

Pressure at infinity on countable Markov shifts

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(Received 4 June 2025 and accepted in revised form 17 November 2025)

Abstract. In this article, we study the pressure at infinity of potentials defined over countable Markov shifts. We establish an upper semi-continuity result concerning the limiting behaviour of the pressure of invariant probability measures, where the escape of mass is controlled by the pressure at infinity. As a consequence, we establish criteria for the existence of equilibrium states and maximizing measures for uniformly continuous potentials. Additionally, we study the pressure at infinity of suspension flows defined over countable Markov shifts and prove an upper semi-continuity result for the pressure map.

Key words: thermodynamic formalism, countable Markov shifts, suspension flows
2020 Mathematics Subject Classification: 37D35, 37B10 (Primary); 37A35, 37A10 (Secondary)

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1. Introduction

Symbolic dynamics has a wide range of applications in the study of smooth dynamical systems. For instance, a well-known result of Bowen [Bo1] states that any uniformly



hyperbolic diffeomorphism on a compact manifold admits a Markov partition. The Markov partition provides a finite-to-one semiconjugacy between a subshift of finite type and most of the original system. Several applications to the study of the ergodic theory and thermodynamic formalism of the diffeomorphism follow from the study of symbolic dynamics (see [Bo3]). Similarly, a uniformly hyperbolic flow on a compact manifold can be coded as a suspension flow over a subshift of finite type (see [Bo2, Ra]). In particular, this applies to the geodesic flow on a closed negatively curved manifold. For striking applications of the coding to the counting of periodic orbits, see [La, PP].

More recently, Sarig [Sa5] constructed Markov partitions for surface diffeomorphisms with positive topological entropy. Specifically, for any given $\epsilon > 0$, he constructed a countable Markov shift wherein every ergodic measure with entropy greater than ϵ can be lifted to the symbolic space. Remarkable dynamical results have been obtained making use of this coding. For instance, Buzzi, Crovisier and Sarig [BCS] proved that a positive entropy C^∞ diffeomorphism of a closed surface admits at most finitely many ergodic measures of maximal entropy. It is worth pointing out that the use of countable Markov shifts to code the dynamics is natural in the non-uniformly hyperbolic setting, as subsets with sufficient hyperbolicity are non-compact in phase space. In recent years, several other non-uniformly hyperbolic dynamical systems have been coded with countable Markov shifts, or suspension flows over a countable Markov shift in the case of flows. For instance, see the work of Lima and Matheus [LM], Ben Ovadia [Be], Lima and Sarig [LS], to mention a few.

In a slightly different direction, inducing schemes have been widely used in the study of the thermodynamic formalism of the geometric potential, for instance, see [BT, PS, Y]. In this case, the lifting process allows to transfer information to the full shift on a countable alphabet, which is an example of particular interest in this work.

1.1. Thermodynamic formalism. In this article, we study various aspects of the thermodynamic formalism of countable Markov shifts (CMSs). This subject has been extensively investigated by Gurevich [Gu], Gurevich and Savchenko [GS], Walters [W], Mauldin and Urbański [MU1], Sarig [Sa1, Sa2, Sa3], Sarig and Buzzi [BS], among others. Countable Markov shifts are non-compact generalizations of subshifts of finite type. In contrast to their compact counterparts, they can have infinite topological entropy and may not be locally compact.

Let (Σ, σ) be a one-sided countable Markov shift. Equivalently, Σ is the space of infinite one-sided walks on a directed graph with countably many vertices and $\sigma : \Sigma \rightarrow \Sigma$ is the shift map. We refer to a function $\phi : \Sigma \rightarrow \mathbb{R}$ as a potential. The *measure-theoretic pressure* of ϕ is defined by

$$P(\phi) = \sup_{\mu \in \mathcal{M}(\sigma)} \left\{ h_\mu(\sigma) + \int \phi \, d\mu : \int \phi \, d\mu > -\infty \right\},$$

where $\mathcal{M}(\sigma)$ is the space of invariant probability measures of (Σ, σ) and $h_\mu(\sigma)$ denotes the measure theoretic entropy of μ . Let

$$\mathcal{M}_\phi(\sigma) = \left\{ \mu \in \mathcal{M}(\sigma) : \int \phi \, d\mu > -\infty \right\}.$$

We say that $\mu \in \mathcal{M}_\phi(\sigma)$ is an *equilibrium state* of ϕ if $P(\phi) = h_\mu(\sigma) + \int \phi \, d\mu$. In physical terms, an equilibrium state maximizes the free energy of the system (see [Ru]). We are interested in properties of the pressure, equilibrium states and the pressure map of ϕ :

$$\mu \in \mathcal{M}_\phi(\sigma) \mapsto h_\mu(\sigma) + \int \phi \, d\mu.$$

An important difference between the thermodynamic formalism of subshifts of finite type and countable Markov shifts is that, in the latter, regular potentials may not admit equilibrium states. In its simplest form, a countable Markov shift can have finite entropy but no measure of maximal entropy (see [Gu, Rt]).

In general, by the definition of the measure-theoretic pressure, we are guaranteed of the existence of sequences of measures $(\mu_n)_n$ in $\mathcal{M}_\phi(\sigma)$ such that

$$P(\phi) = \lim_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) + \int \phi \, d\mu_n \right).$$

For subshifts of finite type and continuous potentials ϕ , it is well known that every limit measure of the sequence $(\mu_n)_n$ is an equilibrium state. This follows directly from the upper semicontinuity of the entropy map and the weak* compactness of $\mathcal{M}(\sigma)$. We are interested in the behaviour of the sequence $(\mu_n)_n$ for a general countable Markov shift and a potential ϕ with mild regularity assumptions. More precisely, consider the following class of potentials:

$$\mathcal{H} = \{ \phi \in C_{uc}(\Sigma) : \sup(\phi), \text{var}_2(\phi), P(\phi) < \infty \},$$

where $C_{uc}(\Sigma)$ is the space of uniformly continuous potentials on Σ , equipped with the metric d defined in §2, and $\text{var}_n(\phi)$ denotes the n th variation of ϕ . We will study the problem of existence and non-existence of equilibrium states for potentials in $\mathcal{H}$.

1.2. Compactness in the space of invariant measures. For a transitive countable Markov shift (Σ, σ) , the space of invariant probability measures $\mathcal{M}(\sigma)$ is non-compact. For countable Markov shifts satisfying a combinatorial property known as the $\mathcal{F}$ -property, this issue can be addressed by considering the *space of invariant sub-probability measures*, which we denote by $\mathcal{M}_{\leq 1}(\sigma)$, endowed with the *topology of convergence on cylinders*, which provides a compactification of $\mathcal{M}(\sigma)$ (see [IV, Theorem 1.2] and §2.2).

A sequence $(\mu_n)_n$ is said to *converge on cylinders* to μ if $\lim_{n \rightarrow \infty} \mu_n(C) = \mu(C)$ for every cylinder set $C \subseteq \Sigma$. The *topology of convergence on cylinders*, or *cylinder topology*, is the topology that induces this mode of convergence. A key difference between the weak* topology and the cylinder topology is that, in the latter, sequences of invariant probability measures can converge to sub-probability measures, it is possible to lose mass.

Countable Markov shifts with finite entropy and those that are locally compact have the $\mathcal{F}$ -property. In the latter case, the cylinder topology coincides with the vague topology, and the compactness of the space of invariant sub-probability measures follows from the Riesz representation theorem. However, the full shift on a countable alphabet does not have the $\mathcal{F}$ -property. For countable Markov shifts without the $\mathcal{F}$ -property, the space of invariant

probability measures cannot be compactified using the cylinder topology. This is because there exist sequences of invariant probability measures that converge to functionals that are finitely additive but not countably additive (see [IV, Proposition 4.19]).

To address the absence of a natural compactification for the space of invariant probability measures, we establish the following compactness result.

THEOREM 1.1. *Let (Σ, σ) be a transitive countable Markov shift and $\phi \in \mathcal{H}$. Let $(\mu_n)_n$ be a sequence in $\mathcal{M}_\phi(\sigma)$ such that $\liminf_{n \rightarrow \infty} \int \phi d\mu_n$ is finite. Then, $(\mu_n)_n$ has a subsequence that converges on cylinders to a measure $\mu \in \mathcal{M}_{\leq 1}(\sigma)$, where $\int \phi d\mu > -\infty$.*

Theorem 1.1 is a key result that we use in some technical lemmas and applications. As noted above, it is needed to handle systems without the $\mathcal{F}$ -property, particularly those with infinite topological entropy.

1.3. Pressure at infinity and the upper semicontinuity of the pressure map. We study the pressure at infinity of a potential, generalizing the concept of entropy at infinity of (Σ, σ) , as studied in [Bu, ITV, Rt], to non-zero potentials. We define the *measure theoretic pressure at infinity* as follows:

$$P_\infty(\phi) = \sup_{(\mu_n)_n \rightarrow 0} \limsup_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) + \int \phi d\mu_n \right),$$

where the supremum runs over all sequences $(\mu_n)_n$ in $\mathcal{M}_\phi(\sigma)$ that converge on cylinders to the zero measure. If there is no such sequence, we set $P_\infty(\phi) = -\infty$. In §3, we define the *topological pressure at infinity* of ϕ , denoted by $P_\infty^{\text{top}}(\phi)$, which is analogous to the topological pressure, or Gurevich pressure, of ϕ . We prove a variational principle for the pressures at infinity.

THEOREM 1.2. *Let (Σ, σ) be a transitive countable Markov shift and $\phi : \Sigma \rightarrow \mathbb{R}$ a potential with summable variations and finite pressure. Then, $P_\infty(\phi) = P_\infty^{\text{top}}(\phi)$.*

The following result relates the escape of mass phenomenon with the upper semi-continuity of the pressure map. For a potential $\phi : \Sigma \rightarrow \mathbb{R}$ with finite pressure, we define

$$s_\infty(\phi) = \inf\{t \in [0, 1] : P(t\phi) < \infty\}.$$

THEOREM 1.3. *Let (Σ, σ) be a transitive countable Markov shift and $\phi \in C_{uc}(\Sigma)$ a potential such that $\text{var}_2(\phi)$ and $P(\phi)$ are finite. Let $(\mu_n)_n$ be a sequence in $\mathcal{M}_\phi(\sigma)$ that converges on cylinders to $\lambda\mu$, where $\lambda \in [0, 1]$ and $\mu \in \mathcal{M}_\phi(\sigma)$. Suppose that $\liminf_{n \rightarrow \infty} \int \phi d\mu_n > -\infty$ or that $s_\infty(\phi) < 1$. Then,*

$$\limsup_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) + \int \phi d\mu_n \right) \leq \lambda \left(h_\mu(\sigma) + \int \phi d\mu \right) + (1 - \lambda)P_\infty(\phi).$$

Moreover, the inequality is sharp.

