Bull. Aust. Math. Soc. **84** (2011), 177–185
doi:10.1017/S0004972711002024

ON HOMOLOGICAL PROPERTIES FOR SOME MODULES OF UNIFORMLY CONTINUOUS FUNCTIONS OVER CONVOLUTION ALGEBRAS

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(Received 22 January 2010)

Abstract

For a locally compact group G, let LUC(G) denote the space of all left uniformly continuous functions on G. Here, we investigate projectivity, injectivity and flatness of LUC(G) and its dual space LUC$(G)^*$ as Banach left modules over the group algebra as well as the measure algebra of G.

2010 *Mathematics subject classification*: primary 43A07; secondary 43A15, 43A20, 46H05, 46H25.

Keywords and phrases: amenability, flatness, injectivity, projectivity, Banach module, locally compact group.

1. Introduction and preliminaries

Let G denote a locally compact group with identity element e, modular function Δ, and a fixed left Haar measure λ. As usual, let $M(G)$ denote the measure algebra of G as defined in [4], endowed with the total variation norm $\|\cdot\|$ and the convolution product of measures, and let $\delta_x \in M(G)$ denote the Dirac measure at $x \in G$. Let also $L^1(G)$ denote the group algebra of G as defined in [4], equipped with the norm $\|\cdot\|_1$ and the convolution product $*$ of functions on G. Then $L^1(G)$ is a Banach $M(G)$-bimodule with the module actions defined by

$$(\phi \cdot \mu)(x) = \int_G \phi(xy^{-1})\Delta(y^{-1})\, d\mu(y)$$

and

$$(\mu \cdot \phi)(x) = \int_G \phi(y^{-1}x)\, d\mu(y)$$

for all $\phi \in L^1(G)$, $\mu \in M(G)$ and locally almost all $x \in G$. Now, let $L^\infty(G)$ denote the Lebesgue space as defined in [4], equipped with the essential supremum norm $\|\cdot\|_\infty$. Then $L^\infty(G)$ is the dual bimodule of the Banach $M(G)$-bimodule $L^1(G)$ under the pairing

$$\langle f, \phi \rangle = \int_G f(x)\phi(x)\, d\lambda(x)$$

for all $\phi \in L^1(G)$ and $f \in L^\infty(G)$. We denote by $\mathrm{LUC}(G)$ the space of all bounded left uniformly continuous functions on G and by $C_0(G)$ the space of all continuous functions on G vanishing at infinity. Then $\mathrm{LUC}(G)$ and $C_0(G)$ are closed submodules of the Banach $M(G)$-bimodule $L^\infty(G)$. Hence, the dual spaces $\mathrm{LUC}(G)^*$ of $\mathrm{LUC}(G)$ and $M(G)$ of $C_0(G)$ are Banach $M(G)$-bimodules with the dual actions; the $M(G)$-module actions of $M(G)$ are just the convolution product in $M(G)$.

In particular, $L^\infty(G)$ is a Banach $L^1(G)$-bimodule for which the left and right module actions of $L^1(G)$ on $L^\infty(G)$ are given by the formulae

$$\phi \cdot f = f * \widetilde{\phi} \quad \text{and} \quad f \cdot \phi = \frac{1}{\Delta}\widetilde{\phi} * f$$

for all $f \in L^\infty(G)$ and $\phi \in L^1(G)$, where

$$\widetilde{\phi}(x) = \phi(x^{-1})$$

for all $x \in G$. Moreover, $\mathrm{LUC}(G)$ and $C_0(G)$ are closed submodules of the Banach $L^1(G)$-bimodule $L^\infty(G)$. Thus, $\mathrm{LUC}(G)^*$ is a Banach $L^1(G)$-bimodule with the dual actions

$$\langle \phi \cdot H, h \rangle = \left\langle H, \frac{1}{\Delta}\widetilde{\phi} * h \right\rangle$$

and

$$\langle H \cdot \phi, h \rangle = \langle H, h * \widetilde{\phi} \rangle$$

for all $\phi \in L^1(G)$, $h \in \mathrm{LUC}(G)$ and $H \in \mathrm{LUC}(G)^*$. Similarly, $M(G)$ is a Banach $L^1(G)$-bimodule. Finally, we identify $L^1(G)$ with a closed ideal of $M(G)$ and consider $L^1(G)$ as a closed submodule of $M(G)$ whose module actions are the same as the convolution product $*$.

