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Solvable Fundamental Groups of Algebraic Varieties and Kähler Manifolds

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Abstract. It is shown that if the fundamental group of a normal algebraic variety, respectively Zariski open subset of a compact Kähler manifold, is solvable with a faithful linear representation over $\mathbb{Q}$, respectively polycyclic, then it is virtually nilpotent.

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Our objective, in this paper, is to gain some understanding of those groups which arise as fundamental groups of compact Kähler manifolds, Zariski open subsets of compact Kähler manifolds, or normal complex algebraic varieties. The groups of the first type have come to be known as Kähler groups, and we will refer to those of the second as quasi-Kähler . While the complete structure of these groups seems rather intractable at the moment (see [2] for the state of of the art), the structure of certain subclasses are becoming much clearer. Of specific interest for us is the class of nilpotent groups. Many restrictions on, as well as interesting examples of, nilpotent Kähler groups have been found by Campana [8] and Carlson and Toledo [9]. Although nilpotent quasi-Kähler groups have not been studied systematically, nontrivial constraints can be obtained from Morgan's work [15]. The main conclusion of this paper, is that if one casts the net a little wider, then no really interesting new examples are obtained. In particular, we will show that a polycyclic quasi-Kähler group is virtually nilpotent, which is to say that it contains a nilpotent subgroup of finite index. For algebraic varieties, we can make an even stronger statement that the fundamental group of a normal variety must be virtually nilpotent if it is solvable and possesses a faithful representation into $GL_n(\mathbb{Q})$

The first two sections of this paper are purely group theoretic. We introduce the class of solvable groups of finite rank, which contains the class of solvable subgroups of $GL_n(\mathbb{Q})$ (and polycyclic groups in particular). Canonically attached to every such group Γ is an algebraic group $H(\Gamma)$ defined over $\mathbb{Q}$, and a representation $\Gamma \to H(\Gamma)$ with Zariski dense image. This generalizes earlier constructions of Malcev [14] and Mostow [17] for nilpotent and polycyclic groups, respectively.

The third section contains the first main theorem, that if the quotient Γ of the fundamental group of a normal variety by a term of the derived series has finite rank, then Γ contains an extension of a nilpotent group by a torsion group as a subgroup of finite index. The basic idea is to choose a finitely generated field of definition, and observe that the Galois group acts on the group of $\mathbb{Q}_l$ points of $H(\Gamma)$ for almost all primes l. This together with certain arithmetic considerations, forces the identity component $H(\Gamma)^{\circ}$ to be unipotent, and the theorem follows easily from this. The fourth section of the paper contains the second main theorem: If π is a quasi-Kähler group, with finitely generated derived group, such that the quotient Γ of π by a term of the derived series has finite rank, then Γ contains a nilpotent by torsion subgroup of finite index. Once again the strategy is to establish the unipotency of $H(\Gamma)^{\circ}$. However, this time it is reduced to a homological statement which is shown to be a consequence a generalization of a theorem of Beauville [5] obtained by the first author [3].

The reader will certainly have noticed that the two theorems are very similar in content, but quite different in methods of proof. This by itself should not come as a surprise; a number of results in algebraic geometry, such as the quasiunipotence of local monodromy, have been proved by both arithmetic and transcendental methods. In fact, the style of the first argument is quite similar to that of Grothendieck's proof this theorem. What does seem a bit curious is the slight disparity of the results. There exists finitely presented solvable subgroups of $\mathrm{GL}_n(\mathbb{Q})$ which are not virtually nilpotent or even virtually polycylic. For example, for each prime p, the group

$$\left\{ \begin{pmatrix} 1 & a & b & c \\ 0 & d & e & f \\ 0 & 0 & g & h \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle|\, a, b, c, e, f, h \in \mathbb{Z}\left[\frac{1}{p}\right], d, g \in \mathbb{Z}\left[\frac{1}{p}\right]^{*} \right\}$$

has this property [1]. Such groups cannot be fundamental groups of normal varieties by the first theorem, but the second theorem gives no information. We leave open the question of whether such groups can be quasi-Kähler. It is worth noting that a second proof that nonvirtually nilpotent solvable subgroups of $\mathrm{GL}_n(\mathbb{Q})$ are not fundamental groups of smooth projective varieties can be obtained by combining the above arguments with those of Simpson [20]. Once again, Simpson's arguments are arithmetic in nature and do not apply to nonalgebraic Kähler manifolds.

1. Preliminaries on Algebraic Groups

A general reference for this section is (7). Let $DG = D^1G$ be the derived subgroup of a group G, and set $D^iG = DD^{i-1}G$. If $F \subset E$ is an extension of fields, and G is an algebraic group defined over F, let $G_E = G \times_{\mathrm{spec}\, F} \mathrm{spec}\, E$. For the remainder

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of this section, G will denote an algebraic group over a field F of characteristic 0, and $U(G)$ will be its unipotent radical. We will make use of the following result of Mostow [16]

THEOREM 1.1. *If G is as above, then the exact sequence*:

$$1 \to U(G) \to G \to G/U(G) \to 1$$

is split, and any two splittings are conjugate by an F rational point of $U(G)$.

Let V be the centralizer of $U(G)$ in G, and let $W = V \cap U(G)$. Clearly W is the unipotent radical of V. It follows that $1 \to W \to V \to V/W \to 1$ has a unique splitting $s\colon V/W \to V$, because W central in V. Put $N(G) = s(V/W)$. By construction $N(G)$ is reductive and invariant under all automorphisms of G; in particular, it is normal in G.

LEMMA 1.2 [17, Lemma 4.6]. *Every normal reductive algebraic subgroup of G is contained in $N(G)$.*

DEFINITION 1.3. An algebraic group G is minimal if $N(G)$ is trivial. Let $G_{\min} = G/N(G)$; then this is a minimal algebraic group. An algebraic group G is minimal if and only if the centralizer of $U(G)$ is contained in $U(G)$, as we see from the definition of $N(G)$.

LEMMA 1.4. *Let $f\colon G \to G'$ be a homomorphism of algebraic groups for which $f(G)$ is normal in G'. Then $f(N(G)) \subseteq N(G')$ and, consequently, f induces a homomorphism $f_{\min}\colon G_{\min} \to G'_{\min}$. If f is an injection* (*respectively surjection*), *then $f_{\min}$ is also an injection* (*respectively, surjection*).