Theorem 1.3 is a generalization of [ITV, Theorem 1.1], where the authors proved that the result holds when (Σ, σ) has finite topological entropy and $\phi = 0$. In our case, the finite entropy condition is replaced by a finite pressure assumption. In the context of homogeneous dynamics and for the geodesic flow on non-compact pinched negatively curved manifolds, similar results have been obtained in [EK, EKP, ELMV, GST, RV1, V]. The pressure at infinity has also been studied in the context of countable Markov shifts in [RS] and for the geodesic flow on non-compact pinched negatively curved manifolds in [GST], where a similar upper semicontinuity result was obtained.

1.4. *Some applications.* As a consequence of our methods, we derive applications to the thermodynamic formalism of countable Markov shifts, ergodic optimization and the thermodynamic formalism of suspension flows over countable Markov shifts.

1.4.1. *Existence and non-existence of equilibrium states.* In [Sa3], Sarig introduced the class of strongly positive recurrent (SPR) potentials. The thermodynamic formalism of SPR potentials closely resembles that of Hölder potentials in subshifts of finite type. For SPR potentials, a spectral gap holds on suitable Banach spaces and the Ruelle–Perron–Frobenius (RPF) measure exhibits exponential decay of correlations (see [CS, Sa3]).

More recently, it was shown in [RS] that if ϕ is weakly Hölder, bounded above and has finite pressure, then ϕ is SPR if and only if $P_\infty(\phi) < P(\phi)$. More generally, we say that $\phi \in C_{uc}(\Sigma)$ is SPR if $P(\phi)$ is finite and $P_\infty(\phi) < P(\phi)$.

The following result is a consequence of Theorems 1.1 and 1.3. It provides a criterion for the existence of equilibrium states for potentials in $\mathcal{H}$ (without assuming summable variations) and describes what happens when no equilibrium state exists.

THEOREM 1.4. *Let (Σ, σ) be a transitive CMS and $\phi \in \mathcal{H}$. Let $(\mu_n)_n$ be a sequence in $\mathcal{M}_\phi(\sigma)$ such that $P(\phi) = \lim_{n \rightarrow \infty} (h_{\mu_n}(\sigma) + \int \phi d\mu_n)$. Then, we have the following.*

- (1) *If ϕ is SPR and either $s_\infty(\phi) < 1$ or $\liminf_{n \rightarrow \infty} \int \phi d\mu_n > -\infty$, then $(\mu_n)_n$ has a subsequence that converges in the weak* topology to an equilibrium state of ϕ . In particular, ϕ has an equilibrium state.*
- (2) *If ϕ does not have an equilibrium state, then there exists a subsequence $(\mu_{n_k})_k$ that either converges to the zero measure or satisfies that $\lim_{k \rightarrow \infty} \int \phi d\mu_{n_k} = -\infty$.*

1.4.2. *Ergodic optimization.* For $\phi \in C_{uc}(\Sigma)$, define

$$\beta(\phi) = \sup_{\mu \in \mathcal{M}(\sigma)} \int \phi d\mu.$$

We say that $\mu \in \mathcal{M}(\sigma)$ is a maximizing measure for ϕ if $\beta(\phi) = \int \phi d\mu$. Maximizing measures are a central object of study in ergodic optimization. Natural questions in this context include what conditions ensure the existence of a maximizing measure and, if such measures exist, what can be said about their dynamical properties and support. For a general account of ergodic optimization, we refer the reader to [B, Je].

For a continuous and compact dynamical system, the existence of maximizing measures for continuous potentials is a standard result, which follows from the weak* compactness of the space of invariant probability measures and the continuity of the integral map. In our setting, however, $\mathcal{M}(\sigma)$ is non-compact and this same argument does not apply. In fact, it is relatively straightforward to construct potentials that do not admit any maximizing measure (see Example 7.4). In general, hypotheses on the potential are necessary to guarantee the existence of a maximizing measure. This problem has been studied for the renewal shift in [Io]. For different classes of coercive potentials, the existence of maximizing measures was established in [BF, FV, JMU].

In analogy to the pressure at infinity, we define

$$\beta_{\infty}(\phi) = \sup_{(\mu_n)_n \rightarrow 0} \limsup_{n \rightarrow \infty} \int \phi d\mu_n,$$

where the supremum runs over all sequences in $\mathcal{M}(\sigma)$ that converge on cylinders to the zero measure. If there is no such sequence, we set $\beta_{\infty}(\phi) = -\infty$. The quantity $\beta_{\infty}(\phi)$ was introduced in [RV2] in the context of the geodesic flow on non-compact pinched negatively curved manifolds. In [RV2], the authors proved that for uniformly continuous potentials that are bounded above, the condition $\beta_{\infty}(\phi) < \beta(\phi)$ implies the existence of a maximizing measure for ϕ . In §7, we prove an inequality that relates the upper semi-continuity of the integral map $\mu \mapsto \int \phi d\mu$ to $\beta_{\infty}(\phi)$, which is analogous to Theorem 1.3 (see Theorem 7.7). As an application, we obtain a criterion for the existence of a maximizing measure.

THEOREM 1.5. *Let (Σ, σ) be a transitive CMS and $\phi \in \mathcal{H}$ a potential such that $\beta_{\infty}(\phi) < \beta(\phi)$. Then, ϕ has a maximizing measure.*

A finer description, analogous to Theorem 1.4, is given in Theorem 7.8. The assumptions in Theorem 1.5 allow for the consideration of potentials with any behaviour at infinity. For a coercive potential, we have $\beta_{\infty}(\phi) = -\infty$. However, in Proposition 7.9, we prove that zero temperature limits of potentials in $\mathcal{H}$ with summable variations and such that $\beta_{\infty}(\phi) < \beta(\phi)$ are maximizing measures (see also [FV, Theorem 1]).

1.4.3. Suspension flows. In §8, we study aspects of the thermodynamic formalism of suspension flows defined over countable Markov shifts, a subject studied in [BI, Ke, JJ].

Suspension flows over a countable Markov shift are continuous-time dynamical systems where the base dynamics is a countable Markov shift, and a roof function $\tau : \Sigma \rightarrow (0, \infty)$ specifies the return time to the base. We denote the suspension flow associated with (Σ, σ) and τ by $(Y, \Theta = (\theta_t)_{t \geq 0})$. The space of flow-invariant probability (respectively, sub-probability) measures on Y is denoted by $\mathcal{M}(\Theta)$ (respectively, $\mathcal{M}_{\leq 1}(\Theta)$). In [IV, §6], the authors defined a topology on $\mathcal{M}_{\leq 1}(\Theta)$ that is analogous to the cylinder topology in $\mathcal{M}_{\leq 1}(\sigma)$, which we still refer to as the cylinder topology (for details, see §8).

We consider the following class of roof functions:

$$\Psi = \{\psi \in C_{uc}(\Sigma) : \inf \psi > 0, \text{var}_2(\psi) < \infty, P(-\psi) < \infty\}.$$

If $\tau \in \Psi$, then the topological entropy of (Y, Θ) is finite. Under this assumption, we prove a result similar to Theorem 1.1 (see also Theorem 2.3).

THEOREM 1.6. *Let (Σ, σ) be a transitive countable Markov shift with a roof function $\tau \in \Psi$. Let (Y, Θ) be the associated suspension flow. Then, $\mathcal{M}_{\leq 1}(\Sigma, \sigma, \tau)$ is compact with respect to the cylinder topology. Furthermore, $\mathcal{M}(\Theta)$ is dense in $\mathcal{M}_{\leq 1}(\Theta)$.*

We define the pressure of $\phi : Y \rightarrow \mathbb{R}$ by the formula $P^\Theta(\phi) = \sup_{\nu \in \mathcal{M}(\Theta)} (h_\nu(\Theta) + \int \phi d\nu)$, where $h_\nu(\Theta)$ is the entropy of ν with respect to the time one map θ_1 . We have added the superscript Θ to emphasize that the dynamical system under consideration is the suspension flow (Y, Θ) . We define the pressure at infinity of ϕ as

$$P_\infty^\Theta(\phi) = \sup_{(v_n)_{n \rightarrow 0}} \limsup_{n \rightarrow \infty} \left(h_{v_n}(\Theta) + \int \phi dv_n \right),$$

where the supremum runs over all sequences $(v_n)_n$ in $\mathcal{M}(\Theta)$ that converge on cylinders to the zero measure. If there is no such sequence, we set $P_\infty^\Theta(\phi) = -\infty$. For suspension flows over countable Markov shifts the entropy at infinity was first defined and studied in [IRV]. In this context, we prove a result analogous to Theorem 1.3. Define the following class of potentials:

$$\mathcal{H}_Y = \{ \phi \in C_b(Y) : \Delta\phi \in C_{uc}(\Sigma), \text{var}_2 \Delta\phi < \infty \},$$

where $C_b(Y)$ is the space of continuous and bounded functions on Y .

THEOREM 1.7. *Let $\tau \in \Psi$ and $\phi \in \mathcal{H}_Y$. Let $(v_n)_n$ be a sequence in $\mathcal{M}(\Theta)$ that converges on cylinders to $\lambda\nu$, where $\lambda \in [0, 1]$ and $\nu \in \mathcal{M}(\Theta)$. Then,*

$$\limsup_{n \rightarrow \infty} \left(h_{v_n}(\Theta) + \int \phi dv_n \right) \leq \lambda \left(h_\nu(\Theta) + \int \phi d\nu \right) + (1 - \lambda) P_\infty^\Theta(\phi).$$

Moreover, this inequality is sharp.

In this context, we say that $\phi : Y \rightarrow \mathbb{R}$ is SPR if $P^\Theta(\phi)$ is finite and $P_\infty^\Theta(\phi) < P^\Theta(\phi)$. At the end of §8, we prove that if $\tau \in \Psi$ and $\phi \in \mathcal{H}_Y$ is SPR, then ϕ has an equilibrium state (see Theorem 8.6).