Dales and Polyakov [2] have characterized projectivity, injectivity and flatness of certain Banach left $L^1(G)$-modules; see also the recent works [1, 7]. In this work, we intend to characterize these homological properties for $\mathrm{LUC}(G)$ and $\mathrm{LUC}(G)^*$ as Banach left $L^1(G)$-modules and $M(G)$-modules in terms of some topological and algebraic properties of G.

2. The results

For two Banach spaces E and F, let $B(E, F)$ denote the Banach space of all bounded operators from E into F. An operator $T \in B(E, F)$ is called *admissible* if

$$T \circ S \circ T = T$$

for some $S \in B(F, E)$. In the case where A is a Banach algebra and E and F are Banach left A-modules, ${}_A B(E, F)$ denotes the closed linear subspace of $B(E, F)$ of

https://doi.org/10.1017/S0004972711002024 Published online by Cambridge University Press

all left A-module morphisms. An operator $T \in {}_A B(E, F)$ is a *retraction* if there exists $S \in {}_A B(F, E)$ with

$$T \circ S = I_F,$$

the identity operator on F; in this case, F is called a *retract* of E. A Banach left A-module P is called *projective* if for each Banach left A-modules E and F, each admissible epimorphism $T \in {}_A B(E, F)$ and each $S \in {}_A B(P, F)$, there exists $R \in {}_A B(P, E)$ such that

$$T \circ R = S.$$

We can now state our first result which characterizes projectivity of $\mathrm{LUC}(G)^*$ as a Banach left $L^1(G)$-module.

THEOREM 2.1. *Let G be a locally compact group. If $LUC(G)^*$ is a projective Banach left $L^1(G)$-module, then G is discrete and contains no infinite amenable subgroup.*

PROOF. Suppose that $\mathrm{LUC}(G)^*$ is a projective Banach left $L^1(G)$-module. On the one hand, $M(G)$ is a projective Banach left $L^1(G)$-module if and only if G is discrete; see [2, Theorem 2.6]. On the other hand, each retraction of a projective Banach left $L^1(G)$-module is projective; see [3]. To show that G is discrete we only need to prove that $M(G)$ is a retraction of the Banach left $L^1(G)$-module $\mathrm{LUC}(G)^*$.

To that end, define $\mathcal{Q} : M(G) \longrightarrow \mathrm{LUC}(G)^*$ to be the map that sends a measure μ in $M(G)$ to the integration functional

$$h \mapsto \int_G h \, d\mu$$

for all $h \in \mathrm{LUC}(G)$. This is well defined because h is continuous and corresponds to choosing the extension u of δ_e to be the functional on $\mathrm{LUC}(G)$ that evaluates a function at $e \in G$; in fact, $\mathcal{Q}(\mu) = \mu \cdot u$ for all $\mu \in M(G)$. Clearly, $\mathcal{Q}$ is a left $L^1(G)$-module morphism. Now, let

$$\mathcal{P} : \mathrm{LUC}(G)^* \longrightarrow M(G)$$

be the restriction map, and note that $\mathcal{P}$ is a left $L^1(G)$-module morphism. One can easily check that for every $\mu \in M(G)$ and $h \in C_0(G)$, the function $h \cdot \mu \in C_0(G)$ is given by

$$(h \cdot \mu)(x) = \int_G h(yx) \, d\mu(y)$$

for all $x \in G$, and we therefore have

$$\langle \mu \cdot u, h \rangle = \langle u, h \cdot \mu \rangle = (h \cdot \mu)(e) = \langle \mu, h \rangle.$$

That is, $\mathcal{Q}$ is a right inverse for $\mathcal{P}$, and thus $M(G)$ is a retraction of $\mathrm{LUC}(G)^*$.

In particular, since we now know that G is discrete,

$$\mathrm{LUC}(G)^* = L^\infty(G)^*.$$

So, the second part follows from the fact that G contains no infinite amenable subgroup if $L^\infty(G)^*$ is a projective Banach left $L^1(G)$-module; see [2, Theorem 2.7]. □

Let A be a Banach algebra and E be a Banach left A-module. Then the space $A \widehat{\otimes} E$ is a Banach left A-module with the action $a \cdot (b \otimes \xi) = ab \otimes \xi$ for all $a, b \in A$ and $\xi \in E$. Define the left A-module morphism $\pi : A \widehat{\otimes} E \longrightarrow E$ by the formula

$$\pi(a \otimes \xi) = a \cdot \xi$$

for $\xi \in E$ and $a \in A$. It is shown in [3, Proposition IV.1.1], that if E is essential as a Banach left A-module (that is, the linear span of $A \cdot E$ is dense in E), then E is projective if and only if the canonical morphism $\pi \in {}_A B(A \widehat{\otimes} E, E)$ is a retraction.

