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SLANT SUBMANIFOLDS OF COMPLEX PROJECTIVE AND COMPLEX HYPERBOLIC SPACES

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Abstract. An interesting class of submanifolds of Hermitian manifolds is the class of slant submanifolds which are submanifolds with constant Wirtinger angle. In [**1**–**4**,**7**,**8**] slant submanifolds of complex projective and complex hyperbolic spaces have been investigated. In particular, it was shown that there exist many proper slant surfaces in CP^2 and in CH^2 and many proper slant minimal surfaces in $\mathbf{C}^2$. In contrast, in the first part of this paper we prove that there do not exist proper slant minimal surfaces in CP^2 and in CH^2. In the second part, we present a general construction procedure for obtaining the explicit expressions of such slant submanifolds. By applying this general construction procedure, we determine the explicit expressions of special slant surfaces of CP^2 and of CH^2. Consequently, we are able to completely determine the slant surface which satisfies a basic equality. Finally, we apply the construction procedure to prove that special θ-slant isometric immersions of a hyperbolic plane into a complex hyperbolic plane are not unique in general.

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1. Introduction. Let M be a Riemannian manifold and $\tilde{M}$ an almost Hermitian manifold with almost complex structure J. An isometric immersion $f: M \to \tilde{M}$ of M in $\tilde{M}$ is called holomorphic if at each point $p \in M$ we have $J(T_pM) = T_pM$, where T_pM denotes the tangent space of M at p [**9**]. The immersion is called totally real if $J(T_pM) \subset T_p^{\perp}M$ for each $p \in M$, where $T_p^{\perp}M$ is the normal space of M at p.

Let $\tilde{M}^m(4\epsilon)$ denote a Kählerian m-manifold of constant holomorphic sectional curvature 4ϵ and $f: M \to \tilde{M}^m(4\epsilon)$ an isometric immersion. We denote by $\langle\ ,\ \rangle$ the inner product for M as well as for $\tilde{M}^m(4\epsilon)$.

For any vector X tangent to M, we put $JX = PX + FX$, where PX and FX denote the tangential and normal components of JX, respectively. For each nonzero vector X tangent to M at p, the angle $\theta(X)$, $0 \leq \theta(X) \leq \frac{\pi}{2}$, between JX and T_pM is called the Wirtinger angle of X. An immersion $f: M \to \tilde{M}^m(4\epsilon)$ is called *slant* if the Wirtinger angle θ is a constant [**2**]. The Wirtinger angle θ of a slant immersion is called the *slant angle*. A slant submanifold with slant angle θ is called θ-*slant*. Holomorphic and totally real immersions are slant immersions with slant angle 0 and $\frac{\pi}{2}$, respectively. A slant immersion is called *proper slant* if it is neither

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holomorphic nor totally real. There exist ample examples of n-dimensional proper slant submanifolds in complex-space-forms of complex dimension n (see, for instance, [**2**,**4**,7,**8**,**11**]).

Let M be a proper θ-slant surface in a Kählerian surface $\tilde{M}^2$. If e_1 is a unit vector tangent to M, we choose a canonical orthonormal basis $\{e_1, e_2, e_3, e_4\}$ defined by

$$e_2 = (\sec\theta)Pe_1, \quad e_3 = (\csc\theta)Fe_1, \quad e_4 = (\csc\theta)Fe_2. \tag{1.1}$$

We call such an orthonormal basis an *adapted orthonormal basis* [**2**].

In [**3**] the first author proved that the squared mean curvature H^2 and the Gauss curvature K of a proper slant surface M in a complex space form $\tilde{M}^2(4\epsilon)$ satisfy the following basic inequality:

$$H^2(p) \geq 2K(p) - 2(1 + 3\cos^2\theta)\epsilon, \tag{1.2}$$

at each point $p \in M$. The equality sign of (1.2) holds at a point $p \in M$ if and only if, with respect to some suitable adapted orthonormal basis $\{e_1, e_2, e_3, e_4\}$ at p, the shape operator of M at p takes the following form:

$$A_{e_3} = \begin{pmatrix} 3\lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}. \tag{1.3}$$

A slant surface M in a Kählerian surface $\tilde{M}^2$ is said to be *special slant* if, with respect to some suitable adapted orthonormal frame $\{e_1, e_2, e_3, e_4\}$, the shape operator of M takes the following special form:

$$A_{e_3} = \begin{pmatrix} c\lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}, \tag{1.4}$$

for some constant c and some function λ.

In contrast to the fact that there exist ample examples of proper slant minimal surfaces in $\mathbf{C}^2$, we prove in section 3 that every proper slant surface in CP^2 and in CH^2 is non-minimal. In section 4, we present a general construction procedure for obtaining the explicit expressions of slant submanifolds of complex projective spaces and of complex hyperbolic spaces. In section 5 we apply the procedure to construct the explicit expressions of special slant surfaces and apply it to determine completely the slant surface which satisfies the equality case of the basic inequality (1.2). In the last section, we establish a non-congruent result by applying the construction procedure. In fact, we prove that, up to rigid motions of $CH^2(-4)$, there exist more than one special θ-slant isometric immersions for each $\theta \in (0, \frac{\pi}{2})$ from a surface of constant negative Gauss curvature $-4\cos^2\theta$ into $CH^2(-4)$ whose shape operators satisfy (1.4) with $c = 2$.

2. Basic formulas. Let $f : M \to \tilde{M}^m(4\epsilon)$ be an isometric immersion of a Riemannian n-manifold into $\tilde{M}^m(4\epsilon)$. Denote by h and A the second fundamental form and the shape operator of f, and by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M and $\tilde{M}^m(4\epsilon)$, respectively. The Gauss and Weingarten formulas of M in $\tilde{M}$ are given respectively by

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$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2.2}$$

where X, Y are vector fields tangent to M and ξ is normal to M. The second fundamental form h and the shape operator A are related by $\langle A_\xi X, Y\rangle = \langle h(X, Y), \xi\rangle$. The mean curvature vector $\vec{H}$ of the immersion is defined by $\vec{H} = \frac{1}{n}\operatorname{trace} h$.

Denote by R and $\tilde{R}$ the Riemann curvature tensors of M and $\tilde{M}^m(4\epsilon)$, respectively. The *equation of Gauss* is given by

$$\begin{aligned}\tilde{R}(X, Y; Z, W) =& R(X, Y; Z, W) + \langle h(X, Z), h(Y, W)\rangle \\ &- \langle h(X, W), h(Y, Z)\rangle,\end{aligned} \tag{2.3}$$

for vectors X, Y, Z, W tangent to M and ξ, η normal to M.

