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FOURIER TRANSFORMS AND HARMONIC FUNCTIONS

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Abstract

The purpose of this paper is to present a novel proof of a well-known relationship between functions in harmonic subspaces of $L^2(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)$ and their Fourier transforms. The proof uses a characterisation of spherical harmonics given by Hecke and a method developed by the author in a previous paper.

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The purpose of this paper is to establish a well-known result concerning the behaviour of the Fourier transform in $\mathbf{R}^n$ with regard to the decomposition of $L^2(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)$ into harmonic subspaces. The approach is, I believe, novel in that firstly it uses exclusively a characterisation of spherical harmonic functions given by Hecke ([2] page 849 ff) and secondly it uses a method, derived from Tate's Thesis (see [1] page 305 ff) and developed in [4], bringing in a functional equation to obtain the final result. Thus this paper falls neatly into two parts, the first using Hecke's characterization of spherical harmonic functions to establish the necessary lemmas, the second, introducing an appropriate zeta function and establishing the desired functional equation.

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Let us begin by recalling the definition of a (solid) spherical harmonic function of degree k on $\mathbf{R}^n$.

DEFINITION. A spherical harmonic function of degree k is a homogeneous polynomial P_k of degree k, in n variables such that

$$\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} P_k(\mathbf{x}) = 0.$$

Hecke gave the following characterization of such functions:
$P_k(\mathbf{x})$ is a linear combination of functions of the type

$$\left(\sum_{i=1}^{n} x_i l_i \right)^k$$

where $\sum_{i=1}^{n} l_i^2 = 0$.

(By an abuse of notation I shall write $\sum_{i=1}^{n} x_i l_i \equiv \mathbf{x} \cdot \mathbf{l}$ even though $\mathbf{l} \in \mathbf{C}^n$.)

The following three results will now establish the invariance of the various harmonic subspaces under Fourier transforms.

LEMMA 1. *Let Ω_n be the unit sphere in $\mathbf{R}^n$, $\mathbf{u}, \mathbf{v} \in \Omega_n$, $\mathbf{z} \in \mathbf{R}^n$ and $d\mathbf{u}$ an invariant measure on Ω_n. Then for any continuous function, F, on $[-1, 1]$, the function*

$$f(\mathbf{v}, \mathbf{z}) = \int_{\Omega_n} F(\mathbf{u} \cdot \mathbf{v})(\mathbf{u} \cdot \mathbf{z})^k \, d\mathbf{u}$$

is a homogeneous polynomial in $\mathbf{z}$ whose value depends only on $\mathbf{v} \cdot \mathbf{z}$ and $\mathbf{z} \cdot \mathbf{z}$ that is, there exist constants C_{mp} depending only on F such that

$$f(\mathbf{v}, \mathbf{z}) = \sum_{m+2p=k} C_{mp} (\mathbf{v} \cdot \mathbf{z})^m (\mathbf{z} \cdot \mathbf{z})^p. \tag{1}$$

PROOF (see [3] page 6 ff). Clearly $f(\mathbf{v}, \mathbf{z})$ is a homogeneous polynomial in $\mathbf{z}$. Now let $\sigma \in 0(n, \mathbf{R})$ be an orthogonal transformation of $\mathbf{R}^n$. Then $f(\sigma\mathbf{v}, \sigma\mathbf{z}) = f(\mathbf{v}, \mathbf{z})$ since $d\mathbf{u}$ is an invariant measure. Given some fixed $\mathbf{e} \in \Omega_n$ there is a $\sigma \in 0(n, \mathbf{R})$ such that $\sigma\mathbf{v} = \mathbf{e}$. Then

$$f(\mathbf{v}, \mathbf{z}) = f(\mathbf{v}, (\mathbf{z} \cdot \mathbf{v})\mathbf{v} + \lambda\mathbf{w}) = f(\mathbf{e}, (\mathbf{z} \cdot \mathbf{v})\mathbf{e} + \lambda\mathbf{w}')$$

where $\mathbf{z} = (\mathbf{z} \cdot \mathbf{v})\mathbf{v} + \lambda\mathbf{w}$ and $\mathbf{w}' = \sigma\mathbf{w}$, so that $\mathbf{w} \cdot \mathbf{y} = \mathbf{w}' \cdot \mathbf{e} = 0$, $|\mathbf{w}| = |\mathbf{w}'| = 1$ and $\lambda^2 = \mathbf{z} \cdot \mathbf{z} - (\mathbf{v} \cdot \mathbf{z})^2$.

