varphi(I)$.

Remarks 1. The map $\check{\varphi}$ is supremum-preserving.

2. The maps $\hat{\varphi}$ and $\check{\varphi}$ are not (in general) inverses of each other, but they determine each other by simple formulas. Namely, for $I \in \mathfrak{I}(C)$:

$$\check{\varphi}(I) = \inf\{J \in \mathcal{I}(D) \mid I \subseteq \hat{\varphi}(J)\}.$$

Analogously, for $J \in \mathcal{I}(D)$:

$$\hat{\varphi}(J) = \sup\{I \in \mathfrak{I}(C) \mid \check{\varphi}(I) \subseteq J\}.$$

Also, the connection between $\hat{\varphi}$ and $\check{\varphi}$ can be expressed in the following formula: for $I \in \mathfrak{I}(C)$, $J \in \mathfrak{I}(D)$:

$$I \subseteq \hat{\varphi}(J) \iff \check{\varphi}(I) \subseteq J$$

3. Mortensen gives the intrinsic description of the maps $\hat{\varphi}$: these are the infimumpreserving maps from $\mathcal{I}(D)$ to $\mathcal{I}(C)$ which are continuous in the Hausdorff metric on subsets of [0, 1]. We don't know such an intrinsic definition for the maps $\check{\varphi}$.

Suppose that *A* and *B* are as above. We use the following notation: φ_{ij} denotes the (given) homomorphism between the finite stage algebras A_i and A_j , φ_i denotes the homomorphism from A_i to the limit algebra A, ψ_{ij} and ψ_i have the same meaning for B_i and *B*.

Assume that there is a lattice isomorphism $\Psi \colon \mathfrak{I}(A) \to \mathfrak{I}(B)$. From these data we get the following diagram for the lattices:

where again the horizontal arrows come from the structure of *A* and *B*, while the vertical arrow represents the given isomorphism.

3.2 New Metrics on the Lattices $\mathcal{J}(A_n)$, $\mathcal{J}(A)$, $\mathcal{J}(B_m)$, $\mathcal{J}(B)$

We choose the new metrics as follows. Find a countable dense set $D_n = \{d_{n,1}, d_{n,2}, ...\}$ in the unit ball of each algebra A_n , so that the union D of images of all D_n 's in A is dense in the unit ball of A.

Let $l: \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ be the function of "counting by diagonals", defined as follows: l(n,m) = (n+m-1)(n+m-2)/2 + n.

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(1)

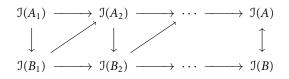
For $I, J \in \mathcal{J}(A)$ and $d \in D$ let $||d||_I = ||d + I||$ in A/I. Let $D_d(I, J) = |||d||_I - ||d||_J|$. Finally, let $D(X, Y) = \sum_{n,m} D_{d_{n,m}}(I, J) \cdot 2^{-l(n,m)}$.

Analogously, for $I, J \in \mathcal{J}(A_k)$ and $d \in D_k$ let $||I||_d = ||d + I||$ in A/I and $D_d(I, J) = ||I||_d - ||J||_d|$. Then, let $D(X, Y) = \sum_{n \le k, \text{ all } m} D_{d_{n,m}}(I, J) \cdot 2^{-l(n,m)}$. Here, the elements from D(n) with n < k are identified with their images in A_k .

We choose the metrics for $\mathcal{J}(B_n)$ and $\mathcal{J}(B)$ in an analogous way.

3.3 Building a "Forwards" Intertwining Map

We will complete the diagram (1) to get the following intertwining diagram:



where the intertwining maps are being built inductively in a special way.

For simplicity we will always assume that all "finite stage" algebras A_i and B_i are isomorphic to $C([0, 1], \mathcal{O}_2)$.

We begin with building a single intertwining map.