THEOREM 2.2. *Let G be a compact group. Then $LUC(G)^*$ is a projective Banach left $M(G)$-module.*

PROOF. It is clear that if G is compact, then $\mathrm{LUC}(G) = C_0(G)$ and so $\mathrm{LUC}(G)^* = M(G)$. So, the result follows from the fact that $M(G)$ is always a projective Banach left $M(G)$-module; indeed, if δ_e is the Dirac measure at $e \in G$, then the left $M(G)$-module morphism

$$\rho : \mu \mapsto \mu \otimes \delta_e, \quad M(G) \to M(G) \widehat{\otimes} M(G),$$

is a right inverse for the canonical morphism $\pi : M(G) \widehat{\otimes} M(G) \to M(G)$. □

We conjecture that the converse of Theorem 2.2 is also true. In proving the next result, we need the following consequence from [3, Corollary IV.4.5]. But first we recall that a Banach space E has *approximation property* if the identity operator on E can be approximated in the compact-open topology by finite dimensional operators.

PROPOSITION 2.3. *Let A be a Banach algebra and E be a Banach left A-module. Suppose that one of A and E has approximation property. Then for each $\xi \in E \setminus \{0\}$, there is $T \in {}_A B(E, A^b)$ such that $T(\xi) \neq 0$, where A^b is the unitization of A. In the case where E is essential, we may suppose that $T \in {}_A B(E, A)$.*

Using this result, we describe projectivity of $\mathrm{LUC}(G)$ as a Banach left $L^1(G)$- or $M(G)$-module.

THEOREM 2.4. *Let G be a locally compact group. Then the following statements are equivalent.*

(a) *$LUC(G)$ is a projective Banach left $L^1(G)$-module.*
(b) *$LUC(G)$ is a projective Banach left $M(G)$-module.*
(c) *G is compact.*

PROOF. (a) ⇒ (b). Suppose that $\mathrm{LUC}(G)$ is a projective Banach left $L^1(G)$-module. Then $\mathrm{LUC}(G)$ is a projective Banach left $M(G)$-module if we show that for each pair of Banach left $M(G)$-modules E and F, each admissible epimorphism $T \in {}_{M(G)} B(E, F)$ and each $S \in {}_{M(G)} B(\mathrm{LUC}(G), F)$ there exists $R \in {}_{M(G)} B(\mathrm{LUC}(G), E)$ such that

$$T \circ R = S.$$

https://doi.org/10.1017/S0004972711002024 Published online by Cambridge University Press

Since $\mathrm{LUC}(G)$ is a projective Banach left $L^1(G)$-module, there exists a left $L^1(G)$-morphism $R : \mathrm{LUC}(G) \longrightarrow E$ with

$$T \circ R = S.$$

To that end, we only need to show that R is a left $M(G)$-morphism. Choose a bounded approximate identity $(e_\gamma)_{\gamma\in\Gamma}$ in $L^1(G)$, and note that $(e_\gamma)_{\gamma\in\Gamma}$ is a left approximate identity for the Banach left $L^1(G)$-module $\mathrm{LUC}(G)$. Then for each $\mu \in M(G)$ and $h \in \mathrm{LUC}(G)$,

$$\begin{aligned} R(\mu \cdot h) &= \lim_\gamma R(\mu \cdot e_\gamma \cdot h) \\ &= \lim_\gamma \mu \cdot e_\gamma \cdot R(h) \\ &= \lim_\gamma \mu \cdot R(e_\gamma \cdot h) \\ &= \mu \cdot R\Big(\lim_\gamma e_\gamma \cdot h\Big) \\ &= \mu \cdot R(h). \end{aligned}$$