For the second fundamental form h, define the covariant derivative $\bar{\nabla} h$ of h with respect to the connection on $TM \oplus T^\perp M$ by

$$(\bar{\nabla}_X h)(Y, Z) = D_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.4}$$

The *equation of Codazzi* is given by

$$(\tilde{R}(X, Y)Z)^\perp = (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z), \tag{2.5}$$

where $(\tilde{R}(X, Y)Z)^\perp$ denotes the normal component of $\tilde{R}(X, Y)Z$.

Let $(\tilde{M}^{2m+1}, g, \phi, \xi)$ be a $(2m+1)$-dimensional almost contact metric manifold with Riemannian (or a pseudo-Riemannian) metric g, the almost contact $(1, 1)$-tensor ϕ, and the structure vector field ξ. An immersion $f : N \to \tilde{M}^{2m+1}$ of a manifold N into $\tilde{M}^{2m+1}$ is called *contact θ-slant* if (i) the structure vector field ξ of $\tilde{M}^{2m+1}$ is tangent to $f_*(TN)$ and (ii) for each nonzero vector X tangent to $f_*(T_pN)$ and perpendicular to ξ, the angle $\theta(X)$ between $\phi(X)$ and $f_*(T_pN)$ is independent of the choice of X.

3. A non-minimality theorem. There exist ample examples of proper slant minimal surfaces in $\mathbf{C}^2$ and there also exist many examples of proper slant surfaces in complex projective plane CP^2 and in complex hyperbolic plane CH^2 (see, for instance [**2**,**3**,**7**,**8**]). In this section we prove the following non-minimality for proper slant surfaces in CP^2 and in CH^2.

THEOREM 3.1. *Every proper slant surfaces in a complex space form $\tilde{M}^2(4\epsilon)$ with $\epsilon \neq 0$ is non-minimal.*

Proof. Suppose that M is a proper slant minimal surface in a complex space form $\tilde{M}^2(4\epsilon)$ with $\epsilon \neq 0$. Denote by θ the slant angle. Then $\theta \in (0, \frac{\pi}{2})$.

Let

$$e_1,\ e_2 = (\sec\theta)Pe_1,\quad e_3 = (\csc\theta)Fe_1,\quad e_4 = (\csc\theta)Fe_2 \tag{3.1}$$

be an adapted orthonormal local frame of M in $\tilde{M}^2(4\epsilon)$.

For any normal vector η, we put $J\eta = t\eta + f\eta$, where $t\eta$ and $f\eta$ denote the tangential and the normal components of η, respectively. Then (3.1) yields

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$$te_3 = -\sin\theta e_1, \quad te_4 = -\sin\theta e_2, \quad fe_3 = -\cos\theta e_4, \quad fe_4 = \cos\theta e_3. \tag{3.2}$$

Since the almost complex structure J on $\tilde{M}^2(4\epsilon)$ is parallel, (2.1) and (2.2) yield

$$\begin{aligned}\nabla_X(PY) + h(X, PY) - A_{FY}X + D_X(FY)\\ = P(\nabla_X Y) + F(\nabla_X Y) + th(X, Y) + fh(X, Y).\end{aligned} \tag{3.3}$$

Comparing the normal components of both sides of (3.3) yields

$$D_X(FY) = F(\nabla_X Y) + fh(X, Y) - h(X, PY). \tag{3.4}$$

Hence, we find

$$D_{e_1}e_3 = (\csc\theta)\{\omega_1^2(e_1)Fe_2 + h_{11}^3 fe_3 + h_{11}^4 fe_4 - \cos\theta(h_{12}^3 e_3 + h_{12}^4 e_4)\}, \tag{3.5}$$

where $h_{ij}^r = \langle h(e_i, e_j), e_r\rangle$, $i, j = 1, 2;\ r = 3, 4$.

On the other hand, since [**2**]

$$A_{FX}Y = A_{FY}X, \tag{3.6}$$

we have $h_{12}^3 = h_{11}^4$ and $h_{12}^4 = h_{22}^3$. Thus, (3.1), (3.2), (3.5) and the minimality of M yield $\omega_3^4(e_1) = \omega_1^2(e_1)$. Similarly, we have $\omega_3^4(e_2) = \omega_1^2(e_2)$. Hence, we get

$$\omega_3^4 = \omega_1^2. \tag{3.7}$$

Let p be a non-totally geodesic point in M. We define a function γ_p by

$$\gamma_p : UM_p \to \mathbf{R} : v \mapsto \gamma_p(v) = \langle h(v, v), Fv\rangle, \tag{3.8}$$

where $UM_p =: \{v \in T_pM : \langle v, v\rangle = 1\}$. Since UM_p is a compact set, there exists a vector v in UM_p such that γ_p attains its absolute minimum at v. Since p is a non-totally geodesic point, it follows from (3.6) that $\gamma_p \neq 0$. By linearity, we have $\gamma_p(v) < 0$. Because γ_p attains an absolute minimum at v, it follows that $\langle h(v, v), Fw\rangle = 0$ for all w orthogonal to v. So, v is an eigenvector of the symmetric operator A_{Fv}. Thus, by choosing an orthonormal basis $\{e_1, e_2\}$ of T_pM with $e_1 = v$, we obtain

$$h(e_1, e_1) = -\mu Fe_1, \quad h(e_1, e_2) = \mu Fe_2, \quad h(e_2, e_2) = \mu Fe_1 \tag{3.9}$$

for some real number μ. If p is a totally geodesic point, (3.9) holds trivially for any adapted orthonormal basis at p. Consequently, there exists a local adapted orthonormal frame e_1, e_2, e_3, e_4 such that the second fundamental form h of the proper slant minimal surface M in $\tilde{M}^2(4\epsilon)$ satisfies

$$h(e_1, e_1) = -\lambda e_3, \quad h(e_1, e_2) = \lambda e_4, \quad h(e_2, e_2) = \lambda e_3 \tag{3.10}$$

where $\lambda = \mu\sin\theta$. Using (2.4), (3.7) and (3.10), we find

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$$(\bar{\nabla}_{e_2} h)(e_1, e_1) = -(e_2\lambda)e_3 - 3\lambda\omega_1^2(e_2)e_4, \tag{3.11}$$
$$(\bar{\nabla}_{e_1} h)(e_1, e_2) = (e_1\lambda)e_4 - 3\lambda\omega_1^2(e_1)e_3, \tag{3.12}$$
$$(\bar{\nabla}_{e_1} h)(e_2, e_2) = (e_1\lambda)e_3 + 3\lambda\omega_1^2(e_2)e_4, \tag{3.13}$$
$$(\bar{\nabla}_{e_2} h)(e_1, e_2) = (e_2\lambda)e_4 - 3\lambda\omega_1^2(e_2)e_3. \tag{3.14}$$

On the other hand, it is easy to see that the normal component of $\tilde{R}(e_2, e_1)e_1$ and $\tilde{R}(e_1, e_2)e_2$ are given by

$$(\tilde{R}(e_2, e_1)e_1)^{\perp} = 3\epsilon \sin\theta\cos\theta, \quad (\tilde{R}(e_1, e_2)e_2)^{\perp} = -3\epsilon\sin\theta\cos\theta. \tag{3.15}$$

Thus, by the equation of Codazzi, (3.11), (3.12) and (3.15) we obtain

$$e_2\lambda = 3\lambda\omega_1^2(e_1) - 3\epsilon\sin\theta\cos\theta, \tag{3.16}$$
$$e_2\lambda = 3\lambda\omega_1^2(e_1) + 3\epsilon\sin\theta\cos\theta. \tag{3.17}$$

Combining (3.16) and (3.17) yields $\epsilon\sin\theta\cos\theta = 0$, which is a contradiction. □

As an immediate consequence of Theorem 3.1, we obtain the following.