Now the subgroup of $0(n, \mathbf{R})$ which fixes $\mathbf{e}$ is transitive on the set of elements $\{\mathbf{w}' \mid \mathbf{w}' \cdot \mathbf{e} = 0, |\mathbf{w}'| = 1\}$. Thus $f(\mathbf{v}, \mathbf{z})$ does not depend on $\mathbf{w}'$ but only on $\mathbf{v} \cdot \mathbf{z}$ and $\mathbf{z} \cdot \mathbf{z}$.

Corollary 1 (Funk-Hecke Formula, see [3] page 20). *Let $P_k(\mathbf{x})$ be a spherical harmonic function and F a continuous function on* $[-1, 1]$. *Then there is a constant C_F such that*

$$\int_{\Omega_n} F(\mathbf{u}\cdot\mathbf{v})P_k(\mathbf{u})\,d\mathbf{u} = C_F P_k(\mathbf{v}).$$

Proof. Considered as an identity for $\mathbf{z}$ in $\mathbf{R}^n$, (1) remains valid for all $\mathbf{z}$ in $\mathbf{C}^n$, by analytic continuation. Let $\mathbf{z} = \mathbf{l}$ where $\Sigma l_i^2 = 0$. Then

$$\int_{\Omega_n} F(\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{l})^k\,d\mathbf{x} = C_F(\mathbf{v}\cdot\mathbf{l})^k.$$

The general result follows by linearity.

This corollary leads us to our first theorem.

Theorem 1. *If $f \in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ is such that $f(\mathbf{r}) = P_k(\mathbf{r})g(r)$ where $r = |\mathbf{r}|$ then*

$$\hat{f}(\mathbf{r}) = P_k(\mathbf{r})h(r).$$

Proof.

$$\begin{aligned}\hat{f}(\mathbf{x}) &= \int_{\mathbf{R}^n} f(\mathbf{r})e^{-2\pi i\mathbf{r}\cdot\mathbf{x}}\,d\mathbf{r}\\ &= \int_{\mathbf{R}^n} P_k(\mathbf{r})g(r)e^{-2\pi i\mathbf{r}\cdot\mathbf{x}}\,d\mathbf{r}\\ &= \int_0^\infty r^k g(r)r^{n-1}\,dr\int_{\Omega_n} P_k(\mathbf{u})e^{-2\pi i r x\mathbf{u}\cdot\mathbf{v}}\,d\mathbf{u}\end{aligned}$$

where $d\mathbf{r} = r^{n-1}\,dr\,d\mathbf{u}$ with $d\mathbf{u}$ an invariant measure on Ω_n and $\mathbf{r} = r\mathbf{u}$, $\mathbf{x} = x\mathbf{v}$ with $\mathbf{u}, \mathbf{v} \in \Omega_n$. But then by the above corollary we have that

$$\int_{\Omega_n} P_k(\mathbf{u})e^{-2\pi i r x\mathbf{u}\cdot\mathbf{v}}\,d\mathbf{u} = P_k(\mathbf{v})\Phi(rx)$$

for some function Φ. Thus

$$\begin{aligned}\hat{f}(\mathbf{x}) &= P_k(\mathbf{v})\int_0^\infty r^k g(r)\Phi(rx)r^{n-1}\,dr\\ &= P_k(\mathbf{x})h(x)\end{aligned}$$

as desired.

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The above three results are roughly parallel to those of [6] pages 146–149, where a different characterization of spherical harmonics is used.

Lastly as in [4] we will need to know the Fourier transform of at least one function. In this I shall follow Schoeneberg [5] page 206.