For a given finite stage A_i and a given positive number δ we will choose a certain finite subset $F \subset \mathcal{I}(A_i)$ as follows. Elements of F correspond to the open intervals in the spectrum, such that the union of all the intervals is the whole segment [0, 1], every interval has the length of δ , every interval is contained in the union of its neighbors, and the length of the intersection of any two neighboring intervals is at most $2\delta/3$.

Elements of *F* has a natural order, we will define them by f_1, f_2, \ldots, f_k .

It follows that every interval is contained in a compact set (denoted by K_i) which is contained in the union of its neighbors.

Proposition 3 Let F be a finite subset in $\mathbb{J}(A_n)$ as above and ε be a positive number. Let G be another finite set in $\mathbb{J}(B_{m_0})$ (see the diagram below).

There exist $m > m_0$ *and a map* $\Phi \colon F \to \mathcal{J}(B_m)$ *satisfying the following properties:*

1. In the the following diagram

the square is commutative on elements from F up to ε ; 2. for every $i \leq k$, there exists a compact set M_i such that

$$\Phi(f_i) \subset M_i \subset \sup\{\Phi(f_{i-1}), \Phi(f_{i+1})\}$$

where the ideals are identified with the corresponding open subsets of [0, 1];

3. for every $f \in F$ and every $g \in G$ such that $\Psi \circ \check{\varphi}_n(f) \subseteq \check{\psi}_{m_0}(g)$ one has: $\Phi(f) \subseteq \check{\psi}_{m_0,m}(g)$ (*i.e.*, if the image of f at infinity is contained in the image of g, then the same inclusion holds at the *m*-th stage).

Proof We will construct the images of the elements from *F* by several successive "adjustments".

One may assume that *G* consists of a single element *g*. To satisfy condition (3.) it's enough to construct the image of *f* within the image of *g*. To satisfy also condition (1.), it's enough to choose *m* sufficiently large, so that for all $f \in F$ the distance between $\dot{\psi}_m \circ \hat{\psi}_m \circ \Psi \circ \check{\varphi}_n(f)$ and $\check{\varphi}_n(f)$ is smaller than $\varepsilon/2$, and the distance between $\hat{\psi}_m \circ \Psi \circ \check{\varphi}_n(f)$ and $\hat{\psi}_m \circ \Psi \circ \check{\varphi}_n(f) \cap \check{\psi}_{m_0,m}(g)$ is also smaller than $\varepsilon/2$. So, the "first" approximation for $\Psi(f)$ will be $\hat{\psi}_m \circ \Psi \circ \check{\varphi}_n(f) \cap \check{\psi}_{m_0,m}(g)$. All successive approximations will be made within it, to preserve condition (3.).

Now we will choose the images of the elements of *F* to satisfy condition (2.). Here and further $||x||_I$ stands for the norm of x + I in A/I, as before. For each *i*, let x_i be a positive element in $\sup\{f_{i-1}, f_{i+1}\}$ and a_i be a positive number such that $K_i = \{I \in \mathcal{I}(A_n) \mid ||x||_I \ge a_i\}$. (Such x_i and a_i evidently exist.) Let $x'_i = \varphi_n(x_i)$. Let K'_i be the subset of the primitive spectrum of *A* defined by $K'_i = \{I \in \text{Prim } A \mid ||x'_i||_I \ge a_i\}$. By [1, 3.3.7], K'_i is compact in the Jacobson topology. Moreover, one checks immediately that $\check{\varphi}_m(f_i) \subset K'_i \subset \sup\{\check{\varphi}_m(f_{i-1}),\check{\varphi}_m(f_{i+1})\}$. (Here again the ideals are identified with the corresponding open subsets in the spectrum of *A*.)

