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REMARKS ON $\mathcal{Z}$-STABLE PROJECTIONLESS C*-ALGEBRAS

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Abstract. It is shown that $\mathcal{Z}$-stable projectionless C*-algebras have the property that every element is a limit of products of two nilpotents. This is then used to classify the approximate unitary equivalence classes of positive elements in such C*-algebras using traces.

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1. Introduction. Let us denote by $\mathcal{Z}$ the Jiang-Su C*-algebra [**8**]. A C*-algebra A is called $\mathcal{Z}$-stable if $A \cong A \otimes \mathcal{Z}$. A theme that has unfolded in the past decade in the field of "structure and classification of C*-algebras" is how the $\mathcal{Z}$-stable C*-algebras have good properties that separate them from "pathological" examples such as those found by Villadsen, Rørdam, Toms and others (see [**13**, **3**, **6**]). The results of this note contribute to the study of $\mathcal{Z}$-stable projectionless C*-algebras. Henceforth, the term "projectionless" is used to designate the C*-algebras none of whose quotients contains a non-zero projection. We prove the following theorem.

THEOREM 1.1. *Let A be a $\mathcal{Z}$-stable projectionless C*-algebra. Then for every $x \in A$ there exist nilpotent elements y_n, z_n, with $n = 1, 2, \ldots,$ in the hereditary subalgebra generated by x, such that $y_n z_n \to x$.*

Theorem 1.1 implies that for every closed hereditary subalgebra $B \subseteq A$, with A $\mathcal{Z}$-stable and projectionless, we have $B \subseteq \overline{\mathrm{GL}(B^\sim)}$ (Corollary 3.2). We say in this case that A "almost has stable rank one". Recall that a unital C*-algebra is said to have stable rank one if its invertible elements are dense in the algebra, while a non-unital C*-algebra has stable rank one if its unitization has stable rank one (i.e., $A^\sim \subseteq \overline{\mathrm{GL}(A^\sim)}$). Since stable rank one passes to hereditary subalgebras, almost stable rank one is a weakening of stable rank one.

In [**13**, Theorem 6.7], Rørdam shows that a finite simple unital $\mathcal{Z}$-stable C*-algebra has stable rank one. It remains an open question whether the assumption that the algebra is unital can be dropped in this theorem. Nevertheless, the almost stable rank one property established here (in the simple case) is a partial extension of Rørdam's theorem to non-unital C*-algebras.

The almost stable rank one property has several interesting consequences. It implies, for example, the equivalence of (i) and (ii) in the following theorem:

THEOREM 1.2. *Let A be a $\mathcal{Z}$-stable projectionless C*-algebra. Let $a, b \in A_+$. The following are equivalent:*

(i) *a is Cuntz smaller than b,*
(ii) $\overline{aA}$ *embeds in* $\overline{bA}$ *as a right A-Hilbert C*-module,*
(iii) $d_\tau(a) \leqslant d_\tau(b)$ *for all lower semicontinuous 2-quasitraces* $\tau\colon A_+ \to [0,\infty]$.

Furthermore, if $d_\tau(a) = d_\tau(b)$ *for all lower semicontinuous 2-quasitraces* τ *then* $\overline{aA} \cong \overline{bA}$ *as right A-Hilbert C*-modules.*

The hypothesis of $\mathcal{Z}$-stability cannot be dropped in Theorem 1.2, even in the simple nuclear case, as demonstrated in [**15**]. In [**11**], Nawata uses Theorem 1.2 in the investigation of Picard groups of $\mathcal{Z}$-stable projectionless simple C*-algebras.

The classification result from [**12**] is also applicable under the almost stable rank one property. We thus deduce the following theorem:

THEOREM 1.3. *Let A be a* $\mathcal{Z}$*-stable projectionless C*-algebra. Let* $a, b \in A_+$*. The following are equivalent:*

(i) *a is approximately unitarily equivalent to b.*
(ii) $\tau(f(a)) = \tau(f(b))$ *for any lower semicontinuous 2-quasitrace* $\tau\colon A_+ \to [0,\infty]$ *and all* $f \in C_0(0,1]_+$.
(iii) $d_\tau((a-t)_+) = d_\tau((b-t)_+)$ *for all* $t \geqslant 0$ *and all* τ *as in (ii).*

In the next section, we prove Theorem 1.1. Then, in the subsequent section, we prove Theorems 1.2 and 1.3.