1.5. Organization of the article.

- Section 2. We review known facts about the thermodynamic formalism of countable Markov shifts, the topology of convergence on cylinders, the entropy at infinity of a CMS and, finally, a useful recoding that will be needed later in the article.
- Section 3. We study the topological pressure at infinity and establish a formula in terms of the topological pressure. We also prove Theorem 1.2, which provides the variational principle for pressures at infinity.
- Section 4. We characterize the existence of sequences of invariant probability measures that converge on cylinders to the zero measure, in terms of a combinatorial property of the graph associated with a countable Markov shift.
- Section 5. We consider a Markov potential with values in $\mathbb{N}$, construct a new countable Markov shift and calculate its entropy at infinity. We prove Theorem 1.1.

- Section 6. Using the countable Markov shift constructed in §5, we prove Theorem 1.3, the main upper semi-continuity result for the pressure map.
- Section 7. We apply the previously obtained results to prove Theorem 1.4. We prove that $\beta_\infty(\phi)$ is the asymptotic slope of the pressure at infinity and establish an upper semi-continuity result for the integral map which is analogous to Theorem 1.3. Furthermore, we prove Theorem 1.5 and obtain a result concerning zero temperature limits of a suitable class of potentials.
- Section 8. We prove Theorems 1.6 and 1.7.
- Appendix A. We state a condition that we interpret as tightness in the cylinder topology and use it to prove a tightness result for the full shift on a countable alphabet.

2. Preliminaries

In this section, we review some facts about countable Markov shifts and their thermodynamic formalism. Good references on this topic include [Sa6] and [MU2, §2].

2.1. Countable Markov shifts. Let S be an alphabet with countably many symbols and $M = (M_{i,j})_{i,j \in S}$ a $S \times S$ matrix with entries 0 or 1. The symbolic space associated to S and M is given by

$$\Sigma = \{(x_1, x_2, \dots) \in S^{\mathbb{N}} : M_{x_i, x_{i+1}} = 1 \text{ for all } i \in \mathbb{N}\}.$$

We endow S with the discrete topology and $S^{\mathbb{N}}$ with the product topology. On Σ , we consider the topology induced by the natural inclusion $\Sigma \subseteq S^{\mathbb{N}}$. The shift map $\sigma : \Sigma \rightarrow \Sigma$ is given by $\sigma(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$. The dynamical system (Σ, σ) is called a countable Markov shift (CMS).

Remark 2.1. We will always assume that S is infinite. Since $\mathbb{N}$ has a natural order, it is sometimes convenient to identify S with $\mathbb{N}$. Depending on the context and convenience, we might use either S or $\mathbb{N}$ as our alphabet.

Given $x \in \Sigma$, we denote by x_i to its i th coordinate; equivalently, the first coordinate of $\sigma^{i-1}(x)$. A cylinder of length m is a set of the form

$$[a_1, \dots, a_m] = \{x \in \Sigma : x_i = a_i \text{ for all } i \in \{1, \dots, m\}\}.$$

An admissible word is a sequence $a_1 \cdots a_n$ of symbols in S such that $M_{a_i, a_{i+1}} = 1$ for every $i \in \{1, \dots, n-1\}$. A cylinder $[a_1, \dots, a_n]$ is non-empty if and only if $a_1 \cdots a_n$ is an admissible word. The collection of cylinders is a basis for the topology on Σ .

We define a metric d on Σ by declaring $d(x, y) = 0$ if $x = y$; $d(x, y) = 1$ if $x_1 \neq y_1$; and $d(x, y) = 2^{-k}$ if k is the length of the longest cylinder containing x and y . Note that balls in the metric d are cylinder sets. Given a potential $\phi : \Sigma \rightarrow \mathbb{R}$, we define its n -variation as

$$\text{var}_n(\phi) = \sup\{|\phi(x) - \phi(y)| : \text{for all } x, y \in \Sigma \text{ such that } d(x, y) \leq 2^{-n}\}.$$

Let $S_n\phi(x) = \sum_{k=0}^{n-1} \phi(\sigma^k x)$ be the Birkhoff sums of ϕ .

We say that ϕ is bounded away from zero if $\inf \phi > 0$. The space of continuous functions on Σ is denoted by $C(\Sigma)$. The space of bounded and continuous

functions on Σ is denoted by $C_b(\Sigma)$. For $\phi \in C_b(\Sigma)$, we consider the C^0 -norm $\|\phi\|_0 = \max_{x \in \Sigma} |\phi(x)|$, which defines the C^0 -topology. The space of uniformly continuous functions on Σ with respect to d is denoted by $C_{uc}(\Sigma)$. Note that $\phi \in C_{uc}(\Sigma)$ if and only if $\lim_{k \rightarrow \infty} \text{var}_k(\phi) = 0$. Let $C_{b,uc}(\Sigma) = C_b(\Sigma) \cap C_{uc}(\Sigma)$. We say that ϕ has summable variations if $\sum_{n \geq 2} \text{var}_n(\phi) < \infty$. We say that ϕ is weakly Hölder if there exist $\lambda \in (0, 1)$ and a constant $C > 0$ such that $\text{var}_n(\phi) \leq C\lambda^n$ for every $n \geq 2$. A function ϕ is called locally constant depending on n coordinates if $\text{var}_n(\phi) = 0$. We say that ϕ is locally constant if $\text{var}_n(\phi) = 0$ for some $n \in \mathbb{N}$.

In §1, we defined

$$\mathcal{H} = \{\phi \in C_{uc}(\Sigma) : \sup(\phi), \text{var}_2(\phi), P(\phi) < \infty\}.$$

Note that locally constant potentials are dense in $\mathcal{H}$ (uniformly continuous potentials can be C^0 -approximated by locally constant potentials). We are interested in the study of the thermodynamic formalism of potentials in $\mathcal{H}$.

We say that (Σ, σ) is topologically transitive (or transitive) if for each pair $(a, b) \in S \times S$, there exists an admissible word starting with a and ending with b . We say that (Σ, σ) is topologically mixing (or mixing) if for each pair $(a, b) \in S \times S$, there exists a number $N(a, b)$ such that for every $n \geq N(a, b)$, there is an admissible word of length n starting with a and ending with b .

A directed graph can be associated with a CMS. If (Σ, σ) is the CMS with alphabet S and transition matrix M , consider the graph $G = (V, E)$, where the set of vertices V is identified with S , and $(i, j) \in E$ if and only if $M_{i,j} = 1$. Points in Σ correspond to infinite forward paths over G and the shift map represents the motion along these paths. Note that (Σ, σ) is transitive if and only if G is strongly connected.

2.2. Topologies on the space of invariant measures. We denote by $\mathcal{M}_{\leq 1}(\sigma)$ the space of invariant sub-probability measures of (Σ, σ) . More precisely, $\mu \in \mathcal{M}_{\leq 1}(\sigma)$ if μ is a non-negative countably additive Borel measure such that $\mu(\Sigma) \in [0, 1]$ and $\mu(A) = \mu(\sigma^{-1}A)$ for every Borel set $A \subseteq \Sigma$. The mass of $\mu \in \mathcal{M}_{\leq 1}(\sigma)$ is given by $|\mu| := \mu(\Sigma)$. We denote by $\mathcal{M}(\sigma)$ the space of invariant probability measures of (Σ, σ) , that is, elements in $\mathcal{M}_{\leq 1}(\sigma)$ with mass equal to one. The zero measure is the measure that assigns zero to every Borel set and it is the unique element in $\mathcal{M}_{\leq 1}(\sigma)$ with zero mass.

In this article, we are interested in the behaviour of the pressure under limits of invariant probability measures. To describe such limits, we use two related topologies on $\mathcal{M}_{\leq 1}(\sigma)$.

We say that $(\mu_n)_n$ converges in the weak* topology to μ if $\lim_{n \rightarrow \infty} \int f d\mu_n = \int f d\mu$ for every $f \in C_b(\Sigma)$. If $(\mu_n)_n$ is a sequence in $\mathcal{M}(\sigma)$, then any weak* limit is also in $\mathcal{M}(\sigma)$. We say that $(\mu_n)_n$ converges on cylinders to μ if $\lim_{n \rightarrow \infty} \mu_n(C) = \mu(C)$ for every cylinder $C \subseteq \Sigma$. This notion of convergence induces the topology of convergence on cylinders, or cylinder topology, on $\mathcal{M}_{\leq 1}(\sigma)$. This topology is metrizable, for instance, consider the metric $\rho : \mathcal{M}_{\leq 1}(\sigma) \times \mathcal{M}_{\leq 1}(\sigma) \rightarrow \mathbb{R}_{\geq 0}$, given by

$$\rho(\mu_1, \mu_2) = \sum_{i \in \mathbb{N}} \frac{1}{2^i} |\mu_1(C_i) - \mu_2(C_i)|, \quad (1)$$

where $(C_i)_i$ is an enumeration of the set of cylinders of Σ .

Remark 2.2. If $(\mu_n)_n$ is a sequence in $\mathcal{M}(\sigma)$ that converges on cylinders to μ , and μ is a probability measure, then $(\mu_n)_n$ converges weak* to μ (see [IV, Lemma 3.7]).

We say that (Σ, σ) has the $\mathcal{F}$ -property if, for every $a \in S$ and $n \in \mathbb{N}$, there are finitely many admissible words of length n that start and end with a . Equivalently, there are only finitely many periodic points of a given period in $[a]$.

THEOREM 2.3. [IV, Theorem 1.2] *Let (Σ, σ) be a transitive CMS with the $\mathcal{F}$ -property. Then, $\mathcal{M}_{\leq 1}(\sigma)$ is a compact metric space with respect to the cylinder topology. Furthermore, $\mathcal{M}(\sigma)$ is dense in $\mathcal{M}_{\leq 1}(\sigma)$.*

Remark 2.4. If (Σ, σ) has finite topological entropy, then it satisfies the $\mathcal{F}$ -property. In particular, Theorem 2.3 applies for finite entropy CMS. However, if (Σ, σ) does not have the $\mathcal{F}$ -property, then $\mathcal{M}_{\leq 1}(\sigma)$ is non-compact (see [IV, Proposition 4.19]).