Proof. As $f(N(G))$ is normal and reductive in $f(G)$, we see that $f(N(G)) \subseteq N(f(G))$. As $N(f(G))$ is invariant under all automorphisms of $f(G)$, and in particular the restriction of inner automorphisms of G' to $f(G)$, it follows that $N(f(G)) \subseteq N(G')$. This proves the first assertion.

Because $H = f(G) \cap N(G')$ is normal in the reductive group $N(G')$, we see that H is reductive. Also, H is normal in $f(G)$, and this implies that $H = N(f(G))$, Thus $\ker(f) = \{1\}$ implies $\ker(f_{\min}) = \{1\}$.

LEMMA 1.5. *Let G be a minimal algebraic group defined over a field F of characteristic* 0. *There is an affine algebraic group A, defined over F, that acts on G so that for all fields $E \supseteq F$, $A(E) \to \mathrm{Aut}(G_E)$ is an isomorphism.*

Proof. If $G = U(G)$, the automorphism of G are just automorphisms of its Lie algebra, and these evidently form an affine algebraic group, denoted by $\mathrm{Aut}(G)$.

In the general case, by 1.1 we can express G as a semidirect product of M and $U(G)$, where M is reductive. The homomorphism $\rho\colon M \to \mathrm{Aut}(U(G))$ is faithful, because G is minimal. Let X be the normalizer of $\rho(M)$ in $\mathrm{Aut}\, U(G)$. Thus X is an algebraic group, and X acts naturally on $U(G)$ and M and, hence, on G, their semidirect product. Now G acts on itself by conjugation. Thus Y, the semidirect product of X and G, acts on G. For any field $E \supset F$, we see $X(E) \to \operatorname{Aut} G_E$ is one to one and its image equals $\{\phi \in \operatorname{Aut} G_E \mid \phi(M_E) = M_E\}$. But, if $\phi \in \operatorname{Aut}(G_E)$, $\phi(M_E)$ is a conjugate of M_E, and we deduce that $Y(E) \to \operatorname{Aut} G_E$ is surjective for all fields E containing F. Finally, the coordinate ring R of G is generated as an F-algebra by a finite-dimensional Y-stable subspace $V \subset R$. Put $K = \ker(Y \to \mathrm{GL}(V))$ and let $A = Y/K$. We see that A is the desired algebraic group.

The algebraic group A in the previous lemma will be henceforth denoted by $\operatorname{Aut} G$.

LEMMA 1.6. *If G is a solvable minimal algebraic group, then the action of* $\operatorname{Aut}(G)^o$ *on $G^o/U(G)$ is trivial.*

Proof. This follows immediately from the fact that the action of a connected algebraic group on a torus is trivial.

LEMMA 1.7. *Let Λ be a directed set, and $\{G_\lambda\}_{\lambda\in\Lambda}$ a directed system of minimal algebraic groups, such that for each $\lambda \leqslant \mu$, we have $G_\lambda \subseteq G_\mu$ and $U(G_\lambda) = U(G_\mu)$. Then there is a minimal algebraic group G and a monomorphism $f_\lambda\colon G_\lambda \to G$ for each $\lambda \in \Lambda$, so that $f_\lambda = f_\mu|_{G_\lambda}$ whenever $\lambda \leqslant \mu$.*

Proof. Let $U = U(G_\lambda)$ for all $\lambda \in \Lambda$. Let S_λ be the set of closed subgroups $M \subseteq G_\lambda$ such that $M \to G_\lambda/U$ is an isomorphism. Then U acts transitively on S_λ. Also $M \to M \cap G_\lambda$ gives a U-equivariant morphism from S_μ to S_λ whenever $\lambda \leqslant \mu$. Now choose $\lambda_0 \in \Lambda$ so that $\dim S_{\lambda_0} \geqslant \dim S_\lambda$ for all $\lambda \in \Lambda$. From the above, we see that $S_\mu \to S_{\mu_0}$ is a bijection if $\lambda_0 \leqslant \mu$. It follows that there is a collection $\{M_\lambda \mid \lambda \in \Lambda\}$ with $M_\lambda \in S_\lambda$, and $M_\mu \cap G_\lambda = M_\lambda$ whenever $\lambda \leqslant \mu$.

If $\rho_\lambda\colon M_\lambda \to \operatorname{Aut} U$ denotes the conjugation action of M_λ on U, we have seen that ρ_λ is one to one because G_λ is minimal. Also, the inequality $\lambda \leqslant \mu$ implies that $\rho_\lambda(M_\lambda) \subseteq \rho_\mu(M_\mu)$. Let M be the Zariski closure of $\cup\{\rho_\lambda(M_\lambda) \mid \lambda \in \Lambda\}$. This is reductive. Let G be the semidirect product of M and U, and define $f_\lambda\colon G_\lambda \to M$ by $f_\lambda(u) = u$ for all $u \in U$, and $f_\lambda = \rho_\lambda$ on M_λ. This completes the proof.

Let G be a connected solvable group defined over F. Let $U = U(G)$ and T a maximal torus. As noted previously, $\operatorname{Aut} U$ is an algebraic group which coincides with the group of automorphisms of the Lie algebra N of U. There is a homomorphism of algebraic groups $T \to \operatorname{Aut} U$ given by conjugation. G is the semidirect product of T with U.

LEMMA 1.8. *With the previous notation, suppose that T acts trivially* (*by conjugation*) *on U/DU then G is isomorphic to the product of U and T, and is therefore nilpotent.*

Proof. We have to show that T acts trivially on U. Let S be the subgroup of $\operatorname{Aut}(N)$ of automorphisms σ satisfying $(1-\sigma)(N) \subseteq [N, N]$. The elements of S are unipotent. By assumption, the image of the homomorphism $T \to \operatorname{Aut}(U) = \operatorname{Aut}(N)$ lies in S and so the map must be trivial.

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There is a homomorphism of algebraic groups $G/DG \to \mathrm{Aut}\, DG/D^2G$ given by conjugation.

LEMMA 1.9. *With the previous notation, suppose that image of G/DG in* $\mathrm{Aut}\, DG/D^2G$ *is unipotent. Then G is the product of U with T.*

Proof. Note that G/DG is a product of a reductive group, which is isomorphic to T, and a unipotent group U', which is isomorphic to the image of U under the projection $G \to G/DG$. The action of T on U' by conjugation is of course trivial. Consider the exact sequence $DG/D^2G \to U/DU \to U' \to 0$. By assumption, the image of $T \subset G/DG$ in $\mathrm{Aut}\, DG/D^2G$ is unipotent, and therefore trivial. Thus T acts trivially on the image of DG/D^2G in U/DU as well as on it the cokernel. Therefore T acts trivially on U/DU, and the lemma follows from the previous one.