COROLLARY 3.2. *Let M be a slant surface of a complex space form $\tilde{M}^2(4\epsilon)$. If M is minimal, then $\epsilon = 0$, or M is a holomorphic, or M is totally real.*

4. A general construction procedure. In this section we fix notation and at the time present a general construction procedure to obtain the explicit expression of a slant submanifold in a complex projective space or in a complex hyperbolic space via Hopf's fibration. The method presented in this section is different from the one used in [**5**,**10**] which represents totally real submanifolds in complex projective or complex hyperbolic spaces.

CASE (1): Slant submanifolds in $CP^m(4)$. Consider the complex number $(m+1)$-space $\mathbf{C}^{m+1}$. Let S^{2m+1} denote the unit hypersphere centered at the origin and $\mathbf{C}^* = \{\lambda \in \mathbf{C} : \lambda\bar{\lambda} = 1\}$. Then we have a $\mathbf{C}^*$-action on S^{2m+1} defined by $z \mapsto \lambda z$. At $z \in S^{2m+1}$ the vector $V = iz$ is tangent to the flow of the action. The quotient space $S^{2m+1}/\sim$ under the identification induced from the action is $CP^m(4)$ with constant holomorphic sectional curvature 4. The almost complex structure J on $CP^m(4)$ is induced from the complex structure J on $\mathbf{C}^{m+1}$ via the Hopf fibration: $\pi : S^{2m+1} \to CP^m(4)$. On S^{2m+1} consider the Sasakian structure obtained from the projection of J of $\mathbf{C}^{m+1}$ on the tangent bundle of S^{2m+1} and with the structure vector field $\xi = V = iz$.

Let $f: M \to CP^m(4)$ be an isometric immersion. Then $\hat{M} = \pi^{-1}(M)$ is a principal circle bundle over M with totally geodesic fibers and the lift $\hat{f}: \hat{M} \to S^{2m+1}$ of f is an isometric immersion such that the diagram:

$$\begin{array}{ccc} \hat{M} & \xrightarrow{\hat{f}} & S^{2m+1} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow{f} & CP^m(4) \end{array} \tag{4.1}$$

commutes.

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Conversely, if $\psi : \hat{M} \to S^{2m+1}$ is an isometric immersion which is invariant under the action of $\mathbf{C}^*$, there is a unique isometric immersion $\psi_\pi : \pi(\hat{M}) \to CP^m(4)$ such that the associated diagram (4.1) commutes. We simply call the immersion $\psi_\pi : \pi(\hat{M}) \to CP^m(4)$ *the projection* of $\psi : \hat{M} \to S^{2m+1}$.

Let $\hat{\nabla}$ and $\tilde{\nabla}$ denote the Levi-Civita connections of S^{2m+1} and $CP^m(4)$ respectively and denote by * the horizontal lift. Follows from $\langle X, Y\rangle = \langle X^*, Y^*\rangle$ and $(JX)^* = \phi X^*$ for X, Y tangent to M, we obtain the following.

Lemma 4.1. *The isometric immersion* $f : M \to CP^m(4)$ *is* θ*-slant if and only if* $\hat{f} : \hat{M} \to S^{2m+1}$ *is contact* θ*-slant.*

Denote by h and $\hat{h}$ the second fundamental forms of f and $\hat{f}$, respectively. Then

$$\hat{h}(X^*, Y^*) = (h(X, Y))^*, \quad \hat{h}(X^*, V) = (FX)^*, \quad \hat{h}(V, V) = 0 \tag{4.2}$$

for X, Y tangent to M, where FX is the normal component of JX in $CP^m(4)$.

It follows from Lemma 4.1 that in order to obtain the explicit expression of a desired θ-slant submanifold of $CP^m(4)$ with second fundamental form h, it is sufficient to construct a contact θ-slant submanifold of S^{2m+1} whose second fundamental form satisfies $\pi_* \hat{h} = h$, and vice versa.

Case (2): Slant submanifolds in $CH^m(-4)$. Consider the complex number $(m+1)$-space $\mathbf{C}_1^{m+1}$ with the pseudo-Euclidean metric $g_0 = -dz_0 d\bar{z}_0 + \sum_{j=1}^m dz_j d\bar{z}_j$. Put $H_1^{2m+1} = \{z = (z_0, z_1, \dots, z_m) : \langle z, z\rangle = -1\}$, where $\langle\ ,\ \rangle$ denotes the inner product on $\mathbf{C}_1^{m+1}$ induced from g_0. We have a $\mathbf{C}^*$-action on H_1^{2m+1} defined by $z \mapsto \lambda z$. At $z \in H_1^{2m+1}$, iz is tangent to the flow of the action. The orbit is given by $z_t = e^{it} z$ with $\frac{dz_t}{dt} = iz_t$ which lies in the negative-definite plane spanned by z and iz. The quotient space $H_1^{2m+1}/\sim$ is the complex hyperbolic space $CH^m(-4)$. The almost complex structure J on $CH^m(-4)$ is induced from the canonical complex structure J on $\mathbf{C}_1^{m+1}$ via the totally geodesic fibration: $\pi : H_1^{2m+1} \to CH^m(-4)$. On $H_1^{2m+1} \subset \mathbf{C}_1^{m+1}$ consider the Sasakian structure obtained from the projection of the J of $\mathbf{C}_1^{m+1}$ onto the tangent bundle of H_1^{2m+1} and with $\xi = V = Jz$.

Let $f : M \to CH^m(-4)$ be an isometric immersion. Then $\hat{M} = \pi^{-1}(M)$ is a principal circle bundle over M with totally geodesic fibers and the lift $\hat{f} : \hat{M} \to H_1^{2m+1}$ of f is an isometric immersion such that the diagram:

$$\begin{tikzcd} \hat{M} \arrow[r, "\hat{f}"] \arrow[d, "\pi"'] & H_1^{2m+1} \arrow[d, "\pi"] \\ M \arrow[r, "f"] & CH^m(-4) \end{tikzcd} \tag{4.3}$$

commutes.