Let $C_0(\Sigma)$ be the space of test functions for the cylinder topology. This space is defined as the C^0 -closure of the set of functions that can be written as a finite sum of characteristic functions of cylinders (for a detailed description, see [IV, §3.4]). For instance, the function $V = \sum_{k \in \mathbb{N}} (1/k) 1_{[k]}$ is in $C_0(\Sigma)$. A sequence $(\mu_n)_n$ in $\mathcal{M}(\sigma)$ converges on cylinders to $\mu \in \mathcal{M}_{\leq 1}(\sigma)$ if and only if $\int f d\mu_n \rightarrow \int f d\mu$ for every $f \in C_0(\Sigma)$.

We conclude this subsection with an inequality that will be needed later in the article.

LEMMA 2.5. *Let $\phi \in C_{uc}(\Sigma)$ be a non-negative function. Let $(\mu_n)_n$ be a sequence in $\mathcal{M}(\sigma)$ that converges on cylinders to $\mu \in \mathcal{M}_{\leq 1}(\sigma)$. Then,*

$$\liminf_{m \rightarrow \infty} \int \phi d\mu_m \geq \int \phi d\mu.$$

Proof. Since $\phi \in C_{uc}(\Sigma)$, we have that $\text{var}_m(\phi) \rightarrow 0$ as $m \rightarrow \infty$. Let $m \in \mathbb{N}$ be large enough such that $\text{var}_m(\phi) < \infty$. Define $\phi_m(x) = \sup\{\phi(y) : y \in [x_1, \dots, x_m]\}$, where $x = (x_1, x_2, \dots)$. Note that ϕ_m is non-negative, $\text{var}_m(\phi_m) = 0$, and $\|\phi - \phi_m\|_0 \leq \text{var}_m(\phi)$. In particular, $\|\phi - \phi_m\|_0 \rightarrow 0$ as $m \rightarrow \infty$. Let $(C_i^m)_i$ be an enumeration of the set of cylinders of length m . Define $A_k^m = \bigcup_{i=1}^k C_i^m$. It follows that

$$\liminf_{n \rightarrow \infty} \int \phi_m d\mu_n \geq \lim_{n \rightarrow \infty} \int \phi_m 1_{A_k^m} d\mu_n = \int \phi_m 1_{A_k^m} d\mu.$$

Sending $k \rightarrow \infty$, we get that $\liminf_{n \rightarrow \infty} \int \phi_m d\mu_n \geq \int \phi_m d\mu$. Finally, send $m \rightarrow \infty$ and use that $|\int \phi d\nu - \int \phi_m d\nu| \leq \|\phi - \phi_m\|_0$ for every $\nu \in \mathcal{M}_{\leq 1}(\sigma)$. $\square$

2.3. Entropy at infinity. Let (Σ, σ) be a transitive CMS with alphabet $\mathbb{N}$. Fix $a \in \mathbb{N}$. Let $\text{Per}_a(n) = \{x \in [a] : \sigma^n(x) = x\}$. We define the topological entropy (or entropy) of (Σ, σ) by the formula $h_{\text{top}}(\sigma) = \limsup_{n \rightarrow \infty} (1/n) \log \#\text{Per}_a(n)$, which is independent of a . Let

$$\text{Per}_a(q, M, n) = \left\{ x \in \text{Per}_a(n) : \#\{k \in \{1, \dots, n\} : x_k \leq q\} \leq \frac{n}{M} \right\},$$

where $q, M \in \mathbb{N}$. Define

$$\delta_\infty(M, q) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \# \text{Per}_a(q, M, n) \quad \text{and} \quad \delta_\infty = \inf_q \inf_M \delta_\infty(M, q).$$

We refer to δ_∞ as the topological entropy at infinity of (Σ, σ) . It follows from the transitivity of (Σ, σ) that δ_∞ is independent of a . Note that $\text{Per}(q, M, n) \subseteq \text{Per}(q', M', n)$, whenever $M \geq M'$ and $q \geq q'$. In particular, we have that $\delta_\infty = \lim_{q \rightarrow \infty} \lim_{M \rightarrow \infty} \delta_\infty(M, q)$.

Define

$$h_\infty = \sup_{(\mu_n)_n \rightarrow 0} \limsup_{n \rightarrow \infty} h_{\mu_n}(\sigma),$$

where the supremum runs over all sequences $(\mu_n)_n$ converging on cylinders to the zero measure. If there is no such sequence, we set $h_\infty = -\infty$ (we consider this definition to be consistent with the definition of the pressure at infinity). The quantity h_∞ is called the measure theoretic entropy at infinity of (Σ, σ) . It is proved in [ITV, Theorem 1.4] that if (Σ, σ) has finite entropy, then $h_\infty = \delta_\infty$. We refer to this property as the variational principle for the entropies at infinity.

Remark 2.6. If (Σ, σ) has infinite topological entropy, then $\delta_\infty = \infty$ (see [ITV, Proposition 8.3]). If there exists a sequence of invariant probability measures that converges to the zero measure, then $h_\infty = \infty$ (see Lemma 3.5), otherwise, $h_\infty = -\infty$. In the infinite entropy setting, it is possible that there is no sequence in $\mathcal{M}(\sigma)$ that converges on cylinders to the zero measure (see [IV, Example 4.17]). We will discuss this issue in §4.

The entropy at infinity is related to the upper semi-continuity of the entropy map when escape of mass is allowed, as we can see in the next result.

THEOREM 2.7. [ITV, Theorem 1.1] *Let (Σ, σ) be a transitive CMS with finite entropy. Let $(\mu_n)_n$ be a sequence in $\mathcal{M}(\sigma)$ that converges on cylinders to $\lambda\mu$, where $\mu \in \mathcal{M}(\sigma)$ and $\lambda \in [0, 1]$. Then,*

$$\limsup_{n \rightarrow \infty} h_{\mu_n}(\sigma) \leq \lambda h_\mu(\sigma) + (1 - \lambda) \delta_\infty.$$

Moreover, the inequality is sharp.

2.4. Thermodynamic formalism. Let (Σ, σ) be a transitive CMS. Fix $a \in S$. For a potential $\phi : \Sigma \rightarrow \mathbb{R}$, we define

$$Z_n(\phi, a) = \sum_{\sigma^n x = x} e^{S_n \phi(x)} 1_{[a]}(x) \quad \text{and} \quad P^{\text{top}}(\phi, a) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log Z_n(\phi, a).$$

If $P^{\text{top}}(\phi, a)$ is independent of $a \in S$, we refer to this value as the topological pressure, or Gurevich pressure, of ϕ . The topological pressure of ϕ is denoted by $P^{\text{top}}(\phi)$. If ϕ has summable variations, then the topological pressure is well defined. Furthermore, if (Σ, σ) is mixing, then the limsup is a limit (see [Sa1, Theorem 1]). Note that the topological pressure of the zero potential is the topological entropy of (Σ, σ) .

Define

$$Z_n^*(\phi, a) = \sum_{\sigma^n x = x} e^{S_n \phi(x)} 1_{\psi_a = n}(x) \quad \text{and} \quad P^*(\phi, a) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log Z_n^*(\phi, a),$$

where $\psi_a(x) = 1_{[a]}(x) \inf\{n \geq 1 : \sigma^n x \in [a]\}$. Extending notions studied for Markov chains, Sarig [Sa1] classified potentials according to their recurrence properties.

Definition 2.8. Let (Σ, σ) be a transitive CMS and ϕ a potential with summable variations and finite pressure. Set $\lambda = \exp P^{\text{top}}(\phi)$.

- (1) If $\sum_{n \geq 1} \lambda^{-n} Z_n(\phi, a)$ diverges, we say that ϕ is recurrent.
- (2) If $\sum_{n \geq 1} \lambda^{-n} Z_n(\phi, a)$ converges, we say that ϕ is transient.
- (3) If ϕ is recurrent and $\sum_{n \geq 1} n \lambda^{-n} Z_n^*(\phi, a)$ converges, we say that ϕ is positive recurrent.
- (4) If ϕ is recurrent and $\sum_{n \geq 1} n \lambda^{-n} Z_n^*(\phi, a)$ diverges, we say that ϕ is null recurrent.

The transfer operator L_ϕ associated with ϕ is given by

$$L_\phi f(x) = \sum_{\sigma y = x} e^{\phi(y)} f(y),$$

which also induces a dual operator L_ϕ^* acting on the space of Borel measures. Sarig [Sa1, Sa2] proved that ϕ is recurrent if and only if there exists a positive continuous function h , and a conservative measure ν finite and positive on cylinders such that $L_\phi h = e^{P^{\text{top}}(\phi)} h$ and $L_\phi^* \nu = e^{P^{\text{top}}(\phi)} \nu$. The measure $h d\nu$ is the Ruelle–Perron–Frobenius measure (RPF measure) associated to ϕ . Moreover, ϕ is positive recurrent if and only if $h d\nu$ is a finite measure. In this case, we define $\mu_\phi = (\int h d\nu)^{-1} h d\nu$. If ϕ is positive recurrent and $\int \phi d\mu_\phi > -\infty$, then μ_ϕ is an equilibrium state of ϕ . A fundamental result in this setting is that a potential with summable variations has at most one equilibrium state (see [BS, Theorem 1.2]). However, if ϕ has an equilibrium state, then ϕ is positive recurrent and μ_ϕ is the unique equilibrium state of ϕ .

Example 2.9. Let $(\mathbb{N}^{\mathbb{N}}, \sigma)$ be the full shift on a countable alphabet and $\phi : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{R}$ a potential such that $\text{var}_1(\phi) = 0$. In this case, $\phi(x) = a(x_1)$, where x_1 is the first coordinate of $x \in \mathbb{N}^{\mathbb{N}}$ and $a : \mathbb{N} \rightarrow \mathbb{R}$ is a function. Then, $P(\phi) = \log(\sum_{i \in \mathbb{N}} e^{a(i)})$.

2.4.1. Variational principle. The measure-theoretic pressure of ϕ is defined by the formula

$$P(\phi) = \sup_{\mu \in \mathcal{M}_\phi(\sigma)} \left(h_\mu(\sigma) + \int \phi d\mu \right),$$

where $\mathcal{M}_\phi(\sigma) = \{\mu \in \mathcal{M}(\sigma) : \int \phi d\mu > -\infty\}$. If (Σ, σ) is mixing and ϕ has summable variations, then $P^{\text{top}}(\phi) = P(\phi)$ (see [Sa1, Theorem 3] and [IJT, Theorem 2.10]). This statement is known as the variational principle for the pressure. In the following remark, we discuss how to deduce the variational principle for transitive CMS from the mixing case, both for completeness and for later use.