2. Proof of Theorem 1.1. Let us first introduce some notation. Let A be a C*-algebra. Let us denote by A_+ the positive elements of A. We will make use of the Cuntz comparison relations on A_+. If $a, b \in A_+$ then we write $a \precsim b$ if $d_n^* b d_n \to a$ for some $d_n \in A$ and $a \sim b$ if $a \precsim b$ and $b \precsim a$.

Let $\mathcal{K}$ denote the C*-algebra of compact operators on a separable Hilbert space. The Cuntz semigroup $\mathrm{Cu}(A)$ is defined as the set $(A \otimes \mathcal{K})_+/\sim$ endowed with a suitable order and addition operation (see [**1**]). Given $a \in (A \otimes \mathcal{K})_+$, we shall denote by $[a] \in \mathrm{Cu}(A)$ its Cuntz equivalence class.

LEMMA 2.1. *Let A be a C*-algebra and let* $x \in A$ *be such that there exist* $e, f \in A_+$ *with the properties that* $ex = xe = x$, $ef = 0$, *and f is a full element of A. Then* $x \otimes 1 \in A \otimes \mathcal{Z}$ *is the product of two nilpotent elements.*

Proof. The relations $ex = xe = x$ imply that x belongs to the Pedersen ideal of A. This and the fact that f is full imply that there exist $a_i, b_i \in A$, with $i = 1, 2, \ldots, m$, such that $x = \sum_{i=1}^m a_i f b_i$. Multiplying by e on the left and on the right if necessary, we may assume that $f a_i = b_i f = 0$ for all i.

Let $n \geqslant 2m$. It is possible to find positive elements $e_1, e_2, \ldots, e_n, d \in \mathcal{Z}$ such that

(1) $\sum_{j=1}^n e_j + d = 1$,
(2) the elements $e_1, e_2, \cdots, e_n$ are pairwise orthogonal,
(3) there exist $(w_j)_{j=1}^{n-1}$ in $\mathcal{Z}$ such that $e_j = w_j w_j^*$ and $e_{j+1} = w_j w_j^*$ for all j,
(4) $[d] \leqslant [e_1]$ in the Cuntz semigroup $\mathrm{Cu}(\mathcal{Z})$ of $\mathcal{Z}$.

(In fact, these elements can be found in the dimension drop algebra $Z_{n-1,n}$, which embeds in $\mathcal{Z}$; see the proof of [**13**, Lemma 4.2].) Since

$$m[e_n + d] \leqslant 2m[e_1] \leqslant [1],$$

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in $\mathrm{Cu}(\mathcal{Z})$ and $\mathcal{Z}$ is a stable rank one C*-algebra, for each $i = 1, \ldots, m$ there exists $v_i \in \mathcal{Z}$ such that $e_n + d = v_i v_i^*$ and the elements $v_i^* v_i \in \mathcal{Z}$ are pairwise orthogonal for all i. Let us now define $\alpha, \beta, \gamma, \delta \in A \otimes \mathcal{Z}$ as follows:

$$\alpha = \sum_{i=1}^{m} a_j f^{1/2} \otimes v_j, \quad \beta = \sum_{i=1}^{m} f^{1/2} b_j \otimes v_j^*,$$
$$\gamma = \sum_{j=1}^{n-1} x \otimes w_j, \qquad \delta = \sum_{j=1}^{n-1} e \otimes w_j^*.$$

We have $\gamma\beta = \alpha\delta = 0$ (by the orthogonality of e and f). Therefore,

$$\begin{aligned}(\gamma + \alpha)(\delta + \beta) &= \gamma\delta + \alpha\beta \\ &= \sum_{j=1}^{n-1} x \otimes e_i + \left(\sum_{i=1}^{m} a_i f b_i\right) \otimes (e_n + d) \\ &= x \otimes 1.\end{aligned}$$