The lattice isomorphism Ψ is a homeomorphism on the level of the primitive spectra with the Jacobson topology. Therefore, the images of all K'_i under Ψ are also compact. Moreover, as the function $I \mapsto ||x||_I$ is lower semi-continuous (see [1, 3.3.2]), every compact set is contained in a compact set of the above type, *i.e.*, there exist positive elements $z'_i \in \sup\{\Psi \circ \check{\varphi}_m(f_{i-1}), \Psi \circ \check{\varphi}_m(f_{i+1})\}$ and positive numbers b_i such that each $\Psi(K'_i)$ is contained in $L'_i = \{I \in \operatorname{Prim} B \mid ||z'_i||_I \ge b_i\}$. One may assume that all z'_i are the images of some elements of some finite stage algebra B_m . (Denote these latter elements by z_i .) Moreover, one may assume that each z_i belongs to the respective "first approximation" for the image of f_i .

Let g_i be a continuous function such that $g_i(\lambda) = 0$ if $\lambda \ge b_i/2$ and $g_i(\lambda) > 0$ if $\lambda > b_i/2$. Let $y_i = g_i(z_i)$ and $y'_i = \psi_m(y_i)$. (Of course $y'_i = g_i(z'_i)$). The ideal in B_m generated by y_i (denote this ideal by Y_i) corresponds to the open set $\{I \in \text{Prim } B_m \mid ||z_i||_I > b_i/2\}$. Let $Y'_i = \check{\psi}_m(Y)$. The ideal Y'_i is generated by y'_i . One can also check that Y'_i corresponds to the open set $\{I \in \text{Prim } B \mid ||z'_i||_I > b_i/2\}$. Hence, Y'_i contains $\Psi \circ \check{\varphi}_n(f_i)$.

Therefore, each z'_i is contained in the ideal generated by y'_{i-1} and y'_{i+1} . Hence, by choosing *m* large enough, one can achieve that each z_i is approximately contained in the ideal generated by y_{i-1} and y_{i+1} , and the discrepancy is less than the smallest of the numbers $b_i/4$. Then, for each z_i there exists an approximation \tilde{z}_i which belongs to the ideal generated by y_{i-1} and y_{i+1} .

Let \tilde{g}_i be a continuous function such that $\tilde{g}_i(\lambda) = 0$ if $\lambda \ge b_i/4$ and $g_i(\lambda) > 0$ if $\lambda > b_i/4$. Let $\tilde{y}_i = \tilde{g}_i(\tilde{z}_i)$ and $\tilde{y}'_i = \psi_m(\tilde{y}_i)$. (Then again $\tilde{y}'_i = \tilde{g}_i(\tilde{z}'_i)$.) The ideal generated by y_i is contained in the ideal generated by \tilde{y}_i . (Denote the latter ideal by \tilde{Y}_i .) It follows that each \tilde{z}_i is contained in the ideal generated by \tilde{y}_{i-1} and \tilde{y}_{i+1} . Now, for each *i* we define $\Phi(f_i)$

to be $(\hat{\psi}_m \circ \Psi \circ \check{\varphi}_n(f_i)) \cap \tilde{Y}_i$. Then condition (2.) is satisfied, with M_i defined as follows:

$$M_i = \{I \in \mathcal{I}(B_m) \mid \|\tilde{z}_i\|_I \ge b_i/4\}$$

Corollary 4 There exists a map $\tilde{\Phi}$: $F \to \mathcal{J}(B_m)$ satisfying all the conditions for Φ in Proposition 3, and in addition such that all the open subsets corresponding to all $\tilde{\Phi}(f_i)$ satisfy the following conditions:

- 1. every such subset is a union of a finite number of intervals;
- 2. endpoints of different subsets don't coincide.

Proof Every open set corresponding to $\Phi(F_i)$ is the union of countably many intervals. One can choose finitely many of them whose union still covers the compact set M_i . Moreover, one can decrease some of the intervals if necessary to make their endpoints different. Take the ideal obtained this way for $\tilde{\Phi}(f_i)$. If the approximations made are close enough, the diagram (3) with $\tilde{\Phi}$ instead of Φ is still approximately commutative.