Remark 2.10. Let (Σ, σ) be a transitive CMS with alphabet S and period p , that is, p is the greatest common divisor of the set of periods of periodic orbits in Σ . Assume $p > 1$ (a CMS is mixing if and only if $p = 1$). Let us define an equivalence relation on S . We say that $a \sim b$ if $a = b$ or if there exists an admissible word $aq_1 \cdots q_{n-1}b$, where n is a multiple of p . There are p equivalence classes $S_1, \dots, S_p$ in the alphabet S . Define $\Sigma_i = \{x \in \Sigma : x_1 \in S_i\}$. It follows that $\Sigma = \bigcup_{i=1}^p \Sigma_i$, $\sigma(\Sigma_i) = \Sigma_{i+1}$, and $\sigma^p(\Sigma_i) = \Sigma_i$ (maybe after reordering the equivalent classes). The dynamical system (Σ_i, σ^p) is a CMS whose alphabet is given by the set of admissible words $\{\tilde{a} = a_1 \cdots a_p : a_1 \in S_i\}$, where $\tilde{a}\tilde{b}$ is admissible if and only if a_pb_1 is admissible in Σ . The period of (Σ_i, σ^p) is equal to one, in other words, it is mixing. The partition $\{\Sigma_i\}_{i=1}^p$ is called the spectral decomposition of Σ .

Note that if we induce on Σ_1 , the first return time is constant and equal to p . Let $\bar{\phi} = S_p\phi$ be the induced potential on Σ_1 , and for $\mu \in \mathcal{M}(\sigma)$, we set $\bar{\mu}(\cdot) = \mu(\Sigma_1 \cap \cdot) / \mu(\Sigma_1)$ (note that $\mu(\Sigma_1) > 0$ for every $\mu \in \mathcal{M}(\sigma)$). The map $\mu \mapsto \bar{\mu}$ is a bijection between $\mathcal{M}(\sigma)$ and $\mathcal{M}(\Sigma_1, \sigma^p)$ with inverse $\nu \mapsto (1/p) \sum_{i=0}^{p-1} \sigma_*^i \nu$. It follows by Kac's and Abramov's formulae that $h_{\bar{\mu}}(\sigma^p) + \int \bar{\phi} d\bar{\mu} = p(h_\mu(\sigma) + \int \phi d\mu)$ for every $\mu \in \mathcal{M}(\sigma)$. In particular, $P(\bar{\phi}) = pP(\phi)$. However, $Z_n(\phi, a)$ is different from zero only if n is a multiple of p ; therefore,

$$P^{\text{top}}(\phi) = \limsup_{k \rightarrow \infty} \frac{1}{kp} \log Z_{kp}(\phi, a).$$

Choose $a \in S_1$. Then,

$$P^{\text{top}}(\phi) = \limsup_{k \rightarrow \infty} \frac{1}{kp} \log Z_{kp}(\phi, a) = \frac{1}{p} \limsup_{k \rightarrow \infty} \frac{1}{k} \log Z_k(\bar{\phi}, a) = \frac{1}{p} P^{\text{top}}(\bar{\phi}).$$

Finally, since (Σ_1, σ^p) is mixing, we have that $P(\bar{\phi}) = P^{\text{top}}(\bar{\phi})$ and, therefore, $P(\phi) = P^{\text{top}}(\phi)$.

Remark 2.11. Observe that $|P^{\text{top}}(\phi, a) - P^{\text{top}}(\phi', a)| \leq \|\phi - \phi'\|_0$ and that $|P(\phi) - P(\phi')| \leq \|\phi - \phi'\|_0$. In particular, the pressure functions $\phi \mapsto P(\phi)$ and $\phi \mapsto P^{\text{top}}(\phi, a)$ are continuous with respect to the C^0 -norm. Since potentials with summable variations are dense in $\mathcal{H}$, we conclude that the topological pressure is well defined for potentials in $\mathcal{H}$ and that the variational principle still holds.

2.5. A recoding. Let (Σ, σ) be a CMS with alphabet S and transition matrix M . In this subsection, we describe a recoding of (Σ, σ) , which will be used in §6.1. Fix $m \in \mathbb{N}$. Define $S_m = \{(a_1, \dots, a_m) \in S^m : \prod_{i=1}^{m-1} M_{a_i, a_{i+1}} = 1\}$ and the $S_m \times S_m$ incidence matrix M_m , given by $(M_m)_{(a_1, \dots, a_m), (b_1, \dots, b_m)} = 1$, if $a_{i+1} = b_i$ for all $1 \leq i \leq m-1$, and zero otherwise. The CMS with alphabet S_m and transition matrix M_m is denoted by (Σ_m, σ_m) .

Let $\pi_m : \Sigma_m \rightarrow \Sigma$ be the canonical map that sends $x = ((x_1, \dots, x_m), (x_2, \dots, x_{m+1}), \dots)$ to $\pi_m(x) = (x_1, x_2, \dots)$. It follows by construction of Σ_m that π_m is a bijection and that $\pi_m \circ \sigma_m = \sigma \circ \pi_m$. Observe that

$$\pi_m([(a_1, \dots, a_m), (a_2, \dots, a_{m+1}), \dots, (a_k, \dots, a_{k+m-1})]) = [a_1, \dots, a_{k+m-1}] \quad (2)$$

for every $k \in \mathbb{N}$. Note that π_m maps a basis of the topology of Σ_m to a basis of the topology of Σ . In particular, π_m is a homeomorphism, and then π_m is a topological conjugacy between (Σ_m, σ_m) and (Σ, σ) . It follows that the push forward $(\pi_m)_* : \mathcal{M}(\sigma_m) \rightarrow \mathcal{M}(\sigma)$ is a homeomorphism with respect to the weak* topologies. The map $(\pi_m)_*$ induces a bijection between $\mathcal{M}_{\leq 1}(\sigma_m)$ and $\mathcal{M}_{\leq 1}(\sigma)$, which is not necessarily a homeomorphism with respect to the cylinder topologies (see Example 2.13). For $\mu \in \mathcal{M}_{\leq 1}(\sigma)$, we set $\tilde{\mu} = (\pi_m)_*^{-1}\mu \in \mathcal{M}_{\leq 1}(\sigma_m)$.

Remark 2.12. Note that π_m maps cylinders of length k in Σ_m to cylinders of length $m + k - 1$ in Σ . In particular, if $(\mu_n)_n$ is a sequence in $\mathcal{M}(\sigma)$ that converges on cylinders to $\mu \in \mathcal{M}_{\leq 1}(\sigma)$, then $(\tilde{\mu}_n)_n$ converges on cylinders to $\tilde{\mu}$. However, if $(\tilde{\mu}_n)_n$ is a sequence in $\mathcal{M}(\sigma_m)$ that converges on cylinders to $\tilde{\mu} \in \mathcal{M}_{\leq 1}(\sigma_m)$, then $\lim_{n \rightarrow \infty} \mu_n(C) = \mu(C)$ for all cylinders $C \subseteq \Sigma$ of length greater than m .

Example 2.13. Consider the full shift $(\mathbb{N}^{\mathbb{N}}, \sigma)$ and the periodic points $p_n = \overline{1n}$. Each periodic point p_n defines a periodic measure $\mu_n = \frac{1}{2}(\delta_{p_n} + \delta_{\sigma(p_n)})$. Observe that $(\mu_n)_n$ satisfies that $\mu_n([1]) = \frac{1}{2}$ and $\lim_{n \rightarrow \infty} \mu_n([a, b]) = 0$ for every $a, b \in \mathbb{N}$. Consider the construction above for $m = 2$ and note that the sequence $(\tilde{\mu}_n)_n$ converges on cylinders to the zero measure, but $(\mu_n)_n$ does not converge to a countably additive measure. In Remark 6.4, we provide a condition that rules out this type of behaviour.

3. Pressure at infinity and the variational principle

In this section, we study the pressure at infinity for potentials defined on countable Markov shifts. We consider two versions of the pressure at infinity: one based on measure-theoretic data and the other on topological data. We will prove Theorem 1.2, which states that both versions coincide for potentials with summable variations and finite pressure. To prove the variational principle for the pressures at infinity, we first derive a formula for the topological pressure at infinity in terms of the topological pressure (see Proposition 3.2). Finally, in the last subsection, we introduce some definitions and prove several useful results needed in later sections.

3.1. Topological pressure at infinity. Let (Σ, σ) be a transitive CMS with alphabet $\mathbb{N}$. Fix $a \in \mathbb{N}$. Recall that $\text{Per}_a(n) = \{x \in \Sigma : x_1 = a, \text{ and } \sigma^n(x) = x\}$, and $\text{Per}_a(q, M, n) = \{x \in \text{Per}_a(n) : \#\{k \in \{1, \dots, n\} : x_k \leq q\} \leq n/M\}$, where $q, M \in \mathbb{N}$.

Definition 3.1. Given a potential $\phi : \Sigma \rightarrow \mathbb{R}$, and $q, M \in \mathbb{N}$, we define

$$Z_n(\phi, a; q, M) = \sum_{x \in \text{Per}_a(q, M, n)} \exp(S_n \phi(x)).$$

If $\text{Per}_a(q, M, n) = \emptyset$, we set $Z_n(\phi, a; q, M) = 0$. Define

$$P_{\infty}^{\text{top}}(\phi, a; q, M) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log Z_n(\phi, a; q, M) \quad \text{and} \\ P_{\infty}^{\text{top}}(\phi, a) = \inf_q \inf_M P_{\infty}^{\text{top}}(\phi, a; q, M).$$

If $P_\infty^{\text{top}}(\phi, a)$ is independent of a , we refer to this value as the topological pressure at infinity of ϕ , which we denote by $P_\infty^{\text{top}}(\phi)$.