Conversely, if $\psi : \hat{M} \to H_1^{2m+1}$ is an isometric immersion which is invariant under the action of $\mathbf{C}^*$, there is a unique isometric immersion $\psi_\pi : \pi(\hat{M}) \to CH^m(-4)$, called the *projection of* ψ, such that the associated diagram commutes.

Similar to Lemma 4.1 we have

Lemma 4.2. *The isometric immersion* $f : M \to CH^m(-4)$ *is* θ*-slant if and only if* $\hat{f} : \hat{M} \to H_1^{2m+1}$ *is contact* θ*-slant.*

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Denote by $\hat{\nabla}$ and $\tilde{\nabla}$ the Levi-Civita connections of H_1^{2m+1} and $CH^m(-4)$, * the horizontal lift, and by h and $\hat{h}$ the second fundamental forms of f and $\hat{f}$, respectively. We also have (4.2). From Lemma 4.2 it follows that in order to obtain the explicit expression of a desired θ-slant submanifold in $CH^m(-4)$, it is sufficient to construct a contact θ-slant submanifold in $H_1^{2m+1} \subset \mathbf{C}^{m+1}$ whose second fundamental form satisfies $\pi_* \hat{h} = h$, and vice versa.

Let M be a θ-slant submanifold of $CP^m(4)$ (respect., of $CH^m(-4)$) and $z : S^{2m+1} \to \mathbf{C}^{m+1}$ (respect., $z : H_1^{2m+1} \to \mathbf{C}_1^{m+1}$) the standard inclusion. Denote by $\breve{\nabla}$ the Levi-Civita connection of $\mathbf{C}^{m+1}$ (respect., of $\mathbf{C}_1^{m+1}$). Then we have [6]

$$\breve{\nabla}_{X^*} Y^* = (\nabla_X Y)^* + (h(X, Y))^* + \langle JX, Y\rangle iz - \varepsilon\langle X, Y\rangle z, \tag{4.4}$$

$$\breve{\nabla}_{X^*} V = \breve{\nabla}_V X^* = (JX)^* \tag{4.5}$$

$$\breve{\nabla}_V V = \varepsilon z, \tag{4.6}$$

for X, Y tangent to M, where $\varepsilon = 1$ if the ambient space is $\mathbf{C}^{m+1}$; $\varepsilon = -1$ if the ambient space is $\mathbf{C}_1^{m+1}$.

In principle, we obtain the representation of a desired θ-slant submanifold by solving system (4.4)–(4.6) of partial differential equations. The general construction procedure goes as follows: First we determine both the intrinsic and extrinsic structures of the θ-slant submanifold in order to obtain the precise form of the differential system (4.4)–(4.6). Next we construct a coordinate system on the associated contact θ-slant submanifold $\hat{M} = \pi^{-1}(M)$. After that we solve the differential system via the coordinate system on $\hat{M}$ to obtain a solution of the system. The solution, say z, of the system gives rise to the explicit expression of the associated contact θ-slant submanifold of S^{2m+1} or of H_1^{2m+1} which in turn provides the representation of the desired θ-slant submanifold via π.

5. Special slant surfaces with *c*≠2. The main purpose of this section is to apply the construction procedure to completely determine the slant surface which satisfies the equality case of the basic inequality (1.2). In order to do so, first we prove the following.

THEOREM 5.1. *For each given $a \in (0, 1)$, let $z : \mathbf{R}^3 \to \mathbf{C}_1^3$ denote the map of $\mathbf{R}^3$ into $\mathbf{C}_1^3$ defined by*

$$\begin{aligned} z(u, v, t) = e^{it}\Bigg(&1 + \frac{\cosh av - 1}{a^2} + \frac{a^2 e^{-av} u^2}{2(4-3a^2)} - iu\frac{\sqrt{1-a^2}(1+e^{-av})}{\sqrt{4-3a^2}}, \\ &\frac{2 - 2a^2 + (2-a^2)e^{-av}}{4-3a^2}u + i\frac{\sqrt{1-a^2}e^{-av}((4-3a^2)(e^{av}-1)(a^2-1+e^{av}) + a^4u^2)}{a^2(4-3a^2)^{3/2}}, \\ &\frac{a\sqrt{1-a^2}(1-e^{-av})}{4-3a^2}u + i\frac{(4-3a^2)(2-2a^2+(2a^2-3)e^{-av}+e^{av}) + a^4e^{-av}u^2}{2a(4-3a^2)^{3/2}}\Bigg). \end{aligned} \tag{5.1}$$

Then we have

(i) *z defines an immersion of $\mathbf{R}^3$ into the anti-de Sitter space time H_1^5,*

(ii) *with respect to the induced metric on $\mathbf{R}^3$, $z : \mathbf{R}^3 \to H_1^5$ is a contact θ-slant isometric immersion with slant angle $\theta = \cos^{-1}\left(a\sqrt{\frac{1-a^2}{4-3a^2}}\right)$.*

(iii) *the image* $z(\mathbf{R}^3)$ *in* H_1^5 *is invariant under the action of* $\mathbf{C}^*$,

(iv) *the projection* $\phi_c : \pi(\mathbf{R}^3) \to CH^2(-4)$ of $z : \mathbf{R}^3 \to H_1^5$ *via* $\pi : H_1^5 \to CH^2(-4)$ *is a special slant isometric immersion with slant angle* θ *given in* (ii) *whose shape operator satisfies*

$$A_{e_3} = \begin{pmatrix} c\lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}, \tag{5.2}$$

with respect to an adapted orthonormal frame filed e_1, e_2, e_3, e_4, *where*

$$c = \frac{8 - 3a^2}{4 - 3a^2} \in (2, 5), \quad \lambda = \sqrt{1 - a^2}, \tag{5.3}$$

(v) $\phi_c : \pi(\mathbf{R}^3) \to CH^2(-4)$ *has nonzero constant mean curvature and constant Gauss curvature, and*

(vi) *up to rigid motions, every proper slant surface with constant mean curvature in* $CH^2(-4)$ *is obtained in the above way if the shape operator satisfies* (5.2) *for some real number* $c \neq 2$.

Proof. Let $z : \mathbf{R}^3 \to \mathbf{C}_1^3$ be the map of $\mathbf{R}^3$ into $\mathbf{C}_1^3$ defined by (5.1). Then $\langle z, z \rangle = -1$. Thus, it defines a map from $\mathbf{R}^3$ into the anti-de Sitter space-time H_1^5. It follows from a direct computation that $z : \mathbf{R}^3 \to H_1^5$ is an immersion. This gives statement (i). Statements (ii), (iii) and (iv) follow from straightforward long computation. Statement (v) follows from statement (iv) and the equation of Gauss.