2. Solvable Groups of Finite Rank

In this section, we shall associate to a solvable group Γ of finite rank, an algebraic group of $H(\Gamma)$ defined over $\mathbb{Q}$ and a homomorphism $i(\Gamma)\colon \Gamma \to H(\Gamma)(\mathbb{Q})$ with Zariski dense image and a torsion subgroup as kernel. While $\Gamma \mapsto H(\Gamma)$ is a not a functor, automorphisms of Γ will extend to automorphisms of $H(\Gamma)$. Furthermore, when Γ is finitely generated, automorphisms of its profinite completion $\hat{\Gamma}$ extend to automorphisms of $H(\Gamma)_{\mathbb{Q}_l}$ for all but finitely many primes l. Our construction of $H(\Gamma)$ is identical to Mostow's [17] in the case where Γ is polycyclic, but our proof that this construction works for solvable groups of finite rank is necessarily a bit more complicated. Also the results on the profinite completion are used crucially when Galois theory is applied. For these reasons, we have chosen to give all the details of the proofs.

Let us recall the standard construction of the proalgebraic hull of $\mathcal{H}(\pi, F)$ associated to a topological group π and a topological field F. One considers the category $C(\pi, F)$ where the objects are pairs (G, f) with G an affine algebraic group defined over F, and $f\colon \pi \to G(F)$ a continuous homomorphism with Zariski dense image. A morphism $(G, f) \to (G', f')$ in our category is simply a commutative diagram

$$\begin{tikzcd} G \arrow[rr, "\phi"] & & G' \\ & \pi \arrow[ul] \arrow[ur] & \end{tikzcd}$$

such that ϕ is homomorphism of algebraic groups. Such a ϕ, if it exists is unique and necessarily an epimorphism because f and f' have Zariski dense images. In particular, by Lemma 1.4, $\phi_{\min}\colon G_{\min} \to G'_{\min}$ is defined. Set

$$\mathcal{H}(\pi, F) = \varprojlim_{(G,f)} G \quad \text{and} \quad H(\pi, F) = \varprojlim_{(G,f)} G_{\min}.$$

Some easy observations follow.

Remark 2.1. $\pi \mapsto \mathcal{H}(\pi, F)$ is a functor, but $\pi \mapsto H(\pi, F)$ is not. However, if $a: \pi \to \pi'$ is a continuous homomorphism with the closure of $a(\pi)$ normal in π', we deduce, from Lemma 1.4 that $H(a, F): H(\pi, F) \to H(\pi', F)$ is defined. In particular, if Γ is a discrete group and $t: \Gamma \to \hat{\Gamma}$ is the homomorphism to its profinite completion, we have a natural epimorphism $H(t, F): H(\Gamma, F) \to H(\hat{\Gamma}, F)$.

Remark 2.2. If $F \to E$ is a continuous homomorphism of fields, we have a functor $C(\pi, F) \to C(\pi, E)$. And this induces an epimorphism $H(\pi, E) \to H(\pi, F)_E$.

Remark 2.3. By construction, we have a continuous homomorphism $i(\pi, F)$ from π to the group of F-rational points of $H(\pi, F)$. With $a: \pi \to \pi'$ as in Remark 2.1, we have $H(a, F) \circ i(\pi, F) = i(\pi', F) \circ a$.

DEFINITION 2.4. A solvable group Γ has finite rank, if there is a decreasing sequence $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_{m+1} = \{1\}$ of subgroups, each normal in its predecessor, such that Γ_i/Γ_{i+1} is Abelian and $\mathbb{Q} \otimes (\Gamma_i/\Gamma_{i+1})$ is finite dimensional for all i. The rank $rk(\Gamma) = \sum_{i=0}^{m} \dim(\mathbb{Q} \otimes (\Gamma_i/\Gamma_{i+1}))$ is clearly independent of the choice of the sequence $\{\Gamma_i\}$.

This is a weakening of the notion of a polycyclic group which, in the above terms, amounts to requiring that each Γ_i/Γ_{i+1} is finitely generated.

For the remainder of this section, all groups considered are solvable of finite rank with discrete topology unless indicated otherwise. We endow $\mathbb{Q}$ with discrete topology, and abbreviate $H(\pi, \mathbb{Q})$, $i(\pi, \mathbb{Q})$, $H(a, \mathbb{Q})$ by $H(\pi)$ etc. The only fields F considered have characteristic zero.

THEOREM 2.5. *Let Γ be a solvable group of finite rank with discrete topology. Then*

(A) *$H(\Gamma)$ is an algebraic group (and not just a proalgebraic group).*
(B) $rk(\Gamma) = \dim U(H(\Gamma))$.
(C) *The kernel of $i(\Gamma): \Gamma \to H(\Gamma)(\mathbb{Q})$ is a torsion group.*
(D) *The image of $i(\Gamma)$ is Zariski dense.*

Part (D) of the theorem is immediate from the construction. When Γ is polycyclic, the theorem is due to Mostow [17, 4.9]. As a corollary we obtain a natural characterization of these groups.

COROLLARY 2.6. *A solvable group Γ has finite rank if and only if there exists a torsion normal subgroup $N \subset \Gamma$ such that Γ/N possesses an embedding into an affine algebraic group defined over $\mathbb{Q}$.*

Proof. One direction follows from the theorem. For the converse, suppose that $N \subseteq \Gamma$ is torsion subgroup and $i: \Gamma \to G(\mathbb{Q})$ a homomorphism into an algebraic group with kernel N. We can assume that G is solvable, after replacing it by the Zariski closure of Γ. Then the sequence $\Gamma_i = i^{-1}(D^i G(\mathbb{Q}))$ has the required properties.

DEFINITION 2.7. $n(\Gamma, F) = \sup\{\dim U(G) \mid (G, f) \in \mathrm{Obj}\, C(\Gamma, F)\}$. Thus $n(\Gamma, F) \in \mathbb{N} \cup \{\infty\}$.

LEMMA 2.8. *If* $1 \to \Gamma' \to \Gamma \to \Gamma'' \to 1$ *is exact*, $n(\Gamma, F) \leqslant n(\Gamma', F) + n(\Gamma'', F)$.

Proof. Let (G, f) be an object of $C(\Gamma, F)$. Now let G' be the Zariski closure of $f(\Gamma')$ and let $G'' = G/G'$. Then f induces $\bar{f}: \Gamma'' \to G''(F)$ and both $(G', f|_{\Gamma'})$ and $(G'', \bar{f})$ are in $C(\Gamma', F)$ and $C(\Gamma'', F)$ respectively. A short exact sequence of algebraic groups induces a short exact sequence of unipotent radicals, so the lemma follows.