3.4 Building the Whole "Forwards" Intertwining Diagram

Starting with $\varepsilon = 1/2$ we get $F_1 \subset \mathcal{J}(A_1)$, as before. Then we choose B_m as in Lemma 3 and renumber it as B_1 . (On this stage, we take $G = \emptyset$.)

Then we choose a finite set (denote it by G_1) in $\mathcal{I}(B_1)$ in the same way as F_1 , but in addition so that for every $I \in F_1$, the ideal $\Phi_1(I)$ is the supremum of some elements of G_1 .

Then we apply the same procedure to $\mathcal{I}(B_1)$ with $\varepsilon = 1/4$. Now we take F_1 for the set G in Proposition 3. We get the following diagram:

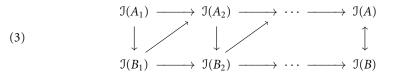
(2)
$$\begin{array}{cccc} \mathbb{J}(A_1) & \longrightarrow & \mathbb{J}(A_2) & \longrightarrow & \mathbb{J}(A) \\ & & & & & \uparrow \\ \mathbb{J}(B_1) & \longrightarrow & \mathbb{J}(B) \end{array}$$

In this diagram, the "horizontal" map $\mathfrak{I}(A_1) \to \mathfrak{I}(A_2) \to \mathfrak{I}(A)$ is approximately equal to the map $\mathfrak{I}(A_1) \to \mathfrak{I}(B_1) \to \mathfrak{I}(B) \to \mathfrak{I}(A)$, which is approximately equal to the map $\mathfrak{I}(A_1) \to \mathfrak{I}(B_1) \to \mathfrak{I}(A_2) \to \mathfrak{I}(A)$. Therefore, the map $\mathfrak{I}(A_1) \to \mathfrak{I}(A_2) \to \mathfrak{I}(A)$ is approximatively equal to the map $\mathfrak{I}(A_1) \to \mathfrak{I}(B_1) \to \mathfrak{I}(B_1) \to \mathfrak{I}(A_2) \to \mathfrak{I}(A)$. Also, the image of every $f \in F_1$ under the latter map is contained in its image under the former map. By construction of the metric on $\mathfrak{I}(A)$ and also because of finite domains of all maps in question, these two maps are approximately equal on some finite stage, *i.e.*, there exists an integer *n* such that the map $\mathfrak{I}(A_1) \to \mathfrak{I}(A_2) \to \mathfrak{I}(A_n)$ is approximately equal to $\mathfrak{I}(A_1) \to \mathfrak{I}(B_1) \to \mathfrak{I}(A_2) \to \mathfrak{I}(A_n)$, with the same condition of inclusion. We renumber A_n as A_2 .

Lemma 5 The triangle in diagram (2) satisfy the following condition: for every $f \in F_1$, the image of f under the map $\mathfrak{I}(A_1) \to \mathfrak{I}(B_1) \to \mathfrak{I}(A_2)$ is contained in the image of f under the map $\check{\varphi}_{1,2}$: $\mathfrak{I}(A_1) \to \mathfrak{I}(A_2)$.

Proof For $f \in F_1$, let its image in $\mathfrak{I}(B_1)$ be the supremum of $g_1, g_2, \ldots, g_k \in \mathfrak{I}(B_1)$. By the construction, the images of all g_1, g_2, \ldots, g_k under $\Psi^{-1} \circ \check{\psi}_1$ are contained in $\check{\varphi}_1(f)$. By Proposition 3, their images under the map $\mathfrak{I}(B_1) \to \mathfrak{I}(A_2)$ are contained in $\check{\varphi}_{1,2}(f)$. But the image of f under the map $\mathfrak{I}(A_1) \to \mathfrak{I}(B_1) \to \mathfrak{I}(A_2)$ is their supremum.

This procedure can be repeated with ε 's summing up to a finite sum, to get the following intertwining diagram:



3.5 Building a Single "Backwards" Map

Let $C, D \in (A_n)_{n=1}^{\infty} \cup (B_n)_{n=1}^{\infty}$. Through this Subsection, we will identify ideals in *C* or *D* with the corresponding open subsets of [0, 1].