LEMMA 2. *Let* $f(\mathbf{r}) = P_k(\mathbf{r})e^{-\pi r^2}$ *where* $r = |\mathbf{r}|$ *and* $P_k(\mathbf{r})$ *is a spherical harmonic function of degree* k. *Then*

$$\hat{f}(\mathbf{r}) = i^{-k}P_k(\mathbf{r})e^{-\pi r^2}.$$

PROOF. Consider the well-known result

$$\int_{\mathbf{R}^n} e^{-\pi r^2}e^{-2\pi i\mathbf{r}\cdot\mathbf{x}}\,d\mathbf{r} = e^{-\pi x^2}. \tag{2}$$

Let

$$D_{\mathbf{l}} \equiv \sum_{j=1}^{n} l_i \frac{\partial}{\partial x_i}$$

be a differential operator with $\sum_{i=1}^{n} l_i^2 = 0$ and apply $D_{\mathbf{l}}^k$ to both sides of (2). We then obtain

$$\int_{\mathbf{R}^n} (\mathbf{r}\cdot\mathbf{l})^k(-2\pi i)^k e^{-\pi r^2}e^{-2\pi i\mathbf{r}\cdot\mathbf{x}}\,d\mathbf{r} = (-2\pi)^k(\mathbf{x}\cdot\mathbf{l})^k e^{-\pi x^2}.$$

Again the general result follows by linearity.

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The only problem which remains is to find the relationship between the functions g and h in Theorem 1. Initially I shall restrict my attention to functions, f, which are Schwartz-Bruhat, that is, infinitely differentiable and such that whenever $\alpha_1 + \alpha_2 + \cdots + \alpha_n = \beta$ and $N = 1, 2, 3, \ldots$

$$\frac{\partial^\beta f(\mathbf{x})}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} = 0\left(|\mathbf{x}|^{-N}\right)$$

as $|\mathbf{x}| \to \infty$. It is well-known that the space of such functions is closed under Fourier transforms and that it is dense in $L^2(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)$. For such functions define a zeta function via

$$\zeta(f, P_k, s) = \int_{\mathbf{R}^n} f(\mathbf{r})\,\overline{P_k(\mathbf{r})}\,r^{s-k-n/2}\,d\mathbf{r}$$

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where $\overline{P}_k$ denotes complex conjugation. It is clear that this converges for all $\mathrm{Re}(s) > -n/2$ and defines an analytic function in this region (recall that $P_k(\mathbf{r}) \sim r^k$, $d\mathbf{r} \sim r^{n-1}\,dr$). The next lemma then follows as in [4].

LEMMA 3. *In the region $-n/2 < \mathrm{Re}(s) < n/2$ we have that*

$$\zeta(f_1, P_k, s)\zeta(\hat{f}_2, P_k, -s) = \zeta(\hat{f}_1, P_k, -s)\zeta(f_2, P_k, s).$$

PROOF. In the stated strip the left hand side is given by

$$\begin{aligned}
&\int_{\mathbf{R}^n} f_1(\mathbf{r}_1)\overline{P_k(\mathbf{r}_1)}\, r_1^{s-k-n/2}\, d\mathbf{r}_1 \int_{\mathbf{R}^n} \hat{f}_2 \overline{P_k(\mathbf{r}_2)}\, r_2^{-s-k-n/2}\, d\mathbf{r}_2 \\
&\quad = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} f_1(\mathbf{r}_1)\hat{f}_2(\mathbf{r}_2)\overline{P_k(\mathbf{r}_1)}\;\overline{P_k(\mathbf{r}_2)}\, r_1^{s-k-n/2} r_2^{-s-k-n/2}\, d\mathbf{r}_1\, d\mathbf{r}_2 \\
&\quad = \int_0^\infty\int_0^\infty\int_{\Omega_n}\int_{\Omega_n} f_1(r_1\mathbf{u}_1)\hat{f}_2(r_2\mathbf{u}_2)\overline{P_k(\mathbf{u}_1)}\;\overline{P_k(\mathbf{r}_2)} \\
&\qquad\qquad \times r_1^{s+n/2-1} r_2^{-s+n/2-1}\, d\mathbf{u}_1\, d\mathbf{u}_2\, dr_2\, dr_2.
\end{aligned}$$