(b) $\Rightarrow$ (c). Suppose that $\mathrm{LUC}(G)$ is projective as a Banach left $M(G)$-module, and let $\varphi \in \mathrm{LUC}(G)$ be a function with compact support such that $0 \le \varphi \le 1$ and $\varphi(e) = 1$. Since $\mathrm{LUC}(G)$ is an essential Banach left $M(G)$-module and $M(G)$ has the approximation property, it follows from Proposition 2.3 that there exists a left $M(G)$-module morphism $T : \mathrm{LUC}(G) \longrightarrow M(G)$ such that

$$T(\varphi) \neq 0.$$

We may suppose that $T(\mathrm{LUC}(G)) \subseteq L^1(G)$; otherwise, we replace T by the map

$$h \longmapsto T(h) \cdot \phi$$

from $\mathrm{LUC}(G)$ into $L^1(G)$ for some function $\phi \in L^1(G)$ with $T(\varphi) \cdot \phi \neq 0$. Choose a natural number n with

$$2(\|T\| + 1) < n\|T(\varphi)\|_1.$$

Then there is a continuous function ψ with compact support such that

$$n\|T(\varphi) - \psi\|_1 < 1.$$

In particular, $\|T(\varphi)\|_1 < 2\|\psi\|_1$.

Now, suppose on the contrary that G is not compact. Then there exists $x_1, \ldots, x_n \in G$ such that the sets $x_i C$ are pairwise disjoint for $i = 1, \ldots, n$, where

$$C := \mathrm{supp}(\varphi) \cup \mathrm{supp}(\psi).$$

So, if we put $\mu = \delta_{x_1} + \cdots + \delta_{x_n}$, then

$$\|\mu * \varphi\|_\infty = 1 \quad \text{and} \quad \|\mu * \psi\|_1 = n\|\psi\|_1.$$

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Thus

$$\begin{aligned} n\|\psi\|_1 &= \|\mu * \psi\|_1 \\ &\le \|T(\mu * \varphi)\|_1 + \|\mu * (T(\varphi) - \psi)\|_1 \\ &\le \|T\|\|\mu * \varphi\|_\infty + n\|T(\varphi) - \psi\|_1 \\ &= \|T\| + 1 \\ &< \frac{n}{2}\|T(\varphi)\|_1. \end{aligned}$$

So, $\|T(\varphi)\|_1 \ge 2\,\|\psi\|_1$. This contradiction completes the proof.

(b) $\Rightarrow$ (c). This is proved in [2, Theorem 3.1]. □

Let A be a Banach algebra. A Banach left A-module I is called *injective* if for each Banach left A-modules E and F, each admissible monomorphism $T \in {}_AB(E, F)$ and each $S \in {}_AB(E, I)$, there exists $R \in {}_AB(F, I)$ such that

$$R \circ T = S.$$

Similar definitions apply for Banach right A-modules.

For each Banach left A-module E, the space $B(A, E)$ is a Banach left A-module with

$$(a \cdot T)(b) = T(ba)$$

for all $a, b \in A$ and $T \in B(A, E)$. Define the left A-module morphism $\Pi : E \longrightarrow B(A, E)$ by the formula

$$\Pi(\xi)(a) = a \cdot \xi$$

for $\xi \in E$ and $a \in A$. It is shown in [3, Proposition III.1.31], that if A is a Banach algebra, and E is faithful as a Banach left A-module (that is, $A \cdot \xi \ne \{0\}$ for all $\xi \in E \setminus \{0\}$), then E is injective if and only if there exists a left A-module morphism $\rho : B(A, E) \longrightarrow E$ with

$$\rho \circ \Pi = I_E.$$

THEOREM 2.5. *Let G be a locally compact group. Then the following statements are equivalent.*

(a) *$LUC(G)$ is an injective Banach left $L^1(G)$-module.*
(b) *$LUC(G)$ is an injective Banach left $M(G)$-module.*
(c) *G is discrete.*

PROOF. (a) $\Rightarrow$ (b). Suppose that E and F are two left $M(G)$-modules, $T : E \longrightarrow F$ is an admissible left $M(G)$-module monomorphism and $S : E \longrightarrow \mathrm{LUC}(G)$ is a left $M(G)$-module morphism. Then the injectivity of $\mathrm{LUC}(G)$ as a left Banach $L^1(G)$-module implies that there exists a left $L^1(G)$-module morphism $R : F \longrightarrow \mathrm{LUC}(G)$ with

$$R \circ T = S.$$

Since $\mathrm{LUC}(G)$ has a left bounded approximate identity $(e_\gamma)_{\gamma\in\Gamma}$ in $L^1(G)$, we have

$$\begin{aligned} R(\mu\cdot\xi) &= \lim_\gamma e_\gamma\cdot R(\mu\cdot\xi)\\ &= \lim_\gamma R(e_\gamma\cdot\mu\cdot\xi)\\ &= \lim_\gamma e_\gamma\cdot\mu\cdot R(\xi)\\ &= \mu\cdot R(\xi)\end{aligned}$$

for all $\mu\in M(G)$ and $\xi\in F$. So, R is also a left $M(G)$-module morphism.