In order to prove statement (vi), we assume that $f : M \to CH^2(-4)$ is a proper slant surface in the complex hyperbolic plane $CH^2(-4)$ whose shape operator satisfies (5.2) for some constant $c \neq 2$ and some function λ. Let $z : \hat{M} \to H_1^5 \subset \mathbf{C}_1^3$ denote the immersion of $\hat{M} = \pi^{-1}(M)$ into $\mathbf{C}_1^3$ obtained from $f : M \to CH^2(-4)$ as mentioned in section 4.

From Theorem 3.1 and the proof of Theorem 5 of [4] we know that $c \in (2, 5)$, the metric tensor of M is given by

$$g = e^{-2ay} dx^2 + dy^2, \quad a = 2\sqrt{\frac{c - 2}{3(c - 1)}} \tag{5.4}$$

where θ is the slant angle given by

$$\theta = \cos^{-1}\left(\frac{1}{3}\sqrt{\frac{(5 - c)(c - 2)}{c - 1}}\right), \tag{5.5}$$

and moreover the shape operator takes the form:

$$A_{e_3} = \begin{pmatrix} c\lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}, \quad \lambda = \sqrt{\frac{5 - c}{3(c - 1)}} \tag{5.6}$$

with respect to an adapted orthonormal frame field e_1, e_2, e_3, e_4 with $e_2 = \partial/\partial y$.

Since $a = 2\sqrt{(c - 2)/3(c - 1)}$, we have $a \in (0, 1)$ and

$$\theta = \cos^{-1}\left(a\sqrt{\frac{1 - a^2}{4 - 3a^2}}\right), \quad \lambda = \sqrt{1 - a^2}. \tag{5.7}$$

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From (5.4) we find

$$\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x}=ae^{-2ay}\frac{\partial}{\partial y},\quad \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y}=-a\frac{\partial}{\partial x},\quad \nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y}=0, \tag{5.8}$$

where ∇ denotes the metric connection of M. Let $e_1=e^{ay}\partial/\partial x$ and $e_2=\partial/\partial y$. Then (5.8) yields

$$\nabla_{e_1}e_1=ae_2,\quad \nabla_{e_1}e_2=-ae_1,\quad \nabla_{e_2}e_1=\nabla_{e_2}e_2=0. \tag{5.9}$$

Denote by $E_1,\dots,E_4$ the horizontal lifts $(e_1)^*,\dots,(e_4)^*$ of $e_1,\dots,e_4$, respectively. Then $E_1,\dots,E_4$ can be regarded in a natural way as vector fields in $\mathbf{C}^3_1$ via the inclusion $H^5_1\subset\mathbf{C}^3_1$.

From (1.1), (4.9), (5.6), and (5.9) we obtain

$$\breve{\nabla}_{E_1}E_1=aE_2+c\lambda E_3+z, \tag{5.10}$$
$$\breve{\nabla}_{E_1}E_2=-aE_1+\lambda E_4+(\cos\theta)iz, \tag{5.11}$$
$$\breve{\nabla}_{E_2}E_1=\lambda E_4-(\cos\theta)iz, \tag{5.12}$$
$$\breve{\nabla}_{E_2}E_2=\lambda E_3+z, \tag{5.13}$$
$$\breve{\nabla}_{V}E_1=\breve{\nabla}_{E_1}V=(\cos\theta)E_2+(\sin\theta)E_3, \tag{5.14}$$
$$\breve{\nabla}_{V}E_2=\breve{\nabla}_{E_2}V=-(\cos\theta)E_1+(\sin\theta)E_4, \tag{5.15}$$
$$\breve{\nabla}_{V}V=-z, \tag{5.16}$$

where $V=iz$ is tangent to $\hat{M}=\pi^{-1}(M)$ and $\breve{\nabla}$ is the metric connection of $\mathbf{C}^3_1$.

Let $F=e^{-ay}\big(E_1-\frac{2}{a}(\cos\theta)V\big)$. Then by using (5.10)–(5.16) we obtain

$$[F,E_2]=[F,V]=[E_2,V]=0.$$

Therefore, there exist coordinates $\{u,v,t\}$ on $\hat{M}=\pi^{-1}(M)$ such that

$$z_u=e^{-av}\left(E_1-\frac{2}{a}(\cos\theta)V\right),\quad z_v=E_2,\quad z_t=V, \tag{5.17}$$

where $z_u=\partial z/\partial u$, $z_v=\partial z/\partial v$ and $z_t=\partial z/\partial t$. From (5.17) we get

$$E_1=e^{av}z_u+\frac{2}{a}(\cos\theta)iz,\quad E_2=z_v,\quad V=z_t. \tag{5.18}$$

Since $iE_1=\cos\theta E_2+\sin\theta E_3$, we have

$$E_3=e^{av}(\csc\theta)iz_u-(\cot\theta)z_v-\left(\frac{2\cot\theta}{a}\right)z. \tag{5.19}$$

Similarly, we have

$$E_4=e^{av}(\cot\theta)z_u+(\csc\theta)iz_v+\left(\frac{2\cos\theta\cot\theta}{a}\right)iz. \tag{5.20}$$

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From (5.10)–(5.20) and a straight-forward computation we obtain

$$z_{uu} = e^{-2av}\left\{e^{av}\left(c\lambda\csc\theta - \frac{4\cos\theta}{a}\right)iz_u + (a - c\lambda\cot\theta)z_v + \left(1 + \frac{4\cos^2\theta}{a^2} - \frac{2c\lambda\cot\theta}{a}\right)z\right\}, \tag{5.21}$$

$$z_{uv} = e^{-av}\left\{(\lambda\cot\theta - a)e^{av}z_u + \left(\lambda\csc\theta - \frac{2\cos\theta}{a}\right)iz_v + \left(\frac{2\lambda\cos\theta\cot\theta}{a} - \cos\theta\right)iz\right\}, \tag{5.22}$$

$$z_{vv} = \lambda e^{av}(\csc\theta)iz_u - \lambda(\cot\theta)z_v + \left(1 - \frac{2\lambda\cot\theta}{a}\right)z, \tag{5.23}$$

$$z_{ut} = iz_u, \qquad z_{vt} = iz_v, \qquad z_{tt} = -z. \tag{5.24}$$

Solving (5.24) yields

$$z(u, v, t) = e^{it}A(x, y), \tag{5.25}$$

for some $\mathbf{C}^3_1$-valued function $A = A(u, v)$. Substituting (5.25) into (5.21)–(5.23) yields

$$A_{uu} = e^{-2av}\left\{e^{av}\left(c\lambda\csc\theta - \frac{4\cos\theta}{a}\right)iA_u + (a - c\lambda\cot\theta)A_v + \left(1 + \frac{4\cos^2\theta}{a^2} - \frac{2c\lambda\cot\theta}{a}\right)A\right\}, \tag{5.26}$$

$$A_{uv} = e^{-av}\left\{(\lambda\cot\theta - a)e^{av}A_u + \left(\lambda\csc\theta - \frac{2\cos\theta}{a}\right)iA_v + \left(\frac{2\lambda\cos\theta\cot\theta}{a} - \cos\theta\right)iA\right\}, \tag{5.27}$$

$$z_{vv} = \lambda e^{av}(\csc\theta)iA_u - \lambda(\cot\theta)A_v + \left(1 - \frac{2\lambda\cot\theta}{a}\right)A. \tag{5.28}$$