Under the transformation $r_1 \to r_1$, $r_2 \to r_1 r_2$ this becomes

$$\int_0^\infty \left(\int_0^\infty \int_{\Omega_n}\int_{\Omega_n} f_1(r_1\mathbf{u}_1)\hat{f}_2(r_1 r_2\mathbf{u}_2)\overline{P_k(\mathbf{u}_1)}\;\overline{P_k(\mathbf{u}_2)}\, r_1^{n-1}\, d\mathbf{u}_1\, d\mathbf{u}_2\, dr_1\right) r_2^{n/2-s-1}\, dr_2.$$

The bracketed expression then gives

$$\begin{aligned}
&\int_0^\infty \int_{\Omega_n}\int_{\Omega_n} f_1(r_1\mathbf{u}_1)\int_{\mathbf{R}^n} f_2(\mathbf{r}_3) e^{-2\pi i r_1 r_2 \mathbf{u}_2\cdot \mathbf{r}_3}\, d\mathbf{r}_3\, \overline{P_k(\mathbf{u}_1)}\;\overline{P_k(\mathbf{u}_2)}\, r_1^{n-1}\, d\mathbf{u}_1\, d\mathbf{u}_2\, dr_1 \\
&\quad = \int_0^\infty\int_0^\infty \int_{\Omega_n}\int_{\Omega_n}\int_{\Omega_n} f(r_1\mathbf{u}_1) f_2(r_3\mathbf{u}_3)\overline{P_k(\mathbf{u}_1)} \\
&\qquad\qquad \times r_1^{n-1} r_3^{n-1}\, \overline{P_k(\mathbf{u}_2)}\, e^{-2\pi i r_1 r_2 r_3 \mathbf{u}_2\cdot\mathbf{u}_3}\, d\mathbf{u}_2\, d\mathbf{u}_1\, d\mathbf{u}_3\, dr_1\, dr_3.
\end{aligned}$$

However by Corollary 1

$$\int_{\Omega_n} e^{-2\pi i r_1 r_2 r_3 \mathbf{u}_2\cdot\mathbf{u}_3}\,\overline{P_k(\mathbf{u}_2)}\, d\mathbf{u}_2 = \Phi(r_1 r_2 r_3)\,\overline{P_k(\mathbf{u}_3)}$$

so that we are left with

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \int_{\Omega_n}\int_{\Omega_n} f_1(r_1\mathbf{u}_1) f_2(r_3\mathbf{u}_3)\overline{P_k(\mathbf{u}_1)}\;\overline{P_k(\mathbf{u}_3)} \\
&\qquad\qquad \times \Phi(r_1 r_2 r_3) r_1^{n-1} r_3^{n-1}\, d\mathbf{u}_1\, d\mathbf{u}_3\, dr_1\, dr_3
\end{aligned}$$

which is clearly symmetric in f_1 and f_2 thus proving the lemma.

Again as in [4] this gives rise to a functional equation.

THEOREM 2. *The function $\zeta(f, P_k, s)$ has an analytic continuation for all $s + k + n/2 \neq 0, -2, -4, \ldots$ and satisfies the functional equation*

$$\zeta(f, P_k, s) = i^k \pi^{-s} \Gamma\left(\tfrac{1}{2}(s + k + \tfrac{1}{2}n)\right) / \Gamma\left(\tfrac{1}{2}(-s + k + \tfrac{1}{2}n)\right) \zeta(\hat{f}, P_k, -s). \tag{3}$$