(b) $\Rightarrow$ (c). Suppose that $\mathrm{LUC}(G)$ is an injective Banach left $M(G)$-module. Then there exists a left $M(G)$-module morphism

$$\rho_G : B(M(G),\mathrm{LUC}(G)) \longrightarrow \mathrm{LUC}(G)$$

such that $\rho_G\circ\Pi_G = I_{\mathrm{LUC}(G)}$, where

$$\Pi_G : \mathrm{LUC}(G) \longrightarrow B(M(G),\mathrm{LUC}(G))$$

is the canonical embedding defined by

$$\Pi_G(h)(\mu) = \mu\cdot h$$

for all $h\in\mathrm{LUC}(G)$ and $\mu\in M(G)$. Now, consider

$$Q : L^\infty(G) \longrightarrow B(M(G),\mathrm{LUC}(G))$$

with

$$Q(f)(\mu) = \mu_a\cdot f$$

for all $f\in L^\infty(G)$ and $\mu\in M(G)$, where μ_a is the absolutely continuous part of μ with respect to the left Haar measure. In particular,

$$Q(h)(\phi) = \Pi_G(h)(\phi)$$

for all $h\in\mathrm{LUC}(G)$ and $\phi\in L^1(G)$. The result will follow if we show that $\rho_G\circ Q : L^\infty(G)\longrightarrow\mathrm{LUC}(G)$ is projection on $\mathrm{LUC}(G)$; see [6, Theorem 4]. To show that

$$k := \rho_G(Q(h)-\Pi_G(h)) = 0$$

for all $h\in\mathrm{LUC}(G)$, choose a left bounded approximate identity $(e_\gamma)_{\gamma\in\Gamma}\subseteq L^1(G)$ for $\mathrm{LUC}(G)$. Since $k\in\mathrm{LUC}(G)$ and ρ_G is a left $M(G)$-module morphism,

$$k = \lim_\gamma e_\gamma\cdot k = \lim_\gamma \rho_G(e_\gamma\cdot Q(h) - e_\gamma\cdot\Pi(h)) = 0;$$

indeed, for each $\mu\in M(G)$ we have $\mu * e_\gamma\in L^1(G)$ for all $\gamma\in\Gamma$, and so

$$(e_\gamma\cdot Q(h) - e_\gamma\cdot\Pi_G(h))(\mu) = (Q(h)-\Pi_G(h))(\mu * e_\gamma) = 0.$$

(c) $\Rightarrow$ (a). This follows from the fact that $L^\infty(G)$ is always an injective Banach left $L^1(G)$-module and that $\mathrm{LUC}(G) = L^\infty(G)$ when G is discrete; see [2, Theorem 2.4]. □

Let A be a Banach algebra and let us recall that a Banach left A-module F is called *flat* if F^* is an injective Banach right A-module. Moreover, a locally compact group G is called *amenable* if there is a positive functional $m \in L^\infty(G)^*$ with $\|m\| = 1$ and $m \cdot \delta_x = m$ for all $x \in G$. The class of amenable groups includes all compact groups and all abelian locally compact groups; however, the discrete free group $\mathbb{F}_2$ on two generators is not amenable; see [8] for more details.

THEOREM 2.6. *Let G be a locally compact group. Then the following statements are equivalent.*

(a) *$LUC(G)$ is a flat Banach left $M(G)$-module.*
(b) *$LUC(G)$ is a flat Banach left $L^1(G)$-module.*
(c) *G is amenable.*

PROOF. (b) ⇔ (c). Suppose that G is amenable. Then by the classical result of Johnson [5], $L^1(G)$ is an amenable Banach algebra; that is,

$$H^1(L^1(G), E^*) = \{0\}$$

for all Banach $L^1(G)$-bimodules E. So, LUC(G) is a flat Banach left $L^1(G)$-module; this follows from the fact that if A is an amenable Banach algebra, then each Banach left or right A-module is flat; see [3, Theorem VII.2.29].