Using (5.4), (5.7) and (5.26)–(5.28), we obtain $A_{uuu} = 0$. Therefore,

$$A(u, v) = E(v) + uF(v) + u^2G(v) \tag{5.29}$$

for some $\mathbf{C}^3_1$-valued functions $E(v)$, $F(v)$ and $G(v)$. From (5.29) we get

$$A_{uu} = 2G(v). \tag{5.30}$$

On the other hand, substituting (5.25) into the right hand side of (5.26) we also have

$$A_{uu} = e^{-2av}\left\{e^{av}\left(c\lambda\csc\theta - \frac{4\cos\theta}{a}\right)i(F + 2uiG) + (a - c\lambda\cot\theta)(E' + uF' + u^2G') + \left(1 + \frac{4\cos^2\theta}{a^2} - \frac{2c\lambda\cot\theta}{a}\right)(E + uF + u^2G)\right\}. \tag{5.31}$$

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By comparing the coefficients of u^2 in (5.30) and in (5.31), we obtain

$$(a - c\lambda \cot\theta)G'(v) = \left(\frac{2c\lambda\cot\theta}{a} - 1 - \frac{4\cos^2\theta}{a^2}\right)G(v). \tag{5.32}$$

Hence, by applying (5.4) and (5.7), we obtain from (5.32) that $G'(v) = -aG(v)$ which implies

$$G(v) = \beta e^{-av}, \tag{5.33}$$

for some constant vector $\beta \in \mathbf{C}^3_1$.

Comparing the coefficients of u in (5.30) and (5.31) and applying (5.33) yield

$$aF'(v) + a^2F(v) = -2\sqrt{(1-a^2)(4-3a^2)}\, i\,\beta. \tag{5.34}$$

Solving (5.34) yields

$$F(v) = \gamma e^{-av} - \frac{2\sqrt{4-7a^2+3a^4}}{a^2}\, i\,\beta, \tag{5.35}$$

for some constant vector $\gamma \in \mathbf{C}^3_1$.

Comparing the coefficients of u^0 in (5.30) and (4.31) and applying (5.33) and (5.35), we obtain

$$E'(v) + aE(v) = -\frac{\sqrt{4-7a^2+3a^4}}{a}\, i\gamma + \frac{(8-6a^2)e^{av}}{a^3}\beta. \tag{5.36}$$

Solving (5.36) yields

$$E(v) = -\frac{\sqrt{4-7a^2+3a^4}}{a^2}\, i\gamma + e^{-av}\delta + \frac{(4-3a^2)e^{av}}{a^4}\beta, \tag{5.37}$$

for some constant vector $\delta \in \mathbf{C}^3_1$. Combining (5.25), (5.29), (5.33), (5.35) and (5.37), we obtain

$$\begin{aligned} z(u, v, t) = e^{it}\big\{&a^{-4}\big(4-3a^2\big)e^{av}\beta - a^{-2}\sqrt{4-7a^2+3a^4}\, i(\gamma + 2u\beta) \\ &+ e^{-av}(\delta + u\gamma + u^2\beta)\big\}. \end{aligned} \tag{5.38}$$

From (5.38) we find

$$z_u(0,0,0) = -2a^{-2}\sqrt{4-7a^2+3a^4}\, i\beta + \gamma, \tag{5.39}$$

$$z_v(0,0,0) = a^{-4}\big(4-3a^2\big)\beta - a\delta, \tag{5.40}$$

$$z_t(0,0,0) = iz(0,0,0). \tag{5.41}$$

By choosing the initial conditions:

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$$\begin{aligned} &z(0,0,0) = (1,0,0), \quad (\text{or equivalently, } z_t(0,0,0) = (i,0,0)),\\ &z_u(0,0,0) = \left(-\frac{2\sqrt{1-a^2}}{\sqrt{4-3a^2}} i, 1, 0\right),\\ &z_v(0,0,0) = \left(0, \frac{a\sqrt{1-a^2}}{\sqrt{4-3a^2}} i, \frac{2-a^2}{\sqrt{4-3a^2}}\right), \end{aligned} \tag{5.42}$$

we obtain from (5.39), (5.40), (5.41) and (5.42) that

$$\beta = \left(\frac{a^2}{2(4-3a^2)}, \frac{a^2\sqrt{1-a^2}}{(4-3a^2)^{3/2}} i, \frac{a^3}{2(4-3a^2)^{3/2}} i\right), \tag{5.43}$$

$$\gamma = \left(-\frac{\sqrt{1-a^2}}{\sqrt{4-3a^2}} i, \frac{2-a^2}{4-3a^2}, -\frac{a\sqrt{1-a^2}}{4-3a^2}\right), \tag{5.44}$$

$$\delta = \left(\frac{1}{2a^2}, \frac{(1-a^2)^{3/2}}{a^2\sqrt{4-3a^2}} i, \frac{2a^2-3}{2a\sqrt{4-3a^2}} i\right). \tag{5.45}$$

Combining (5.38) with (5.43)–(5.45), we obtain (5.1). □

By applying Theorem 5.1 we are able to completely determine the slant surface which satisfies the equality case of the basic inequality (1.2) identically.

THEOREM 5.2. *If M is a proper θ-slant surface in $CH^2(-4)$, then the squared mean curvature and the Gauss curvature of M satisfy*

$$H^2 \geq 2K + 2(1 + 3\cos^2\theta) \tag{5.46}$$

with the equality sign holding identically if and only if, up to rigid motions, M is given by the projection ϕ_3 of the immersion $z : \mathbf{R}^3 \to \mathbf{C}^3_1$ defined by

$$\begin{aligned} z(u,v,t) = e^{it}\Big(&-\tfrac{1}{2} + \tfrac{3}{2}\cosh av + \tfrac{1}{6}u^2 e^{-av} - \tfrac{i}{6}\sqrt{6}u(1+e^{-av}),\\ &\tfrac{1}{3}(1+2e^{-av})u + i\sqrt{6}\big(-\tfrac{1}{3} + \tfrac{1}{4}e^{av} + e^{-av}\big(\tfrac{1}{12} + \tfrac{1}{18}u^2\big)\big),\\ &\frac{\sqrt{2}}{6}(1-e^{-av})u + i\sqrt{3}\big(\tfrac{1}{6} + \tfrac{1}{4}e^{av} + e^{-av}\big(-\tfrac{5}{12} + \tfrac{1}{18}u^2\big)\big)\Big), \quad a = \sqrt{\tfrac{2}{3}} \end{aligned} \tag{5.47}$$

via the hyperbolic Hopf fibration $\pi : H^5_1 \to CH^2(-4)$.