PROOF. Let $f = f_1$ and $f_2(\mathbf{r}) = P_k(\mathbf{r}) e^{-\pi r^2}$ in Lemma 3. Then

$$\begin{aligned}
\zeta(f_2, P_k, s) &= \int_{\mathbf{R}^n} P_k(\mathbf{r}) e^{-\pi r^2} \overline{P_k(\mathbf{r})}\, r^{s-k-n/2}\, d\mathbf{r} \\
&= \int_0^\infty e^{-\pi r^2} r^{s+k+n/2-1}\, dr \int_{\Omega_n} |P_k(\mathbf{u})|^2\, d\mathbf{u} \\
&= C_k \pi^{-(s+k+n/2)/2} \Gamma\left(\tfrac{1}{2}(s + k + \tfrac{1}{2}n)\right)
\end{aligned}$$

with $C_k = \int_{\Omega_n} |P_n(\mathbf{u})|^2\, d\mathbf{u} \neq 0$. Similarly,

$$\zeta(\hat{f}_2, P_k, -s) = C_k i^{-k} \pi^{-(-s+k+n/2)/2} \Gamma\left(\tfrac{1}{2}(-s + k + \tfrac{1}{2}n)\right)$$

using the result of Lemma 1. Thus in the strip $-n/2 < \operatorname{Re}(s) < n/2$ we have that

$$\zeta(f, P_k, s) = i^k \pi^{-s} \Gamma\left(\tfrac{1}{2}(s + k + \tfrac{1}{2}n)\right) \Gamma\left(\tfrac{1}{2}(-s + k + \tfrac{1}{2}n)\right) \zeta(\hat{f}, P_k, -s).$$

However the right hand side is analytic in the region

$$\{s\colon \operatorname{Re}(s) < n/2,\ s + k + n/2 \neq 0, -2, -4, \ldots\}$$

and so gives an analytic continuation of the left hand side for all such s.

Now to determine the relationship between the g and h of Theorem 1 we note that if $f(\mathbf{r}) = P_k(\mathbf{r}) g(r)$ then

$$\zeta(f, P_k, s) = \int_0^\infty g(r) r^{s+k+n/2} \frac{dr}{r} \int_{\Omega_n} |P_k(\mathbf{u})|^2\, d\mathbf{u}$$

which is essentially just the Mellin transform of g. Thus Theorem 2 above simply relates the Mellin transform of g to the Mellin transform of h. Inverting we obtain the desired relationship between g and h.

THEOREM 3. *Let $f \in L^2(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)$ be of the form $f(\mathbf{r}) = P_k(\mathbf{r}) g(r)$, so that $\hat{f}(\mathbf{r}) = P_k(\mathbf{r}) h(r)$. Then*

$$h(r) = 2\pi i^{-k} r^{-n/2-k+1} \int_0^\infty g(t) J_{n/2+k-1}(2\pi r t) t^{n/2+k}\, dt.$$

PROOF. By well-known density results it is sufficient to show this for Schwartz-Bruhat functions f. Taking the inverse Mellin transform of both sides (3) in

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Theorem 2, we have, for some c in the interval $(-\frac{1}{2}n, +\frac{1}{2}n)$,

$$\begin{aligned} h(r) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} i^{-k}\pi^{s}\Gamma\left(\tfrac{1}{2}\left(-s+k+\tfrac{1}{2}n\right)\right)/\Gamma\left(\tfrac{1}{2}\left(s+k+\tfrac{1}{2}n\right)\right) \\ &\qquad \times \int_0^\infty g(t)t^{s+k+n/2}\frac{dt}{t}r^{s-k-n/2}\,ds \\ &= i^{-k}\int_0^\infty g(t)t^{2k+n} \\ &\qquad \times \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\pi^{s}\Gamma\left(\tfrac{1}{2}\left(-s+k+\tfrac{1}{2}n\right)\right)/\Gamma\left(\tfrac{1}{2}\left(s+k+\tfrac{1}{2}n\right)\right)(tr)^{s-k-n/2}\,ds\,\frac{dt}{t} \\ &= 2\pi i^{-k}r^{-n/2-k+1}\int_0^\infty g(t)J_{n/2+k-1}(2\pi rt)t^{k+n/2}\,dt. \end{aligned}$$

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