For the converse, suppose that LUC(G) is flat as a Banach left $L^1(G)$-module; that is, LUC$(G)^*$ is injective as a Banach right $L^1(G)$-module. An argument similar to the proof of Theorem 2.1 shows that the Banach right $L^1(G)$-module $M(G)$ is a retraction of LUC$(G)^*$. Thus $M(G)$ is also an injective Banach right $L^1(G)$-module; this is because each retraction of an injective Banach module is injective; see [3, Proposition III.1.16]. Therefore, G is amenable by [2, Corollary 4.7].

(a) ⇔ (b). Since the inclusion $\theta : L^1(G) \to M(G)$ is a bounded homomorphism and $M(G)$ is a flat Banach left $L^1(G)$-module, (a) implies (b); see [9, Proposition 4.18]. To prove the converse, suppose that LUC$(G)^*$ is an injective Banach right $L^1(G)$-module. We must to prove that for each Banach right $M(G)$-modules E and F, each admissible monomorphism $T \in {}_{M(G)}B(E, F)$ and each $S \in {}_{M(G)}B(E, \text{LUC}(G)^*)$, there exists $R \in {}_{M(G)}B(F, \text{LUC}(G)^*)$ such that

$$R \circ T = S.$$

By (b), there exists a right $L^1(G)$-module morphism $R : F \longrightarrow \text{LUC}(G)^*$ with $R \circ T = S$. Since the Banach left $L^1(G)$-module LUC(G) has bounded left approximate identity $(e_\gamma)_{\gamma\in\Gamma}$ in $L^1(G)$, it follows that

$$\begin{aligned}(R(\xi \cdot \mu) - R(\xi) \cdot \mu)(h) &= \lim_\gamma (R(\xi \cdot \mu) - R(\xi) \cdot \mu)(e_\gamma \cdot h)\\ &= \lim_\gamma (R(\xi \cdot \mu) \cdot e_\gamma - R(\xi) \cdot \mu \cdot e_\gamma)(h)\\ &= \lim_\gamma (R(\xi \cdot \mu \cdot e_\gamma) - R(\xi) \cdot \mu \cdot e_\gamma)(h)\\ &= 0,\end{aligned}$$

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for all $h \in \mathrm{LUC}(G)$, $\mu \in M(G)$ and $\xi \in F$. This implies that R is a right $M(G)$-module morphism and the proof is complete. □

We end this work with the following conjectures for a locally compact group G.

CONJECTURE 2.7. $\mathrm{LUC}(G)^*$ is projective as a Banach left $M(G)$-module if and only if G is compact.

CONJECTURE 2.8. $\mathrm{LUC}(G)^*$ is flat as a Banach left $L^1(G)$ or $M(G)$-module if and only if G is amenable.

Acknowledgements

The authors thank the Center of Excellence for Mathematics at the Isfahan University of Technology. The authors would like to thank the referee for invaluable comments.

References

[1] F. Bahrami, R. Nasr-Isfahani and S. Soltani Renani, 'Homological properties of certain Banach modules over group algebras', *Proc. Edinb. Math. Soc.* (2) **54** (2011), 321–328.
[2] H. G. Dales and M. E. Polyakov, 'Homological properties of modules over group algebras', *Proc. Lond. Math. Soc.* (3) **89** (2004), 390–426.
[3] A. Ya. Helemskii, *The Homology of Banach and Topological Algebras* (Kluwer Academic Publishers Group, Dordrecht, 1989).
[4] E. Hewitt and K. Ross, *Abstract Harmonic Analysis II* (Springer, New York, 1970).
[5] B. E. Johnson, *Cohomology in Banach Algebras*, Memoirs of the American Mathematical Society, 127 (American Mathematical Society, Providence, RI, 1972).
[6] A. T. Lau and V. Losert, 'Complementation of certain subspaces of $L_\infty(G)$ of a locally compact group', *Pacific J. Math.* **141** (1990), 295–310.
[7] R. Nasr-Isfahani and S. Soltani Renani, 'Character contractibility of Banach algebras and homological properties of Banach modules', *Studia Math.* **202** (2011), 205–225.
[8] J. P. Pier, *Amenable Locally Compact Groups*, Pure and Applied Mathematics (John Wiley and Sons, New York, 1984).
[9] M. C. White, 'Injective module for uniform algebras', *Proc. Lond. Math. Soc.* (3) **73** (1996), 155–184.

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