Let us now show that $\gamma + \alpha$ and $\delta + \beta$ are nilpotent elements. We have that $\gamma^n = 0$ and $\alpha^2 = 0$ (since $fa_i = 0$ for all i), Finally, we have that $\alpha\gamma = 0$. Hence, $(\gamma + \alpha)^k = \sum_{i=1}^{k} \gamma^i \alpha^{k-i}$ for all k. Thus, for $k = n + 1$ we get $(\gamma + \alpha)^{n+1} = 0$. Similarly, δ and β are nilpotent and $\delta\beta = 0$. Thus, $\delta + \beta$ is nilpotent. □

Proof of Theorem 1.1 Let us identify A with $A \otimes \mathcal{Z}$. Every element of $A \otimes \mathcal{Z}$ is approximately unitarily equivalent to one of the form $x \otimes 1$ (see the proof of [**3**, Theorem 5.5]). Thus, it suffices to assume that the given element has the form $x \otimes 1 \in A \otimes \mathcal{Z}$.

Let us set $x^*x + xx^* = a$. Since the property of being projectionless passes to hereditary subalgebras, we may assume that a generates A as a hereditary subalgebra (i.e., a is strictly positive). Let us choose an approximate unit $e_n \in C^*(a)$ of A such that $e_{n+1}e_n = e_n$ for all $n \in \mathbb{N}$. Let us set $e_n x e_n = x_n$. Since $x_n \to x$, it suffices to show that $x_n \otimes 1$ is the product of two nilpotents for all $n \in \mathbb{N}$. Let $f_n \in C^*(a)_+$ be such that $e_n f_n = 0$ and $\delta_n a \leqslant e_{n+1} + f_n$ for some scalar $\delta_n > 0$ and all n. The desired conclusion will follow from the previous lemma once we have shown that f_n is a full element for all $n \in \mathbb{N}$. Let us fix $n \in \mathbb{N}$ and let I denote the closed two-sided ideal generated by f_n. Let us suppose for the sake of contradiction that $I \neq A$. In A/I, we have that $\delta_n \bar{a} \leqslant \bar{e}_{n+1}$, where $\bar{a}, \bar{e}_{n+1} \in A/I$ denote the images of a and e_{n+1}. This implies that 0 is an isolated point of the spectrum of $\bar{a}$, which in turn implies that A/I contains a non-zero projection. But this contradicts the assumption that A is projectionless. Thus, f_n is full for all $n \in \mathbb{N}$. □

3. Proofs of Theorems 1.2 and 1.3.

DEFINITION 3.1. Let A be a C*-algebra. Let us say that A almost has stable rank one if for every closed hereditary subalgebra $B \subseteq A$ we have $B \subseteq \overline{\mathrm{GL}(B^\sim)}$.

COROLLARY 3.2. *Let A be a $\mathcal{Z}$-stable projectionless C*-algebra. Then A almost has stable rank one.*

Proof. This follows from Theorem 1.1 and the fact that $x + \lambda \cdot 1$ is invertible if x is nilpotent and $\lambda \in \mathbb{C}\backslash\{0\}$. □

Before proving Theorems 1.2 and Theorem 1.3, let us introduce notation and make some preparatory remarks.

Let A be a C*-algebra. Let us denote by $\mathrm{QT}_2(A)$ the cone of lower semicontinuous 2-quasitraces on the C*-algebra A. A lower semicontinuous 2-quasitrace $\tau \in \mathrm{QT}_2(A)$ induces a dimension function d_τ on the positive elements of A given by $d_\tau(a) = \lim_n \tau(a^{1/n})$ for $a \in A^+$. The value of d_τ on a depends only on the Cuntz class of a.