Since $\text{Per}_a(q, M, n) \subseteq \text{Per}_a(q', M', n)$, whenever $M \geq M'$ and $q \geq q'$, it follows that

$$P_\infty(\phi, a) = \lim_{q \rightarrow \infty} \lim_{M \rightarrow \infty} P_\infty(\phi, a; q, M).$$

In the next proposition, we establish a useful formula for the topological pressure at infinity of a potential. Let $\text{Per}_a(q, M, n)^c = \text{Per}_a(n) \setminus \text{Per}_a(q, M, n)$.

PROPOSITION 3.2. *Let $\phi : \Sigma \rightarrow \mathbb{R}$ be a potential with summable variations. Let $V = \sum_{k=1}^\infty 1/k 1_{[k]} \in C_0(\Sigma)$. Then,*

$$P_\infty^{\text{top}}(\phi, a) = \lim_{t \rightarrow \infty} P(\phi - tV).$$

Proof. We will first prove that $P_\infty^{\text{top}}(\phi, a) = P_\infty^{\text{top}}(\phi - tV, a)$ for every $t \geq 0$. Observe that if $x \in \text{Per}_a(q, M, n)$, then $S_n V(x) \leq (n/M) + (n/q)$. It follows that

$$Z_n(\phi - tV, a; q, M) \geq Z_n(\phi, a; q, M) \exp\left(-nt\left(\frac{1}{M} + \frac{1}{q}\right)\right),$$

and therefore,

$$P_\infty^{\text{top}}(\phi - tV, a; q, M) \geq P_\infty^{\text{top}}(\phi, a; q, M) - t\left(\frac{1}{M} + \frac{1}{q}\right).$$

Sending q and M to infinity, we obtain $P_\infty^{\text{top}}(\phi - tV, a) \geq P_\infty^{\text{top}}(\phi, a)$. The other inequality follows from $\phi \geq \phi - tV$. We conclude that $P_\infty^{\text{top}}(\phi, a) = P_\infty^{\text{top}}(\phi - tV, a)$.

Since $P_\infty^{\text{top}}(\phi, a) = P_\infty^{\text{top}}(\phi - tV, a) \leq P(\phi - tV)$, we have $P_\infty^{\text{top}}(\phi, a) \leq \lim_{t \rightarrow \infty} P(\phi - tV)$. To complete the proof, we need to show that $\lim_{t \rightarrow \infty} P(\phi - tV) \leq P_\infty^{\text{top}}(\phi, a)$.

Fix $q, M \in \mathbb{N}$. If $x \in \text{Per}_a^c(q, M, n)$, then $\#\{1 \leq k \leq n : x_k \leq q\} > (n/M)$. In particular, $S_n V(x) > (n/Mq)$. Then,

$$\sum_{x \in \text{Per}_a^c(q, M, n)} \exp(S_n(\phi - tV)(x)) \leq \exp\left(-\frac{tn}{Mq}\right) \sum_{x \in \text{Per}_a^c(q, M, n)} \exp(S_n \phi(x)).$$

By definition of the pressure, there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then $\sum_{x \in \text{Per}_a^c(q, M, n)} \exp(S_n \phi(x)) \leq \exp(n(P(\phi) + 1))$. Therefore,

$$\sum_{x \in \text{Per}_a^c(q, M, n)} \exp(S_n(\phi - tV)(x)) \leq \exp\left(n\left(P(\phi) + 1 - \frac{t}{Mq}\right)\right)$$

for every $n \geq n_0$. Finally, for $n \geq n_0$ and $t > qM(P(\phi) + 1 + \kappa)$, we have

$$\begin{aligned} Z_n(\phi - tV, a) &= \sum_{x \in \text{Per}_a(q, M, n)} \exp(S_n(\phi - tV)(x)) + \sum_{x \in \text{Per}_a^c(q, M, n)} \exp(S_n(\phi - tV)(x)) \\ &\leq \sum_{x \in \text{Per}_a(q, M, n)} \exp(S_n \phi(x)) + e^{-\kappa n} \end{aligned}$$

for every $\kappa > 0$. We conclude that $P(\phi - tV) \leq \max\{P_\infty^{\text{top}}(\phi, a; q, M), -\kappa\}$ and, therefore, $\lim_{t \rightarrow \infty} P(\phi - tV) \leq P_\infty^{\text{top}}(\phi, a; q, M)$. $\square$

Remark 3.3. It follows from Proposition 3.2 that if $\phi : \Sigma \rightarrow \mathbb{R}$ has summable variations, then $P_\infty^{\text{top}}(\phi, a)$ is independent of $a \in \mathbb{N}$. In particular, $P_\infty^{\text{top}}(\phi)$ is well defined. This can also be proved directly, similar to the proof of the independence of the base symbol for the topological pressure.

3.2. Measure-theoretic pressure at infinity and the variational principle. In §1, we defined the measure-theoretic pressure at infinity as

$$P_\infty(\phi) = \sup_{(\mu_n)_n \rightarrow 0} \limsup_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) + \int \phi d\mu_n \right),$$

where the supremum runs over all sequences $(\mu_n)_n$ in $\mathcal{M}_\phi(\sigma)$ that converge on cylinders to the zero measure. If there is no such sequence, we set $P_\infty(\phi) = -\infty$. The next two results state some basic properties of the measure-theoretic pressure at infinity.

LEMMA 3.4. *The following hold:*

- (1) $P_\infty(\phi) \leq P(\phi)$;
- (2) $P_\infty(\phi + c) = P_\infty(\phi) + c$ for every $c \in \mathbb{R}$;
- (3) if $\phi \leq \psi$, then $P_\infty(\phi) \leq P_\infty(\psi)$;
- (4) $|P_\infty(\phi) - P_\infty(\psi)| \leq \|\phi - \psi\|_0$;
- (5) the function $t \mapsto P_\infty(t\phi)$ is convex whenever finite;
- (6) $P_\infty(\phi + g) = P_\infty(\phi)$, whenever $g \in C_0(\Sigma)$.

Proof. Parts (1)–(5) follow easily from the definition. For part (6), note that $\lim_{n \rightarrow \infty} \int g d\mu_n = 0$ for every sequence $(\mu_n)_n$ that converges on cylinders to the zero measure. $\square$

LEMMA 3.5. *Suppose that there exists a sequence in $\mathcal{M}(\sigma)$ that converges on cylinders to the zero measure. Let $\phi \in C(\Sigma)$ be a potential such that $P(\phi) = \infty$. Then, $P_\infty(\phi) = \infty$.*

Proof. Since periodic measures are dense in $\mathcal{M}(\sigma)$, there exists a sequence $(\nu_n)_n$ in $\mathcal{M}_\phi(\sigma)$ that converges on cylinders to the zero measure. By the variational principle, there exists a sequence $(\mu_n)_n$ in $\mathcal{M}_\phi(\sigma)$ such that $h_{\mu_n}(\sigma) + \int \phi d\mu_n \geq n^2$. Define $\eta_n = (1 - (1/n))\nu_n + (1/n)\mu_n$. The sequence $(\eta_n)_n$ converges to the zero measure and satisfies that $h_{\eta_n}(\sigma) + \int \phi d\eta_n \geq n$. $\square$

The next proposition is the key ingredient to prove the variational principle for the pressures at infinity.

PROPOSITION 3.6. *Let $\phi : \Sigma \rightarrow \mathbb{R}$ be a potential with summable variations and finite pressure. Let $(\mu_n)_n$ be a sequence in $\mathcal{M}_\phi(\sigma)$ that converges on cylinders to the zero measure. Then,*

$$\limsup_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) + \int \phi d\mu_n \right) \leq P_\infty^{\text{top}}(\phi). \quad (3)$$

If $P_\infty^{\text{top}}(\phi) > -\infty$, then there exists a sequence of invariant probability measures, as described above, for which equality holds.

Proof. Let $V = \sum_{k=1}^\infty (1/k) 1_{[k]}$. Note that $\lim_{n \rightarrow \infty} \int V d\mu_n = 0$. By the standard variational principle, we have that

$$\limsup_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) + \int \phi d\mu_n \right) = \limsup_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) + \int (\phi - tV) d\mu_n \right) \leq P(\phi - tV).$$

In particular, the left-hand side is bounded from above by $\lim_{t \rightarrow \infty} P(\phi - tV)$. Inequality (3) follows from Proposition 3.2.

Let us suppose now that $M := P_\infty^{\text{top}}(\phi) > -\infty$. We will prove that inequality (3) is sharp. It follows from Proposition 3.2 that $P(\phi - tV) \geq M$ holds for all $t \in \mathbb{R}$. The variational principle guarantees the existence of σ -invariant probability measures $(\nu_n)_n$ such that

$$h_{\nu_n}(\sigma) + \int (\phi - nV) d\nu_n \geq P(\phi - nV) - \frac{1}{n}, \quad (4)$$

where $\int (\phi - nV) d\nu_n > -\infty$. In particular, $\int \phi d\nu_n > -\infty$. We claim that $(\nu_n)_n$ converges on cylinders to the zero measure. We argue by contradiction and suppose that $\limsup_{n \rightarrow \infty} \nu_n(C) > 0$ for some cylinder $C \subseteq \Sigma$. In this case, we would have that $\limsup_{n \rightarrow \infty} n \int V d\nu_n = \infty$, which is not possible because inequality (4) implies that $P(\phi) - n \int V d\nu_n \geq M - 1$. However,

$$\begin{aligned} P_\infty^{\text{top}}(\phi) &= \lim_{n \rightarrow \infty} P(\phi - nV) \leq \liminf_{n \rightarrow \infty} \left(h_{\nu_n}(\sigma) + \int \phi d\nu_n - n \int V d\nu_n \right) \\ &\leq \liminf_{n \rightarrow \infty} \left(h_{\nu_n}(\sigma) + \int \phi d\nu_n \right). \end{aligned}$$

In conclusion, $(\nu_n)_n$ is a sequence in $\mathcal{M}_\phi(\sigma)$ that converges on cylinders to the zero measure and $P_\infty^{\text{top}}(\phi) = \lim_{n \rightarrow \infty} h_{\nu_n}(\sigma) + \int \phi d\nu_n$. $\square$

Proof of Theorem 1.2. In Proposition 3.6, we proved that $P_\infty(\phi) \leq P_\infty^{\text{top}}(\phi)$ and that equality holds if $P_\infty^{\text{top}}(\phi) > -\infty$. Observe that inequality $P_\infty(\phi) \leq P_\infty^{\text{top}}(\phi)$ also implies that if $P_\infty^{\text{top}}(\phi) = -\infty$, then $P_\infty(\phi) = -\infty$. $\square$

Remark 3.7. Let ϕ be a potential with summable variations. Note that if ϕ has finite pressure and there is no sequence in $\mathcal{M}(\sigma)$ that converges to the zero measure, then $P_\infty(\phi) = P_\infty^{\text{top}}(\phi) = -\infty$ (see Proposition 3.6). Suppose now that $P(\phi) = \infty$. Then, the following hold.