Let $\varepsilon > 0$, and let $F \subset \mathcal{I}(C)$ be a finite subset as chosen above. This set has a natural order; let $F = \{f_i\}_{i=1}^k$. Let $\Phi: F \to \mathcal{I}(D)$ be an arbitrary map.

We will build the corresponding "backwards" everywhere defined map $\Psi: \mathfrak{I}(D) \to \mathfrak{I}(C)$. Everywhere we identify the ideals with the corresponding open sets—their open supports.

Elements from the image $\Phi(F)$ correspond to open subsets of [0, 1]. By the conditions above, these open sets consist of finite number of open intervals with different endpoints. These intervals break the whole segment [0, 1] into the disjoint union of a finite number of intervals which may be open or closed or half-open. Denote the set of these intervals by R. For each interval $r \in R$, denote the middle point of r by m_r . Let P be the set of all these middle points.

It's enough to define Ψ only on maximal ideals corresponding to the open subsets of the type $S_t = [0,t) \cup (t,1]$ and make sure it is continuous in the Hausdorff metric. (See [3, Proof of Theorem 4.3.1].)

For every $p \in P$, we put $\Psi(S_p)$ to be the union of those elements of *F* whose images do not contain the point *p*. Then, $\Psi(S_p)$ is a certain open set.

Moreover, for neighboring points $p, q \in P$, the images $\Psi(S_p)$ and $\Psi(S_q)$ are at most ε apart in the Hausdorff metric in $\Im(C)$. Indeed, these two images are different by exactly one small interval from F, say f_i . This interval can bring to a large jump with respect to the Hausdorff metric only in one case: namely, if the interval f_i covers a gap. In any other case, the jump would be small. But if this case happens, it means that both $\Psi(S_p)$ and $\Psi(S_q)$ don't contain at least one of the neighbors of f_i . (Because if they contained both of them, the gap wouldn't exist.) Suppose these sets don't contain f_{i-1} . Then, they must contain f_{i-2} (unless we are doing near the left border) because otherwise the gap would be too large to be covered by f_i . But this means that after adding (or before subtracting) f_i , the union of the intervals would contain both f_{i-2} and f_i but not contain f_{i-1} . This is a contradiction: if the images of both f_{i-2} and f_i don't cover a certain point (p or q), the image of f_{i-1} shouldn't do either.

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Finally, we will define Ψ on all remaining S_t 's by interpolation, making it continuous. We will perform the interpolation as follows. Let $p, q \in P$ be two neighboring points, corresponding to the neighboring intervals $p', q' \in R$. Let a be the common endpoint of p' and q'. Assume that $a \in p'$, that $\Psi(S_q) = \Psi(S_p) \cup (b, d)$, and that $\Psi(S_p) \cap (b, d) = (b, c)$. (All other cases are considered analogously.) For all $t \in (p, a]$ we put $\Psi(S_t) \equiv \Psi(S_p)$, and for $t \in (a, q)$ we define $\Psi(S_t) \equiv \Psi(S_p) \cup (b, c + (d - c)(t - a)/(q - a))$. One checks that this is a continuous interpolation such that the resulting backwards map Ψ satisfies the following property: for every $f \in F$: $\Phi(f) = \inf\{I \mid I \subset \Psi(f)\}$. In other words, the "forwards" map $\mathfrak{I}(C) \to \mathfrak{I}(D)$ derived from Ψ as described in Subsection 3.1 extends the map Φ .