LEMMA 2.9. $n(\Gamma, F) \leqslant rk(\Gamma)$, *where* F *is a field of characteristic zero.*

Proof. From the previous lemma, by induction on the length of the derived series of Γ, we are reduced to the case where Γ is Abelian. If $(G, f) \in \mathrm{Obj}\, C(\Gamma, F)$, then the inclusion $U(G) \hookrightarrow G$ is split by a homomorphism $p: G \to U(G)$. But in this case, $U(G)$ is a vector space, spanned by $p(f(\Gamma))$, and so evidently $rk(\mathbb{Q} \otimes \Gamma) \geqslant \dim U(G)$.

LEMMA 2.10. *If* $(G, f) \in \mathrm{Obj} C(\Gamma, F)$ *satisfies* $n(\Gamma, F) = \dim U(G)$, *then* $H(\Gamma, F) \to G_{\mathrm{min}}$ *is an isomorphism. In particular,* $H(\Gamma, F)$ *is an algebraic group and* $\dim U(H(\Gamma, F)) = n(\Gamma, F)$.

Proof. For the first assertion, we need to check that any morphism $\phi: (G', f') \to (G, f)$ induces an isomorphism $\phi_{\mathrm{min}}: G'_{\mathrm{min}} \to G_{\mathrm{min}}$. Now, ϕ being as epimorphism, restricts to an epimorphism of unipotent radicals. This gives $n(\Gamma, F) \geqslant \dim U(G') \geqslant \dim U(G) = n(\Gamma, F)$.

Therefore, $U(G') \to U(G)$ gives an isomorphism of Lie algebras, and is itself an isomorphism. Consequently, $\ker \phi$ is reductive, and ϕ_{min} is an isomorphism.

For the second assertion, we need to know that such a (G, f) exists, and this is assured by the previous lemma.

LEMMA 2.11. *Let* $1 \to \Gamma' \to \Gamma \to \Gamma'' \to 1$ *be exact. Assume that* $i(\Gamma', F)$ *extends to a homomorphism* $j: \Gamma \to G(F)$ *where* G *is an algebraic group that contains* $H(\Gamma', F)$. *Then*

(A) $H(\Gamma', F) \to H(\Gamma, F)$ *is a monomorphism,*
(B) $n(\Gamma, F) = n(\Gamma', F) + n(\Gamma'', F)$,
(C) $1 \to \ker i(\Gamma', F) \to \ker i(\Gamma, F) \to \ker i(\Gamma'', F)$ *is exact.*

Proof. Let $q: \Gamma' \to \Gamma$ and $p: \Gamma' \to \Gamma''$ denote the given homomorphisms. By Remark 2.1, we have $H(q, F): H(\Gamma', F) \to H(\Gamma, F)$ and $H(p, F): H(\Gamma, F) \to$

$H(\Gamma'', F)$. Now $H(p, F)$ is an epimorphism whose kernel contains the normal subgroup image$(H(q, F))$. Thus, if we assume (A), we obtain

$$\begin{aligned} n(\Gamma, F) &= \dim U(H(\Gamma, F)) \\ &\geqslant \dim U(H(\Gamma', F)) + \dim U(H(\Gamma'', F)) \\ &= n(\Gamma', F) + n(\Gamma'', F) \end{aligned}$$

from the previous lemma. But Lemma 2.8 gives the reverse inequality, and this proves part (B). That (A) implies (C) is clear by Remark 2.3, for we have

$$H(p, F) \circ i(\Gamma, F) = i(\Gamma'', F)$$

and

$$H(q, F) \circ i(\Gamma', F) = i(\Gamma, F) \circ q.$$

To check (A), replace G in the lemma by the Zariski closure of $f(\Gamma)$, This makes (G, j) an object of $C(\Gamma, F)$; denote by $k\colon H(\Gamma, F) \to G_{\min}$ the natural homomorphism. Then $k \circ H(q, F)$ is the composite: $H(\Gamma', F) \to G \to G_{\min}$. By Lemma 1.4, this is an inclusion, This completes the proof of the lemma.

LEMMA 2.12. *The (G, j) in the previous lemma exists if*

(A) $\Gamma'' \cong \mathbb{Z}$, *or*
(B) Γ'' *is a Abelian torsion group.*

Proof. Case (A): Here Γ is a semidirect product. Choose $\gamma \in \Gamma$ that maps to a generator of Γ''. Then $\sigma(\delta) = \gamma\delta\gamma^{-1}$, for $\delta \in \Gamma'$ gives an automorphism of Γ', and induces therefore an automorphism $H(\sigma, F)$ of the algebraic group $H(\Gamma', F)$. By Lemma 1.5, $A = \operatorname{Aut} H(\Gamma, F)$ is an algebraic group. Let G be the semidirect product of A and $H(\Gamma, F)$, and define $j\colon \Gamma \to G(F)$ by $j(\delta) = i(\Gamma')(\delta)$ for $\delta \in \Gamma'$ and $j(\gamma) = H(\sigma, F)$.

Case (B): First assume that Γ'' is finite. Let $\rho\colon H(\Gamma', F) \to \mathrm{GL}(V)$ be a faithful representation where V is finite-dimensional vector space defined over F. Consider the induced representation $W = F[\Gamma] \otimes_{F[\Gamma']} V$. Let $G = \mathrm{GL}(W)$, and $j\colon \Gamma \to G$ be the action of Γ on W. Clearly, there is a monomorphism $k\colon H(\Gamma', F) \to \mathrm{GL}(W)$ so that $k \circ i(\Gamma', F) = j|_{\Gamma'}$, so the result follows.

In the general case, let $\mathcal{S} = \{\pi \subseteq \Gamma \mid \pi \supseteq \Gamma',\ \pi/\Gamma' \text{ is finite}\}$. If $\pi_1, \pi_2 \in \mathcal{S}$ and $\pi_1 \subseteq \pi_2$, then π_1/π_2 is finite. Therefore, from the first part of the previous lemma, $H(\pi_1, F) \to H(\pi_2, F)$ has no kernel. From Lemma 1.7, we get

$$\Gamma = \varinjlim_{\pi \in \mathcal{S}} \pi \to \varinjlim_{\pi \in \mathcal{S}} H(\pi, F)(F) \to G(F)$$

and this completes the proof.