- (1) $P_\infty^{\text{top}}(\phi) = \infty$ (see Proposition 3.2). Then, $P(\phi) < \infty$ if and only if $P_\infty^{\text{top}}(\phi) < \infty$.
- (2) If there exists a sequence in $\mathcal{M}(\sigma)$ that converges on cylinders to the zero measure, then $P_\infty(\phi) = \infty$ (see Lemma 3.5). If there is no sequence converging to the zero measure, then by definition, $P_\infty(\phi) = -\infty$.

Remark 3.8. Observe that $|P_\infty^{\text{top}}(\phi, a) - P_\infty^{\text{top}}(\phi', a)| \leq \|\phi - \phi'\|_0$ and that $|P_\infty(\phi) - P_\infty(\phi')| \leq \|\phi - \phi'\|_0$. In particular, the pressure at infinity functions $\phi \mapsto P_\infty(\phi)$ and $\phi \mapsto P_\infty^{\text{top}}(\phi, a)$ are continuous with respect to the C^0 -norm. Since potentials with

summable variations are dense in $\mathcal{H}$, we conclude that the topological pressure at infinity is well defined for potentials in $\mathcal{H}$ and that if $\phi \in \mathcal{H}$, then $P_\infty^{\text{top}}(\phi) = P_\infty(\phi)$.

3.3. Further properties and classes of potentials. In this subsection, we introduce some definitions and establish useful facts for later use.

For a potential $\phi \in C(\Sigma)$ with finite pressure, we define

$$s_\infty(\phi) = \inf\{t \in [0, \infty) : P(t\phi) < \infty\}.$$

By definition, $s_\infty(\phi) \in [0, 1]$. Note that if $\|\phi_1 - \phi_2\|_0 < \infty$, then $s_\infty(\phi_1) = s_\infty(\phi_2)$.

LEMMA 3.9. *Let $\phi \in C(\Sigma)$ be a potential with finite pressure. Suppose that $s_\infty(\phi) < 1$. Let $(\mu_n)_n$ be a sequence in $\mathcal{M}_\phi(\sigma)$ such that $\lim_{n \rightarrow \infty} \int \phi d\mu_n = -\infty$. Then,*

$$\lim_{n \rightarrow \infty} h_{\mu_n}(\sigma) + \int \phi d\mu_n = -\infty.$$

Proof. Since the pressure is convex whenever finite, it follows that $P(t\phi) < \infty$ for every $t \in (s_\infty(\phi), 1]$. Consider $\epsilon > 0$ small such that $P((1 - \epsilon)\phi) < \infty$, and note that

$$h_{\mu_n} + \int \phi d\mu_n \leq P((1 - \epsilon)\phi) + \epsilon \int \phi d\mu_n$$

for every $n \in \mathbb{N}$. □

LEMMA 3.10. *Let $\phi \in C(\Sigma)$ be a potential with finite pressure. Assume that $s_\infty(\phi) < 1$ and that $P_\infty(\phi) > -\infty$. Let $(\mu_n)_n$ be a sequence in $\mathcal{M}_\phi(\sigma)$ that converges on cylinders to the zero measure and such that $\lim_{n \rightarrow \infty} (h_{\mu_n}(\sigma) + \int \phi d\mu_n) = P_\infty(\phi)$. Therefore, $\liminf_{n \rightarrow \infty} \int \phi d\mu_n > -\infty$.*

Proof. We argue by contradiction and assume that $\liminf_{n \rightarrow \infty} \int \phi d\mu_n = -\infty$. In particular, there exists a subsequence whose limit is $-\infty$. The result now follows from Lemma 3.9. □

In §1, we defined

$$\Psi = \{\phi \in C_{uc}(\Sigma) : \inf \phi > 0, \text{var}_2(\phi) < \infty, P(-\phi) < \infty\}.$$

Note that if $\phi \in \Psi$, then $t \mapsto P(-t\phi)$ is strictly decreasing and, therefore, $P(-t\phi)$ is finite for $t \in (s_\infty(-\phi), \infty)$ and $P(-t\phi) = \infty$ if $t \in (0, s_\infty(-\phi))$. Furthermore, $-t\phi \in \mathcal{H}$ for every $t > 0$ and, therefore, Remark 3.8 applies.

Definition 3.11. Let $\phi \in \Psi$. Define

$$h(\phi) = \inf\{t : P_\infty^{\text{top}}(-t\phi) \leq 0\}.$$

We say that ϕ belongs to Ψ_1 if there exists $s > 0$ such that $P_\infty^{\text{top}}(-s\phi) \in (0, \infty)$. We say that $\phi \in \Psi_2$ if ϕ does not belong to Ψ_1 .

Remark 3.12. Note that $P(-t\phi) < \infty$ if and only if $P_\infty^{\text{top}}(-t\phi) < \infty$ (see Remark 3.7(1)). In particular, $h(\phi) \geq s_\infty(-\phi)$. Moreover, if $t > s_\infty(-\phi)$, then $P_\infty^{\text{top}}(-t\phi) = P_\infty(-t\phi)$ (see Remark 3.8). Furthermore, the following hold.

- (1) If $\phi \in \Psi_1$, then there exists a unique real number $s_\phi > s_\infty(-\phi)$ such that $P_\infty^{\text{top}}(-s_\phi\phi) = 0$. In this case, $h(\phi) = s_\phi$.
- (2) If $\phi \in \Psi_2$, then $P_\infty^{\text{top}}(-s\phi) < 0$ for every $s > s_\infty(-\phi)$. In this case, $h(\phi) = s_\infty(-\phi)$.

The quantity $h(\phi)$ has a clear dynamical interpretation, as it represents the entropy at infinity of the suspension flow over (Σ, σ) with roof function ϕ (see §8).

4. SPR potentials and the existence of sequences converging to the zero measure

In his study of phase transitions, Sarig [Sa3] introduced the class of strongly positive recurrent potentials (referred to as SPR). The thermodynamic formalism of SPR potentials is similar to that of Hölder potentials in subshifts of finite type. For instance, a weakly Hölder SPR potential is positive recurrent and the RPF measure has exponential decay of correlations (see [CS, Sa3]). Furthermore, it is proven in [CS, Theorem 2.2] that the space of SPR potentials is open and dense in the space of weakly Hölder potentials with finite pressure (in the C^0 and finer topologies).

Let (Σ, σ) be a CMS with alphabet S and transition matrix M . Let $a \in S$ and define $\Sigma(a)$ as the set of points in $[a]$ that return to $[a]$ infinitely many times. Let $\sigma_a : \Sigma(a) \rightarrow \Sigma(a)$ be the first return map to $\Sigma(a)$. The system $(\Sigma(a), \sigma_a)$ is conjugate to the full shift on a countable alphabet. The Markov partition on $\Sigma(a)$ is given by cylinders of the form $[a, b_1, \dots, b_n]$, where each $b_i \neq a$ and b_na is an admissible word. Denote by $\tau_a : \Sigma(a) \rightarrow \mathbb{N}$ the first return time. The function τ_a is locally constant and depends only on the first coordinate of the Markov structure on $\Sigma(a)$. For a potential $\psi : \Sigma \rightarrow \mathbb{R}$, we denote by $\bar{\psi} : \Sigma(a) \rightarrow \mathbb{R}$ the induced potential, that is, $\bar{\psi}(x) = \sum_{i=0}^{\tau_a(x)-1} \psi(\sigma^i x)$ for every $x \in \Sigma(a)$. Let Φ be the set of weakly Hölder potentials in Σ that are bounded above and have finite pressure.

Remark 4.1. If $\psi \in \Phi$ is recurrent and $P(\psi) = 0$, then $\bar{\psi}$ is weakly Hölder, positive recurrent and $P(\bar{\psi}) = 0$ (see [Sa3, Lemma 3]).

Let $\phi \in \Phi$. Define $p_a^*(\phi) = \sup\{t : P(\overline{\phi + t}) < \infty\}$. It is proved in [Sa3] (see also [CS, §7.3]) that $p_a^*(\phi) = -P^*(\phi, a)$ (see §2.4). The discriminant of ϕ at $a \in S$ is defined as

$$\Delta_a(\phi) = \sup\{P(\overline{\phi + t}) : t < p_a^*(\phi)\}. \quad (5)$$

The following result was proved by Sarig in [Sa3, Theorem 2].

THEOREM 4.2. (Discriminant theorem) *Let (Σ, σ) be a mixing CMS and $\phi \in \Phi$. Let $a \in S$.*

- (1) *The equation $P(\overline{\phi + t}) = 0$ has a unique solution $t = p(\phi)$ if $\Delta_a(\phi) \geq 0$ and no solution if $\Delta_a(\phi) < 0$. Moreover,*

$$P(\phi) = \begin{cases} -p(\phi) & \text{if } \Delta_a(\phi) \geq 0, \\ -p_a^*(\phi) & \text{if } \Delta_a(\phi) < 0. \end{cases}$$

- (2) *ϕ is positive recurrent if $\Delta_a(\phi) > 0$ and transient if $\Delta_a(\phi) < 0$. If $\Delta_a(\phi) = 0$, then ϕ is either positive recurrent or null recurrent.*

We say that $\phi \in \Phi$ is strongly positive recurrent if $\Delta_a(\phi) > 0$. If the zero potential is SPR, we say that (Σ, σ) is strongly positive recurrent. The class of SPR (or stable-positive recurrent) countable Markov shifts has been extensively studied by Gurevich and Savchenko in [GS]. It was proved in [RS, Theorem 8.2] that if $\phi \in \Phi$, then ϕ is SPR if and only if $P_\infty(\phi) < P(\phi)$. It is natural to consider the following definition.

Definition 4.3. We say $\phi \in C(\Sigma)$ is SPR if $P(\phi)$ is finite and $P_\infty(\phi) < P(\phi)$.