Proof. Follows from Theorem 1 of [3] and Theorem 5.1. □

6. Special slant surfaces with $c = 2$. Let U be a simply-connected open subset of the Cartesian 2-plane $\mathbf{R}^2$ and let $E = E(x, y)$ be a positive function on U satisfying the following conditions:

$$\frac{\partial}{\partial x}\left(\frac{1}{E}\frac{\partial}{\partial x}\left(\frac{1}{E}\frac{\partial E}{\partial y}\right)\right) = 0, \quad \frac{\partial E}{\partial y} \neq 0. \tag{6.1}$$

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For a given $\theta \in (0, \frac{\pi}{2})$, we put $G = \frac{1}{2}(\sec\theta)E_y/E$, $E_y = \partial E/\partial y$. Denote by $M(\theta, -1, E)$ the Riemannian surface (U, g) with metric tensor $g = E^2 dx^2 + G^2 dy^2$, where E and G are given as above. Then $M(\theta, -1, E)$ has constant negative Gauss curvature $K = 4\epsilon \cos^2\theta$. Thus, each $M(\theta, -1, E)$ is locally isometric to the hyperbolic plane $H^2(-4\cos^2\theta)$ of curvature $-4\cos^2\theta < 0$.

It was proved in [**4**] that, for each Riemannian surface $M(\theta, -1, E)$, $\theta \in (0, \frac{\pi}{2})$, there exists a special θ-slant isometric immersion $\psi_{\theta,-1,E} : M(\theta, -1, E) \to CH^2(-4)$ of $M(\theta, -1, E)$ into $CH^2(-4)$ whose shape operator takes the forms:

$$A_{e_3} = \begin{pmatrix} -2\sin\theta & 0 \\ 0 & -\sin\theta \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} 0 & -\sin\theta \\ -\sin\theta & 0 \end{pmatrix}, \tag{6.2}$$

with respect to the adapted orthonormal frame filed $\{e_1, e_2, e_3, e_4\}$ with $e_1 = E^{-1}\partial/\partial x$, $e_2 = G^{-1}\partial/\partial y$. It was also proved in [**4**] that $\psi_{\theta,-1,E} : M(\theta, -1, E) \to CH^2(-4)$, $\theta \in (0, \frac{\pi}{2})$, are the only non-minimal proper slant surfaces in a complete simply-connected complex space form satisfying (1.4) with $c = 2$; moreover, from [**4**] we know that ϕ_c (as defined in Theorem 5.1) and $\psi_{\theta,-1,E}$ are the only nonminimal proper special slant surfaces in $CH^2(-4)$ with constant mean curvature.

One purpose of this section is to determine the explicit representation of some of such slant surfaces by applying the general construction method.

Let $U = \{(x, y) \in \mathbf{R}^2 : y > 0\}$ denote the upper half-plane and $E = y$. Then E satisfies condition (6.1). For each given $\theta \in (0, \frac{\pi}{2})$ we define a metric on U by

$$g = y^2 dx^2 + \frac{\sec^2\theta}{4y^2} dy^2. \tag{6.3}$$

We denote this Riemannian 2-manifold (U, g) by M. Clearly, $M = M(\theta, -1, y)$. From (6.3) we get

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = -4y^3\cos^2\theta \frac{\partial}{\partial y}, \quad \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = \frac{1}{y}\frac{\partial}{\partial x}, \quad \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = -\frac{1}{y}\frac{\partial}{\partial y}. \tag{6.4}$$

Let $e_1 = y^{-1}\partial/\partial x$ and $e_2 = 2y\cos\theta\partial/\partial y$. Then (6.4) yields

$$\nabla_{e_1} e_1 = -2\cos\theta e_2, \quad \nabla_{e_1} e_2 = 2\cos\theta e_1, \quad \nabla_{e_2} e_1 = \nabla_{e_2} e_2 = 0. \tag{6.5}$$

Denote by $E_1, \dots, E_4$ the horizontal lifts $(e_1)^*, \dots, (e_4)^*$ of $e_1, \dots, e_4$, respectively. Then $E_1, \dots, E_4$ can be regarded in a natural way as vector fields in $\mathbf{C}^3_1$ via the inclusion $H^5_1 \subset \mathbf{C}^3_1$.

From (1.1), (4.9), (6.2), and (6.5) we obtain

$$\breve{\nabla}_{E_1} E_1 = -2(\cos\theta)E_2 - 2(\sin\theta)E_3 + z, \tag{6.6}$$

$$\breve{\nabla}_{E_1} E_2 = 2(\cos\theta)aE_1 - (\sin\theta)E_4 + (\cos\theta)iz, \tag{6.7}$$

$$\breve{\nabla}_{E_2} E_1 = -(\sin\theta)E_4 - (\cos\theta)iz, \tag{6.8}$$

$$\breve{\nabla}_{E_2} E_2 = -(\sin\theta)E_3 + z, \tag{6.9}$$

$$\breve{\nabla}_V E_1 = \breve{\nabla}_{E_1} V = (\cos\theta)E_2 + (\sin\theta)E_3, \tag{6.10}$$

$$\breve{\nabla}_V E_2 = \breve{\nabla}_{E_2} V = -(\cos\theta)E_1 + (\sin\theta)E_4, \tag{6.11}$$

$$\breve{\nabla}_V V = -z. \tag{6.12}$$

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Let $F = e^{2(\cos\theta)y}(E_1 + V)$. Then by using (6.6)–(6.12) we obtain

$$[F, E_2] = [F, V] = [E_2, V] = 0.$$

Therefore, there exist coordinates $\{u, v, t\}$ on $\hat{M} = \pi^{-1}(M)$ such that

$$\begin{aligned} z_u &= e^{2(\cos\theta)v}E_1 + iz, \quad z_v = E_2, \quad z_t = V, \\ E_1 &= e^{2v\cos\theta}z_u - iz, \quad E_2 = z_v, \quad V = iz. \end{aligned} \tag{6.13}$$

where $z_u = \frac{\partial z}{\partial u}$, $z_v = \frac{\partial z}{\partial v}$, $z_t = \frac{\partial z}{\partial t}$ and $z : \hat{M} \to H_1^5 \subset \mathbf{C}_1^3$ is the immersion from $\hat{M}$ into $\mathbf{C}_1^3$ obtained from $f : M \to CH^2(-4)$ as mentioned in section 4.