3.6 Building the Whole "Backwards" Intertwining Diagram

Proposition 6 Let $C, D \in (A_n)_{n=1}^{\infty} \cup (B_n)_{n=1}^{\infty}$. Suppose that the lattice $\mathfrak{I}(C)$ is equipped with the Hausdorff metric, while the lattice $\mathfrak{I}(D)$ is equipped with an arbitrary metric, in which it is a compact space. Let $\varepsilon > 0$. Let F be the finite subset of $\mathfrak{I}(C)$ representing covering of [0,1] by segments of length ε . Let Ψ_1 and Ψ_2 be two continuous infimum-preserving maps from $\mathfrak{I}(D)$ to $\mathfrak{I}(C)$. Let δ be the modulus of uniform continuity of the map Ψ_1 corresponding to $\varepsilon/2$. Let Θ_1 and Θ_2 be the maps from $\mathfrak{I}(C)$ to $\mathfrak{I}(D)$ corresponding to Ψ_1 and Ψ_2 respectively as in Subsection 3.1. Suppose that for every $f \in F: \Theta_2(f) \subseteq \Theta_1(f)$, and the distance between $\Theta_2(f)$ and $\Theta_1(f)$ is not more than δ . Then for every $I \in \mathfrak{I}(D)$, the distance between $\Psi_1(I)$ and $\Psi_2(I)$ is not more than 2ε .

Proof Let $I \in \mathcal{I}(D)$. Let $J = \Psi_1(I)$ and $K = \Psi_2(I)$. Let $F_J = \{f \in F \mid f \notin J\}$ and $F_K = \{f \in F \mid f \notin K\}$. If $F_J = F_K$ then by definition of the Hausdorff metric, the distance between *J* and *K* is not more than ε . By the condition of inclusion above, we always have that $F_K \subseteq F_J$. Indeed, if an interval *f* isn't contained in *K*, then $\Theta_2(f)$ isn't contained in *I*, therefore $\Theta_1(f)$ (which is larger) isn't contained in *I* either, so *f* isn't contained in *J*.

Now suppose that $f \in F_J \setminus F_K$. If for each such f the set F_K includes at least one of the neighbors of f, then the distance between J and K is not more than 2ε . So, we can assume that F_K includes neither f nor its neighbors. Let g be the supremum of f and its neighbor(s). Then $\Theta_2(g) \subseteq I$. Let $L = \Theta_1(g)$ and $M = \Theta_2(g)$. Then the distance in $\mathcal{I}(D)$ between L and M is not more than δ . Therefore, the distance in $\mathcal{I}(C)$ between $\Psi_1(L)$ and $\Psi_1(M)$ should be no more than ε . But $M \subseteq I$, therefore $\Psi_1(M) \subseteq \Psi_1(I) = J$. In particular, f isn't contained in $\Psi_1(M)$. On the other side, g is contained in $\Psi_1(M)$ is at least $2\varepsilon/3$. This is a contradiction.

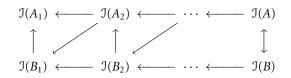
Proposition 7 For every $\varepsilon > 0$ there exists $\delta > 0$ such that for every *n* and every two ideals $I, J \in \mathcal{J}(A_n)$ lying at the distance less than δ from each other in terms of the metric defined in Subsection 3.2, the Hausdorff distance between the preimages of I and J in $\mathcal{J}(A_1)$ is less than ε . (In other words, all the maps $\mathcal{J}(A_n) \to \mathcal{J}(A_1)$ have the common modulus of uniform continuity.)

Proof First, suppose that both $\mathcal{I}(A_n)$ and $\mathcal{I}(A_1)$ are equipped with the metric defined in Subsection 3.2. Then the map $\mathcal{I}(A_n) \to \mathcal{I}(A_1)$ mapping every ideal to its preimage is a

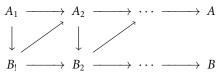
contraction. Indeed, for each $I \in \mathcal{I}(A_n)$, let J be the preimage of I in $\mathcal{I}(A_1)$. The homomorphism $A_1/J \to A_n/I$ (which is induced from the given homomorphism $A_1 \to A_n$) is one-to-one, therefore isometric. Therefore for every $d \in A_1$: $||d||_J = ||d||_I$. (Here d is identified with its image in A_n .) Therefore, the distance between I and J is larger than the distance between their preimages, as the former contains the same terms as the latter does, plus some additional terms.