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We can now prove Theorem 2.5.

Proof. Part (A) has already proved in Lemma 2.10. The theorem is certainly true if $\Gamma \cong \mathbb{Z}$, it is also true if Γ is a commutative torsion group by Lemma 2.9. For the general case, we note that Γ has a filtration: $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_{m+1} = \{1\}$ such that the successive quotients are either Abelian torsion groups or isomorphic to $\mathbb{Z}$. We proceed by induction on m, and so we can assume the theorem for Γ_1. Now Lemmas 2.11, 2.12 give the theorem for Γ.

LEMMA 2.13. *If Γ is solvable of finite rank, then $H(\Gamma, F) \to H(\Gamma)_F$ an isomorphism.*

Proof. Put $G = H(\Gamma)_F$ and let $f: \Gamma \to H(\Gamma)_F(F)$ be given by $f(\gamma) = i(\Gamma)(\Gamma)$ for $\gamma \in \Gamma$. Then $(G, f) \in \operatorname{Obj} C(\Gamma, F)$ and $\dim U(G) = \dim U(H(\Gamma)) = rk\Gamma$ by the theorem. From Lemma 2.10, $H(\Gamma, F) \to G_{\min} = G$ is an isomorphism, and so the lemma is proved.

THEOREM 2.14. *Let Γ be a finitely generated solvable group of finite rank. There is a finite set of prime numbers S so that $H(\Gamma)_{\mathbb{Q}_l} \to H(\hat{\Gamma}, \mathbb{Q}_l)$ is an isomorphism for all primes $l \notin S$.*

Proof. We may regard $H(\Gamma)$ as an algebraic subgroup of $(\mathrm{GL}_n)_{\mathbb{Q}}$. Because Γ is finitely generated, $i(\Gamma)(\Gamma) \subset \mathrm{GL}_n(S^{-1}\mathbb{Z})$ for some finite set of primes S. If l is a prime not in S, then $\mathrm{GL}_n(S^{-1}\mathbb{Z}) \subset \mathrm{GL}_n(\mathbb{Z}_l)$, and the latter is a profinite group. This gives a continuous homomorphism $f_l \colon \hat{\Gamma} \to \mathrm{GL}_n(\mathbb{Z}_l)$. Because $image(i(\Gamma)) \subset H(\Gamma)(\mathbb{Q}_l)$ and the second group is closed in $\mathrm{GL}_n(\mathbb{Q}_l)$, we see that $f_l(\hat{\Gamma}) \subset H(\Gamma)(\mathbb{Q}_l)$. The object $(H(\Gamma)_{\mathbb{Q}_l}, f_l)$ of $C(\hat{\Gamma}, \mathbb{Q}_l)$ gives an epimorphism $H(\hat{\Gamma}, \mathbb{Q}_l) \to H(\Gamma)_{\mathbb{Q}_l}$. By the previous lemma and Remark 2.1, we have $H(\Gamma)_{\mathbb{Q}_l} \to H(\hat{\Gamma}, \mathbb{Q}_l)$, and these arrows are inverses of each other.

LEMMA 2.15. *If Γ is solvable of finite rank and if $H(\Gamma)^{\circ}$ is unipotent, then there are normal subgroups $\Gamma_1 \supseteq \Gamma_2$ of Γ so that*

(a) Γ/Γ_1 *is finite,*
(b) Γ_1/Γ_2 *is nilpotent, and*
(c) Γ_2 *is torsion.*

Proof. We take $\Gamma_1 = i(\Gamma)^{-1}H(\Gamma)^{\circ}(\mathbb{Q})$ and $\Gamma_2 = \ker(i(\Gamma))$. Then $\Gamma_1/\Gamma_2 \subset H(\Gamma)^{\circ}(\mathbb{Q})$ and the latter is nilpotent. From Theorem 2.5 Γ_2 is torsion, and $\Gamma/\Gamma_1 \subset H(\Gamma)/H(\Gamma)^{\circ}(\mathbb{Q})$ is finite.

3. Fundamental Groups of Varieties

Let Y be a normal variety defined over a subfield $K \subset \mathbb{C}$ with a K-rational point y_0. For any field extension $K' \supseteq K$, set $Y_{K'} = Y \times_{\operatorname{spec} K} \operatorname{spec} K'$. Let $\pi_1^{\mathrm{alg}}(X)$ denote the algebraic fundamental group of a connected scheme X (with an unspecified base point), and $\hat{\pi}$ the profinite completion of a group π. Then we have a split

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exact sequence $1 \to \hat{\pi}_1(Y_{\mathbb{C}}^{\mathrm{an}}) \to \pi_1^{\mathrm{alg}}(Y_K) \to \mathrm{Gal}(\bar{K}/K) \to 1$, (where the splitting depends on y_0) (13, IX 6.4, XII 5.2). Thus $\mathrm{Gal}(\bar{K}/K)$ acts continuously on $\hat{\pi}_1(Y_{\mathbb{C}})$, and therefore also on the Abelianization $H_1(Y_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Z}) \otimes \hat{\mathbb{Z}}$ and its pro-l part $H_1(Y_{\mathbb{C}}^{an}, \mathbb{Z}) \otimes \mathbb{Z}_l$.

LEMMA 3.1. *Let Y be a normal variety defined over a finitely generated field $K \subset \mathbb{C}$. Then $H_0(\mathrm{Gal}(\bar{K}/K), H_1(Y_{\mathbb{C}}^{an} \otimes \mathbb{Z}_l))$ is finite.*

Proof. The group $H_1(Y)/(torsion) \otimes \mathbb{Z}_l$ is dual to $H^1_{\mathrm{et}}(Y_{\bar{K}}, \mathbb{Z}_l)$ as a $\mathrm{Gal}(\bar{K}/K)$-module, thus it suffices to prove that the second group has no invariants. Let $p\colon \tilde{Y} \to Y_{K'}$ be a desingularization defined over a finite extension $K' \supseteq K$. By Zariski's main theorem the geometric fibers of p are connected, thus $H^1_{\mathrm{et}}(Y_{\bar{K}}, \mathbb{Z}_l)$ injects into $H^1_{\mathrm{et}}(\tilde{Y}_{\bar{K}}, \mathbb{Z}_l)$, and this is compatible with the $\mathrm{Gal}(\bar{K}/K')$-action. $H^1(\tilde{Y}_{\bar{K}})$ has no $\mathrm{Gal}(\bar{K}/K')$-invariants, because the eigenvalues of the Frobenius at any prime of good reduction have absolute value $q^{1/2}$ or q by [11, Sect. 3.3].