Let $a, b \in A_+$. If A has stable rank one, then $a \precsim b$ if and only if $\overline{aA}$ embeds as a Hilbert C*-module over A in $\overline{bA}$ and $a \sim b$ if and only if $\overline{aA}$ is isomorphic to $\overline{bA}$. This was first shown in [**5**, Theorem 3] using the language of Hilbert C*-modules. It was later re-proven using positive elements in [**4**, Proposition 1] and [**9**, Proposition 1.5]. By inspecting either the proof in [**5**] or in [**4**], it can be readily seen that they rely only on the fact that $B \subseteq \overline{\mathrm{GL}(B^\sim)}$ for every σ-unital hereditary subalgebra B of A. Thus, we arrive at the following proposition:

PROPOSITION 3.3. *Let A be a C*-algebra. Let $a, b \in A_+$. Suppose that A almost has stable rank one. Then $a \precsim b$ if and only if $\overline{aA} \hookrightarrow \overline{bA}$ and $a \sim b$ if and only if $\overline{aA} \cong \overline{bA}$.*

Let $a \in A_+$. The Cuntz semigroup element $[a]$ is called compact if it is compactly contained in itself; i.e., $[a] \ll [a]$. It is shown in [**2**] that if $[a]$ is compact then either 0 is an isolated point in the spectrum of a or A contains a scaling element (see [**2**, Lemma 3.1 and Proposition 3.2]). In either case, A contains a non-zero projection. It follows that if A is projectionless and $a \in A_+$ then neither $[a]$ not its image after passing to a quotient of A can be a non-zero compact element. Thus, $[a]$ is a purely non-compact element in the sense of [**7**].

Proof of Theorem 1.2 The equivalence if (i) and (ii) and the fact that Cuntz equivalence implies isomorphism of the right ideals generated by a and b, follow from Corollary 3.2 and Proposition 3.3. The implication (i)⇒(iii) is well known. Finally, let us show (iii)⇒(i). As it was argued in the previous paragraph, every non-zero positive element of A gives rise to a purely non-compact element of $\mathrm{Cu}(A)$. But it is shown in [**7**, Theorem 6.6] that if A is $\mathcal{Z}$-stable and $[a]$ and $[b]$ are purely non-compact elements such that $d_\tau(a) \leqslant d_\tau(b)$ for any $\tau \in \mathrm{QT}_2(A)$ then $[a] \leqslant [b]$. This concludes the proof. □

Proof of Theorem 1.3 The implications (i)⇒(ii)⇒(iii) are clear. Let us prove (iii)⇒(i). By Theorem 1.2, $[(a-t)_+] = [(b-t)_+]$ for all $t \geqslant 0$. By the classification result [**12**, Theorem 1], (i) will follow once we have shown that for every $x, e \in A$, with e a positive contraction such that $ex = xe = x$, the elements $x^*x + e$ and $xx^* + e$ are stably approximately unitarily equivalent. It is shown in [**12**, Proposition 4 (i)] that C*-algebras of stable rank one have this property. Exactly the same proof applies to a C*-algebra with the almost stable rank one property, and in particular to a $\mathcal{Z}$-stable projectionless C*-algebra. □

It is not true that a $\mathcal{Z}$-stable projectionless C*-algebra must have stable rank one. An example can be obtained as follows:

EXAMPLE 3.4. Let A be a simple $\mathcal{Z}$-stable projectionless C*-algebra such that $\mathrm{K}_1(A) \neq 0$. Examples of such algebras are well known (see [**10**]). Then $C[0,1] \otimes A$ is $\mathcal{Z}$-stable and projectionless. On the other hand, $\mathrm{K}_1(A) \neq 0$ is a known obstruction to

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$C[0, 1] \otimes A$ having stable rank one, since any path connecting two unitaries in $M_n(A)$ with different K_1-class, viewed as an element of $C[0, 1] \otimes M_n(A)$, cannot be in the closure of the invertibles.

It is worth noting that, as shown in [**14**], tensoring an approximately subhomogeneous C*-algebra by the Jacelon–Razak algebra does result in a ($\mathcal{Z}$-stable and projectionless) C*-algebra of stable rank one. The following question remains open:

QUESTION 3.5. Let A be a simple, stably projectionless, and $\mathcal{Z}$-stable C*-algebra. Is A of stable rank one?

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