Since $iE_1 = \cos\theta E_2 + \sin\theta E_3$, we have

$$E_3 = e^{2v\cos\theta}(\csc\theta)iz_u - (\cot\theta)z_v - (\csc\theta)z. \tag{6.14}$$

Similarly, we find

$$E_4 = e^{-2v\cos\theta}(\cot\theta)z_u + (\csc\theta)iz_v - (\cot\theta)iz. \tag{6.15}$$

From (6.6)–(6.15) and a straight-forward computation we obtain

$$z_{uu} = 0, \tag{6.16}$$
$$z_{uv} = (\cos\theta)z_u, \tag{6.17}$$
$$z_{vv} = -e^{-2v\cos\theta}iz_u + (\cos\theta)z_v, \tag{6.18}$$
$$z_{ut} = iz_u, \quad z_{vt} = iz_v, \quad z_{tt} = -z. \tag{6.19}$$

Solving (6.19) yields

$$z(u, v, t) = e^{it}A(x, y), \tag{6.20}$$

for some $\mathbf{C}_1^3$-valued function $A = A(u, v)$. Substituting (6.20) into (6.16)–(6.18) yields

$$A_{uu} = 0, \tag{6.21}$$
$$A_{uv} = (\cos\theta)A_u, \tag{6.22}$$
$$z_{vv} = -e^{-2v\cos\theta}iA_u + (\cos\theta)A_v. \tag{6.23}$$

Solving (6.21)–(6.23) yields

$$z(u, v, t) = e^{it}\left\{\beta + e^{v\cos\theta}(\gamma + 2u\alpha) - ie^{-v\cos\theta}(\sec^2\theta)\alpha\right\}, \tag{6.24}$$

for some vectors $\alpha, \beta, \gamma \in \mathbf{C}_1^3$.

From (6.24) we get

$$z_u(0, 0, 0) = 2\alpha, \tag{6.25}$$
$$z_v(0, 0, 0) = (\cos\theta)\gamma + i(\sec\theta)\alpha, \tag{6.26}$$
$$z_t(0, 0, 0) = iz(0, 0, 0). \tag{6.27}$$

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By choosing the initial conditions:

$$\begin{aligned} &z(0,0,0)=(1,0,0), \quad \text{(or equivalently, } z_t(0,0,0)=(i,0,0)),\\ &z_u(0,0,0)=(i,1,0),\\ &z_v(0,0,0)=(0, i\cos\theta, i\sin\theta),\end{aligned} \tag{6.28}$$

we obtain from (6.25), (6.26), (6.27) and (6.28) that

$$\alpha=\left(\tfrac{i}{2},\tfrac{1}{2},0\right), \tag{6.29}$$

$$\beta=\left(1-\sec^2\theta, -i+i\sec^2\theta, -i\tan\theta\right), \tag{6.30}$$

$$\gamma=\left(\tfrac{1}{2}\sec^2\theta, \tfrac{i}{2}(2-\sec^2\theta), i\tan\theta\right). \tag{6.31}$$

Combining (6.24) with (6.29)–(6.31), we obtain

$$\begin{aligned} z(u,v,t)=e^{it}(&\sec^2\theta\cosh(v\cos\theta)-\tan^2\theta+iue^{v\cos\theta},\\ &ue^{v\cos\theta}+i(e^{v\cos\theta}+\tan^2\theta-\sec^2\theta\cosh(v\cos\theta)), i\tan\theta(e^{v\cos\theta}-1)).\end{aligned} \tag{6.32}$$

It is straight-forward to verify that (6.32) defines a contact θ-slant immersion of $\pi^{-1}(M)$ into H_1^5. The special θ-slant immersion $\psi_{\theta,-1,y}$ of $M(\theta,-1,y)$ into $CH^2(-4)$ is the projection of the contact θ-slant immersion $z:\pi^{-1}(M)\to H_1^5$ induced from (6.32).

In summary, we obtain the following.

PROPOSITION 6.1. (1) *For each given* $\theta\in(0,\frac{\pi}{2})$,

$$\begin{aligned} z(u,v,t)=e^{it}(&\sec^2\theta\cosh(v\cos\theta)-\tan^2\theta+iue^{v\cos\theta},\\ &ue^{v\cos\theta}+i(e^{v\cos\theta}+\tan^2\theta-\sec^2\theta\cosh(v\cos\theta)), i\tan\theta(e^{v\cos\theta}-1)).\end{aligned} \tag{6.33}$$

defines a contact θ-slant immersion into the anti-de Sitter space time H_1^5.

(2) *The projection of the contact θ-slant immersion given by* (6.33) *is a special θ-slant immersion whose second fundamental form satisfies* (6.2).

(3) *Up to rigid motions, the special slant isometric immersion $\psi_{\theta,-1,y}$ is given by the projection of the contact θ-slant immersion defined by* (6.33).

REMARK 6.1. For each $M(\theta,-1,E)$, there exists a special θ-slant isometric immersion $\psi_{\theta,-1,E}$ of $M(\theta,-1,E)$ into $CH^2(-4)$. In particular, there exists a special θ-slant isometric immersion $\psi_{\theta,-1,(x+y)^{-2}}$ from $M(\theta,-1,(x+y)^{-2})$ into $CH^2(-4)$ whose shape operator satisfies (6.2) with respect to $e_1=E^{-1}\partial/\partial x, e_2=G^{-1}\partial/\partial y$, where $G=\sec\theta/(x+y)$. It is easy to verify that $\nabla_{e_2}e_2=(x+y)e_1$. Thus, the integral curves of e_2 are not geodesics in $M(\theta,-1,(x+y)^{-2})$ in general. On the other hand, (6.5) implies that the integral curves of e_2 are always geodesics in $M(\theta,-1,y)$. Since both θ-slant immersions $\psi_{\theta,-1,(x+y)^{-2}}$ and $\psi_{\theta,-1,y}$ satisfy the same form of the shape operator (1.4) with respect to e_1,e_2, these two θ-slant immersions cannot be congruent to each other in $CH^2(-4)$. Consequently, we obtain the following non-uniqueness result.

COROLLARY 6.2. *Up to rigid motions of $CH^2(-4)$, for each $\theta\in(0,\frac{\pi}{2})$ there exist more than one special θ-slant isometric immersions of a surface of constant negative Gauss curvature $-4\cos^2\theta$ into $CH^2(-4)$ whose shape operators satisfy* (6.2).

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