Remark 3.2. The argument can be simplified (and lengthened) in a couple of ways. An appropriate Lefschetz type argument allows one to reduce to the case where Y is a curve where the relevant estimate on eigenvalues of the Frobenius goes back to Weil. Alternatively, when Y is curve, one can deduce the finiteness of $H_0(\mathrm{Gal}(\bar{K}/K), H_1(Y_{\mathbb{C}}^{an} \otimes \mathbb{Z}_l))$ directly from class field theory.

THEOREM 3.3. *Let X be a normal* (*not necessarily complete*) *algebraic variety defined over $\mathbb{C}$. Let $\pi = D^0\pi \supseteq D^1\pi \supseteq \cdots$ be the derived series of $\pi = \pi_1(X, x_0)$. If there is a natural number n so that $\pi/D^n\pi$ is solvable of finite rank, then there are normal subgroups $P \supseteq Q \supseteq D^n\pi$ of π so that*

(a) *π/P is finite,*
(b) *P/Q is nilpotent, and*
(c) *$Q/D^n\pi$ is a torsion group.*

Proof. Put $\Gamma = \pi/D^n\pi$, and $T = H(\Gamma)^0/U(H(\Gamma))$. By Lemma 2.15, the theorem follows once it has been proved that T is trivial. We may assume that X and x_0 are defined over a finitely generated field $K \subset \mathbb{C}$. There is an action of $\mathrm{Gal}(\bar{K}/K)$ on $\hat{\pi}$, and also on $\hat{\Gamma}$, because this is a quotient of $\hat{\pi}$ by the closure of $D^n(\hat{\pi})$. Choose a prime l, so that $\Gamma \to H(\Gamma)(\mathbb{Q})$ extends to a continuous homomorphism $\hat{\Gamma} \to H(\Gamma)(\mathbb{Q}_l)$. For such a prime, $H(\hat{\Gamma}, \mathbb{Q}_l) = H(\Gamma)_{\mathbb{Q}_l}$ by Theorem 2.14. Thus the Galois action on $\hat{\Gamma}$ yields a homomorphism from $\mathrm{Gal}(\bar{K}/K)$ to the group of $\mathbb{Q}_l$-rational points of $G = \mathrm{Aut}\,(H(\Gamma))$. After replacing K by a finite extension, if necessary, we can assume image$(\rho) \subset G^0(\mathbb{Q}_l)$. By Lemma 1.6, the action of $\mathrm{Gal}(\bar{K}/K)$ on T_{Q_l} is trivial. Let $\pi' = \ker[\pi \to (H(\Gamma)/H(\Gamma)^0)(\mathbb{Q})]$. Then $\pi' = \pi_1(Y, y_0)$ where Y is an etale cover of X. The composite $\hat{\pi}' \hookrightarrow \hat{\pi} \to (H(\Gamma)/U(H(\Gamma))(\mathbb{Q}_l)$ factors through $H_1(Y, \mathbb{Z}) \otimes \hat{\mathbb{Z}} = \hat{\pi}'_{\mathrm{ab}} \to T(\mathbb{Q}_l)$, $T(\mathbb{Q}_l)$ contains an open pro-l-group, thus $\hat{\pi}'$ further factors through

$$h\colon H_1(Y) \otimes \hat{\mathbb{Z}} \to H_1(Y) \otimes \mathbb{Z}_l \oplus A \to T(\mathbb{Q}_l),$$

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where A is a finite group. However, $H_0(\mathrm{Gal}(\bar{K}/K), H_1(Y) \otimes \mathbb{Z}_l)$ is finite by Lemma 3.1. So we deduce that the image of $H_1(Y) \otimes \mathbb{Z}_l$ in $T(\mathbb{Q}_l)$ is finite, because $\mathrm{Gal}(\bar{K}/K)$ acts trivially on $T(\mathbb{Q}_l)$. However the image h is Zariski dense. Thus T is finite and connected, and therefore trivial. This proves the theorem.

COROLLARY 3.4. *If the fundamental group of a normal complex variety is solvable and possesses a faithful representation into* $\mathrm{GL}_n(\mathbb{Q})$, *then it is virtually nilpotent, i.e. it must contain a nilpotent subgroup of finite index.*

Proof. We can assume that the fundamental group is torsion free after passing to a subgroup of finite index [19, Lemma 8]. The theorem implies that this must contain a nilpotent group of finite index.

4. Fundamental Groups of Kähler Manifolds

A group Γ will be called quasi-Kähler if it there exists a connected compact Kähler manifold X, and a divisor with normal crossings $D \subset X$, such that $\Gamma \cong \pi_1(X - D)$. (Note that by resolution of singularities [4, 6], it is enough to assume that D is an analytic subset.)

The proof of the following lemma will be given in the appendix.

LEMMA 4.1. *A subgroup of a quasi-Kähler group of finite index is quasi-Kähler.*

LEMMA 4.2. *Let A be a finitely generated Abelian group, and M a nontrivial one dimensional* $\mathbb{C}[A]$*-module. Then* $H^i(A, M) = 0$ *for all* i.

Proof. This is clear for cyclic groups by direct computation. In general, express A as a product of cyclic groups $\prod_i C_i$ and M as a tensor product of C_i-modules, and apply the Künneth formula.

Set $\Gamma^{ab} = \Gamma/D\Gamma$.

LEMMA 4.3. *Suppose that Γ is a finitely generated group and $A = \Gamma/N$ an abelian quotient. Suppose that M is a nontrivial one-dimensional A-module then*

$$H^1(\Gamma, M) \cong \mathrm{Hom}_{\mathbb{Z}[A]}(N^{\mathrm{ab}}, M) \cong \mathrm{Hom}_{\mathbb{Q}[A]}(N^{\mathrm{ab}} \otimes \mathbb{Q}, M).$$

Proof. From the Hochschild–Serre spectral sequence associated to the extension $1 \to N \to \Gamma \to A \to 0$, we obtain an exact sequence

$$0 \to H^1(A, M) \to H^1(\Gamma, M) \to H^0(A, H^1(N, M)) \to H^2(A, M).$$

By the previous lemma, this gives an isomorphism $H^1(\Gamma, M) \cong H^0(A, H^1(N, M))$. Furthermore, $H^1(N, M) \cong \mathrm{Hom}(N, M) = \mathrm{Hom}(N^{ab}, M)$ as A-modules. Therefore $H^0(A, H^1(N, M)) \cong \mathrm{Hom}_{\mathbb{Z}[A]}(N^{ab}, M)$.