Now, it suffices to let δ be the modulus of uniform continuity of the identity map from $\mathcal{I}(A_1)$ with the metrics coming from elements to itself with the Hausdorff metric, corresponding to ε .

Now we will build the "backwards" intertwining diagram analogous to the "forwards" diagram (3). Letting ε be subsequently equal to 1/2, 1/4, 1/8 *etc.*, we find the corresponding values of δ in accordance with Proposition 7. By passing to an appropriate sub-diagram in (3), we can achieve that the tolerances of the triangles are not more than these values of δ . Then, subsequently applying Proposition 6, we obtain the intertwining backwards diagram like in the Mortensen's case:



Applying Mortensen's existence and uniqueness theorems to every intertwining map in the above diagram, we can build the corresponding approximate intertwining of the C^* -algebras:



This gives the isomorphism $\rho: A \to B$.

3.7 $\check{\rho} = \Psi$

Proposition 8 The map $\check{\rho}$: $\mathfrak{I}(A) \to \mathfrak{I}(B)$ arising from the isomorphism ρ as above coincides with the given isomorphism Ψ .

Proof First, we prove that for $I \in \mathcal{J}(A)$, $\check{\rho}(I) \subseteq \Psi(I)$. For this, it's enough to check that $\rho(I) \subseteq \Psi(I)$. Let $x \in I$. Then $\rho(x) \in \check{\rho}(I)$.

May suppose that x is the image of some $y \in A_n$. Moreover, up to arbitrarily small ε , $\rho(x)$ is the image of the same y. Denoting all the images of the element y in all A_m by the same letter y, and denoting all the preimages of I by the same letter Y, we have:

$$||Y||_{y} = 0$$

Passing to the images of y and Y in B_m and denoting them again by the same letters y and Y, we have: $||Y||_y$ is arbitrarily small in B_m for sufficiently large m. On the other

hand, by construction of the intertwining we have: $||Y||_{y} \to ||\Psi(I)||_{\rho(y)}$ as $m \to \infty$. So, $||\Psi(I)||_{\rho(y)} = 0$ and $\rho(y) \in \Psi(I)$. Therefore, $\check{\rho}(I) \subseteq \Psi(I)$.

Now we have:

$$\rho(\rho^{-1}(I)) = I$$
$$\rho(\hat{\rho}(I)) = I$$
$$\check{\rho}(\hat{\rho}(I)) = I$$

So, $\check{\rho}(J) = (\hat{\rho})^{-1}(J)$ for $J \in \mathfrak{I}(B)$. In addition, $\widehat{\rho^{-1}}(I) = \rho(I) = \check{\rho}(I)$. Therefore, $\widehat{\rho^{-1}}(I) \subseteq \Psi(I)$.

Exchanging the places of *A* and *B* we get the same results with ρ^{-1} instead of ρ and Ψ^{-1} instead of Ψ . Hence, for $J \in \mathcal{J}(B)$: $(\check{\rho})^{-1}(J) = \hat{\rho}(J) \subseteq \Psi^{-1}(J)$.

Therefore, all the four maps: $\check{\rho}$, $(\check{\rho})^{-1}$, Ψ , and Ψ^{-1} preserve inclusions. Let $J = \Psi(I)$, $K = \check{\rho}(I)$, $L = \Psi^{-1}(J)$, $M = \Psi^{-1}(K)$, and $N = (\check{\rho})^{-1}(K)$. We have:

1.
$$L = \Psi^{-1}(\Psi(I)) = I$$
 and $N = (\check{\rho})^{-1}(\check{\rho}(I)) = I$;

2.
$$N = (\check{\rho})^{-1}(K) \subseteq \Psi^{-1}(K) = M;$$

3. $M = \Psi^{-1}(K) \subseteq \Psi^{-1}(J) = L.$

Therefore, N = M = L = I and hence J = K.

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