Given a character $\rho \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)$, let $\mathbb{C}_\rho$ denote the associated Γ-module. We define $\Sigma^1(\Gamma)$ to be the set of characters $\rho \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)$ such that $H^1(\Gamma, \mathbb{C}_\rho)$ is nonzero. Let us say that a Γ-module V is quasi-unipotent if there is a subgroup $\Gamma' \subseteq \Gamma$ of finite index whose elements act unipotently on V.

LEMMA 4.4. *Let A be a finitely generated Abelian group and V a finite dimensional $\mathbb{C}[A]$-module. Then A acts quasi-unipotently on V if and only if the only characters* $\rho \in \mathrm{Hom}(A, \mathbb{C}^*)$ *for which* $\mathrm{Hom}_{\mathbb{C}[A]}(V, \mathbb{C}_\rho) \neq 0$ *are torsion characters* (*i.e. elements of* $\mathrm{Hom}(\Gamma, \mathbb{C}^*)$ *of finite order*).

Proof. Define V_ρ to be the generalized eigenspace associated to a character ρ. In other words, V_ρ is the maximal subspace on which $a - \rho(a)$ is nilpotent for all $a \in A$. V is a direct sum of these eigenspaces, thus we can assume that $V = V_\rho \neq 0$. To complete the proof, observe that $\mathrm{Hom}(V_\rho, \mathbb{C}_{\rho'}) \neq 0$ if and only if $\rho = \rho'$, and that V_ρ is quasi-unipotent if and only if ρ is torsion.

LEMMA 4.5. *Let Γ be a finitely generated group and $\Gamma' \subseteq \Gamma$ a subgroup of finite index such that $V = (\Gamma' \cap D\Gamma)^{ab} \otimes \mathbb{Q}$ is finite-dimensional. Then Γ' acts quasi-unipotently on V if and only if* $\Sigma^1(\Gamma') \cap \mathrm{image}(\mathrm{Hom}(\Gamma, \mathbb{C}^*) \to \mathrm{Hom}(\Gamma', \mathbb{C}^*))$ *consists of torsion characters*.

Proof. Set $N = \Gamma' \cap D\Gamma$. Then Γ'/N is isomorphic to the image of Γ' in Γ^{ab} and, therefore, $\mathrm{Hom}(\Gamma'/N, \mathbb{C}^*)$ is coincides with $\mathrm{image}(\mathrm{Hom}(\Gamma, \mathbb{C}^*) \to \mathrm{Hom}(\Gamma', \mathbb{C}^*))$. Thus, Lemma 4.3 implies that

$$S = (\Sigma^1(\Gamma') - \{1\}) \cap \mathrm{image}(\mathrm{Hom}(\Gamma, \mathbb{C}^*) \to \mathrm{Hom}(\Gamma', \mathbb{C}^*))$$

is the set of nontrivial characters $\rho \in \mathrm{Hom}(\Gamma'/N, \mathbb{C}^*)$ for which $\mathrm{Hom}_{\mathbb{C}[A]}(V \otimes \mathbb{C}, \mathbb{C}_\rho) \neq 0$. Thus Γ'/N acts quasi-unipotently on V if and only if S consists of torsion characters by the previous lemma. The Γ' action on V factors through Γ'/N, thus the lemma is proved.

LEMMA 4.6. *Let $K \supseteq \mathbb{Q}$ be a finite extension, and O_K the ring of integers of K. Let $\{\sigma_1, \dots \sigma_n\}$ be the set of all embeddings K into $\mathbb{C}$. Then for any finitely generated group Γ,* $\mathrm{Hom}(\Gamma, U(1)) \cap \bigcap_i \sigma_i \circ \mathrm{Hom}(\Gamma, O_K^*)$ *consists of torsion characters*.

Proof. This follows from Kronecker's theorem that an algebraic integer is a root of unity if and only if all its Galois conjugates have absolute value one.

THEOREM 4.7. *Let Γ be a quasi-Kähler group such that $D\Gamma$ is a finitely generated. Then for any subgroup $\Gamma' \subseteq \Gamma$ of finite index, Γ' acts quasi-unipotently on the finite-dimensional vector space* $(\Gamma' \cap D\Gamma)^{ab} \otimes \mathbb{Q}$.

Proof. By Lemma 4.5, it is enough to show that the intersection S of $\Sigma^1(\Gamma')$ and the image of $\mathrm{Hom}(\Gamma, \mathbb{C}^*)$ consists of torsion characters. The subgroup $\Gamma' \cap D\Gamma \subseteq D\Gamma$ has finite index, and is therefore finitely generated. Thus the set of characters of Γ' which correspond to one-dimensional quotients of $(\Gamma' \cap D\Gamma)^{\mathrm{ab}} \otimes \mathbb{C}$ are defined over the ring of integers of a finite extension K of $\mathbb{Q}$. It then follows from Lemma 4.3 that S is a finite subset of $\mathrm{Hom}(\Gamma, O_K^*)$. S is evidently stable under $\mathrm{Aut}(\mathbb{C})$ and thus lies in $\cap_{\sigma: K \hookrightarrow \mathbb{C}} \sigma \circ \mathrm{Hom}(\Gamma, O_K^*)$.

Theorem V.1.6 of [3] implies that $\Sigma^1(\Gamma')$ is a finite union of translates of subtori of $\mathrm{Hom}(\Gamma')$ by unitary characters. S, which is the intersection of this set with the image $(\mathrm{Hom}(\Gamma, \mathbb{C}^*))$, must clearly inherit a similar structure. In particular, being finite, it follows that S consists of unitary characters. Therefore, the theorem follows from Lemma 4.6.

LEMMA 4.8. *Let Γ be a solvable group of finite rank. Suppose that every subgroup $\Gamma' \subseteq \Gamma$ of finite index acts quasi-unipotently on the finite-dimensional vector space $(\Gamma' \cap D\Gamma)^{ab} \otimes \mathbb{Q}$. Then there exists normal subgroups $\Gamma_1 \supset \Gamma_2$ of Γ so that*

(a) Γ_1 *has finite index,*
(b) Γ_1/Γ_2 *is nilpotent, and*
(c) Γ_2 *is torsion.*

Proof. It suffices to prove that $G = H(\Gamma)^o$ is unipotent by Lemma 2.15 (then Γ_1 can be be taken to be $i(\Gamma)^{-1}(G)$ and $\Gamma_2 = \ker i(\Gamma)$). The unipotency of G will follow from Lemma 1.9, once we show that the action of G/DG on DG/D^2G, by conjugation, is unipotent. The map $(\Gamma_1 \cap D\Gamma)^{ab} \otimes \mathbb{Q} \to DG(\mathbb{Q})/D^2G(\mathbb{Q})$ is compatible with the Γ_1-actions, and is surjective, because the image of $\Gamma_1 \cap D\Gamma$ is Zariski dense in $DG(\mathbb{Q})$. By hypothesis, Γ_1 contains a finite index subgroup Γ'' which acts unipotently on $(\Gamma_1 \cap D\Gamma)^{ab} \otimes \mathbb{Q}$. As Γ'' is Zariski dense in G, the lemma follows.

THEOREM 4.9. *Let π be a quasi-Kähler group. Suppose that $D\pi$ is finitely generated and $\pi/D^n\pi$ is solvable of finite rank for some natural number n. Then there are normal subgroups $P \supseteq Q \supseteq D^n\pi$ of π so that*

(a) π/P *is finite,*
(b) P/Q *is nilpotent, and*
(c) $Q/D^n\pi$ *is a torsion group.*

Proof. The theorem follows from Theorem 4.7 and Lemma 4.8.

COROLLARY 4.10. *A polycyclic quasi-Kähler group is virtually nilpotent.*

Proof. By [18, 4.6], a polycyclic group contains a torsion free subgroup π of finite index. The theorem implies that π must contain a nilpotent subgroup of finite index.

5. Appendix A. Construction of Kähler Metrics

Most of the results in this appendix are well known, but we indicate proofs for lack of a suitable reference. The following is elementary, and left to the reader.

LEMMA A.1. *Let $W = U \oplus V$ be a finite-dimensional $\mathbb{C}$-vector space. Let $Q_{i,t}$, $i = 1, 2$ be two hermitian forms on W which depend continuously on a parameter t varying over a compact set T. Suppose that $Q_{1,t}$ is positive semidefinite and $Q_{1,t}|_U$ and $Q_{2,t}|_V$ are positive definite for all t. Then there exists a constant $x > 0$ such that $xQ_{1,t} + Q_{2,t}$ is positive definite for all t.*

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LEMMA A.2. *Let V be a holomorphic vector bundle over a complex manifold X. Then the hyperplane bundle $O(1)$ on $\pi:\mathbb{P}(V) \to X$ carries a Hermitian metric which restricts to the Fubini-Study metric on every fiber. This metric will be referred to as a Fubini-Study type metric.*

Proof. The line bundle $O(1)$ is a quotient of π^*V. Let h be a Hermitian metric on V. Then the metric on $O(1)$ induced from π^*h has the desired properties.

Given a Hermitian metric h on a line bundle, let $\tilde{c}_1(h)$ denote the first Chern form, given locally by $(i/2\pi)\partial\bar{\partial}\log||s||_h^2$ where s is holomorphic section.

LEMMA A.3. *Let X be compact complex manifold with a Kähler form ω. If V is a holomorphic vector bundle and h a Fubini-Study type metric on $O_{\mathbb{P}(V)}(1)$, then $C\pi^*\omega + \tilde{c}_1(h)$ is a Kähler form on $\pi:\mathbb{P}(V) \to X$ for all $C \gg 0$.*

Proof. The form $C\pi^*\omega + \tilde{c}_1(h)$ is real $(1, 1)$, and positive if $C \gg 0$ by A.1.

LEMMA A.4. *The blow up of a compact Kähler manifold along a submanifold is Kähler.*

Proof. Let $\pi:\tilde{X} \to X$ be the blowup of X along a closed submanifold S, and let E be the exceptional divisor. By construction [12], there is an open tubular neighbourhood U of S, such that the preimage $\tilde{U}$ embeds into $U \times \mathbb{P}(N)$, where N is the normal bundle of S. The restriction of $O(-E)$ to U coincides with the pullback of $O(1)$.

Let h_1 be the restriction of a Fubini-Study type metric to $O(-E)|_{\tilde{U}}$. Choose a C^∞ cutoff function $\rho:\tilde{X} \to [0, 1]$ which vanishes outside $\tilde{U}$ and is identically 1 on a neighbourhood of V of E. Let h_2 be a constant metric on the trivial bundle $O(-E)|_{X-\bar{V}}$. Then $h = \rho h_1 + (1-\rho)h_2$ defines a metric on $O(-E)$. If ω is a Kähler metric on X, then Lemma A.1 shows that $C\pi^*\omega + \tilde{c}_1(h)$ is Kähler for $C \gg 0$.

PROPOSITION A.5. *Suppose that X is a compact Kähler manifold, $D \subset X$ is a divisor with normal crossings, and $Y^\circ \to X - D$ is a finite sheeted unramified cover. Then Y° has a (nonsingular) Kähler compactification Y, such that $Y - Y^\circ$ is a divisor with normal crossings.*

Proof. Let $X^\circ = X - D$ and let n be the number of sheets of $Y^\circ \to X^\circ$. Then there exists a principal S_n-bundle P, such that Y^o is isomorphic to $P \times_{S_n} \{1 \dots n\}$ Let $S_n \to \mathrm{GL}_n(\mathbb{C})$ be the permutation representation associated to the standard basis $\{e_i\}$. Then we obtain a holomorphic vector bundle $V^\circ = P \times_{S_n} \mathbb{C}^n$. There is a holomorphic embedding $Y^\circ \mapsto V^\circ$ induced by the map $\{1 \dots n\} \to \mathbb{C}^n$ given by $i \mapsto e_i$. V° is a flat vector bundle, so it has a natural flat connection ∇. V° extends to a holomorphic bundle V on X with regular singularities with respect to ∇ [10]. Let Y' be the closure of Y° in V. It is easy to check that Y' is an analytic subset of V. We can embed V into the projective space bundle $P = \mathbb{P}(V \oplus O)$. By resolution of singularities [4, 6], there exists a commutative diagram

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$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d] & \tilde{P} \arrow[d] \\ Y' \arrow[r, hook] & P, \end{tikzcd}$$

where $\tilde{P} \to P$ is a composite of blow ups along smooth centers lying over D, Y is nonsingular, and $Y - Y^\circ$ is a divisor with normal crossings. $\tilde{P}$ is Kähler by the previous lemmas, therefore the same is true of Y.

As an immediate corollary, we obtain

LEMMA A.6. *A finite index subgroup of a quasi-Kähler group is quasi-Kähler .*

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