Let $G = (V, E)$ be the directed graph associated with (Σ, σ) . Recall that the set of vertices V is identified with S and $(i, j) \in E$ if and only if $M_{i,j} = 1$. A subset $F \subseteq V$ is called a uniform Rome if there exists $N \in \mathbb{N}$ such that $V \setminus F$ has no paths in G of length greater than N . A finite uniform Rome is a uniform Rome where F is a finite set. In the next result, the equivalence between parts (1) and (2) was proved by Cyr in [Cy, Theorem 2.1], and the equivalence between parts (2) and (3) follows from [CS, Theorem 2.3].

THEOREM 4.4. Let (Σ, σ) be a mixing CMS and G the associated directed graph. Then, the following statements are equivalent.

- (1) The graph G has a finite uniform Rome.
- (2) The set of transient potentials in Φ is empty.
- (3) Every potential in Φ is SPR.

We add another statement to the equivalences mentioned above.

THEOREM 4.5. Let (Σ, σ) be a mixing CMS and G the associated directed graph. Then, the following statements are equivalent.

- (1) The graph G has a finite uniform Rome.
- (2) The set of transient potentials in Φ is empty.
- (3) $P_\infty^{\text{top}}(\phi) = -\infty$ for every $\phi \in \Phi$. In particular, every potential in Φ is SPR.
- (4) There is no sequence in $\mathcal{M}(\sigma)$ that converges on cylinders to the zero measure.

Proof. Let $\phi \in \Phi$. It follows from Proposition 3.6 that if $P_\infty^{\text{top}}(\phi) > -\infty$, then there are sequences of invariant probability measures that converge on cylinders to the zero measure. We conclude that part (4) implies that $P_\infty^{\text{top}}(\phi) = -\infty$ and, therefore, ϕ is SPR. In other words, part (4) implies part (3). By Theorem 4.4, to complete the proof, it is enough to show that part (1) implies part (4). Let $F \subseteq V$ be a finite uniform Rome of $G = (V, E)$. It follows from the definition of uniform Rome that if $\mu \in \mathcal{M}(\sigma)$ is a periodic measure, then $\mu(\bigcup_{s \in F} [s]) \geq 1/(N+1)$. In particular, there are no sequences of periodic measures that converge on cylinders to the zero measure. Since periodic measures are dense in $\mathcal{M}(\sigma)$, we conclude that part (4) holds. $\square$

Remark 4.6. Theorem 4.5 still holds if (Σ, σ) is transitive (see Remark 2.10).

5. A modified countable Markov shift

Let (Σ, σ) be a countable Markov shift with alphabet S and transition matrix M . Let $\tau : \Sigma \rightarrow \mathbb{N}$ be a potential that only depends on the first two coordinates, that is, $\text{var}_2(\tau) = 0$. The potential τ defines a function on $Q := \{(a, b) \in S \times S : M_{a,b} = 1\}$, which we still

denote by τ . With these data, we will construct a countable Markov shift $(\widehat{\Sigma}, \widehat{\sigma})$ with alphabet $\widehat{S}$ and transition matrix $\widehat{M}$ as follows.

- (1) For each $(a, b) \in Q$, we consider a collection of symbols $\{c_1^{a,b}, \dots, c_{\tau(a,b)}^{a,b}\}$, where $c_1^{a,b} = a$. We assume that the symbols $(c_i^{a,b})_{i,a,b}$, where $2 \leq i \leq \tau(a, b)$ and $(a, b) \in Q$, are pairwise different and not in S . Set $\widehat{S} = \bigcup_{(a,b) \in Q} \{c_1^{a,b}, \dots, c_{\tau(a,b)}^{a,b}\}$.
- (2) If $i \in \{1, \dots, \tau(a, b) - 1\}$, we set $\widehat{M}_{c_i^{a,b}, j} = 1$ if and only if $j = c_{i+1}^{a,b}$. Additionally, $\widehat{M}_{c_{\tau(a,b)}^{a,b}, q} = 1$ if and only if $q = b$.

The countable Markov shift $(\widehat{\Sigma}, \widehat{\sigma})$ can be described in simple terms by considering its associated directed graph. Let $G = (V, E)$ be the directed graph associated with (Σ, σ) . The function τ assigns a natural number to each edge $e \in E$. If $\tau(e) > 1$, we subdivide the edge e into $\tau(e)$ sub-edges, preserving the orientation. If $\tau(e) = 1$, we do not modify the edge e . The resulting graph $\widehat{G}$ represents $(\widehat{\Sigma}, \widehat{\sigma})$. Infinite paths in $\widehat{G}$ are in one-to-one correspondence with infinite paths in G , but the dynamics on the edges is slowed down according to the function τ . Note that $(\widehat{\Sigma}, \widehat{\sigma})$ is transitive if and only if (Σ, σ) is transitive.

For $(a, b) \in Q$, we define the admissible word in $(\widehat{\Sigma}, \widehat{\sigma})$,

$$w_{a,b} = c_1^{a,b} \cdots c_{\tau(a,b)}^{a,b}.$$

Note that $w_{a,b}w_{p,q}$ is admissible if and only if $b = p$. We will use the words $(w_{a,b})_{(a,b) \in Q}$ to define points and cylinders in $\widehat{\Sigma}$. These points and cylinders are understood to have entries corresponding to those in the admissible words.

Let $\mathcal{A} = \bigcup_{s \in S} [s] \subseteq \widehat{\Sigma}$ and set $\pi : \Sigma \rightarrow \mathcal{A}$, given by $\pi(a_1, a_2, a_3, \dots) = (w_{a_1, a_2}, w_{a_2, a_3}, \dots)$, which is well defined by construction of $\widehat{\Sigma}$.

LEMMA 5.1. *Let $T : \mathcal{A} \rightarrow \mathcal{A}$ be the induced transformation of $\widehat{\sigma}$ on $\mathcal{A}$. Then, π is a topological conjugacy between (Σ, σ) and $(\mathcal{A}, T)$.*

Proof. It follows by the definition of $\widehat{\Sigma}$ that every $x \in \mathcal{A}$ can be uniquely written as $x = (w_{a_1, a_2}, w_{a_2, a_3}, \dots)$ for some $y = (a_1, a_2, \dots) \in \Sigma$. Equivalently, the map π is a bijection. Moreover, note that $T(x) = (w_{a_2, a_3}, w_{a_3, a_4}, \dots)$ and, therefore, $\pi(\sigma(y)) = T(\pi(y))$. We conclude that π conjugates (Σ, σ) and $(\mathcal{A}, T)$. It remains to prove that π is a homeomorphism. Indeed, observe that $\pi([a_1, a_2, \dots, a_n]) = [w_{a_1, a_2}, w_{a_2, a_3}, \dots, w_{a_{n-1}, a_n}]$ and that the collection of cylinders of the form $[w_{a_1, a_2}, w_{a_2, a_3}, \dots, w_{a_{n-1}, a_n}]$ is a basis of the topology of $\widehat{\Sigma}$. Since π maps a basis of Σ to a basis of $\widehat{\Sigma}$, the claim follows. $\square$

Remark 5.2. The induced transformation $T : \mathcal{A} \rightarrow \mathcal{A}$ is given by $T|_{[a, c_2^{a,b}]} = \widehat{\sigma}^{\tau(a,b)}$. In other words, τ specifies the first return time to $\mathcal{A}$. The dynamical system $(\widehat{\Sigma}, \widehat{\sigma})$ is a discrete suspension flow over $(\mathcal{A}, T)$ or, equivalently, (Σ, σ) . The roof function, which depends only on the first two coordinates, is given by $\tau : \Sigma \rightarrow \mathbb{N}$ and the time-one map of the suspension flow corresponds to $\widehat{\sigma}$. In particular, there is a correspondence between σ -invariant measures on Σ and $\widehat{\sigma}$ -invariant measures on $\widehat{\Sigma}$. In the next subsection, we elaborate on this correspondence, which is explicit in this context, as well as on the relation

between the cylinder topologies. We will use Kac's formula and Abramov's formula to relate the integral and entropy of invariant measures on these two CMS.

5.1. Measures on $(\widehat{\Sigma}, \widehat{\sigma})$ and convergence on cylinders.

LEMMA 5.3. *Let ν be a $\widehat{\sigma}$ -invariant sub-probability measure on $(\widehat{\Sigma}, \widehat{\sigma})$ different from the zero measure. Then, $\nu(\mathcal{A}) > 0$.*

Proof. Since ν has positive mass, there exists a cylinder $[c_i^{a,b}]$ such that $\nu([c_i^{a,b}])$ is positive. If $i = 1$, there is nothing to prove. Assume that $i > 1$, in particular, $\tau(a, b) \geq 2$. In this case, $\sigma^{-(i-1)}[c_i^{a,b}] = [c_1^{a,b}, c_2^{a,b}]$ and, therefore, $\nu([a, c_2^{a,b}]) = \nu([c_i^{a,b}]) > 0$. $\square$

Let $\mathcal{M}(\sigma; \tau) = \{\mu \in \mathcal{M}(\sigma) : \int \tau d\mu < \infty\}$. Define $\varphi : \mathcal{M}(\widehat{\sigma}) \rightarrow \mathcal{M}(\sigma; \tau)$ by the formula

$$\varphi(\nu)(E) = \frac{1}{\nu(\mathcal{A})} \nu(\pi(E)).$$

The function φ is the normalized restriction map to $\mathcal{A}$, where we have used the identification between Σ and $\mathcal{A}$ given by π . It follows from general results in ergodic theory that $\varphi(\nu)$ is a σ -invariant probability measure and that it belongs to $\mathcal{M}(\sigma; \tau)$. Indeed, by Kac's formula, we have that $\int \tau d\varphi(\nu) = 1/\nu(\mathcal{A}) < \infty$ for every $\nu \in \mathcal{M}(\widehat{\sigma})$ (see Lemma 5.3). In general, Kac's formula states that for $f \in L^1(\widehat{\Sigma}, \nu)$, it holds that

$$\int_{\widehat{\Sigma}} f d\nu = \frac{1}{\int \tau d\varphi(\nu)} \int_{\Sigma} \sum_{k=0}^{\tau(x)-1} f(\widehat{\sigma}^k \pi(x)) d\varphi(\nu)(x). \quad (6)$$

The relationship between the measures ν and $\varphi(\nu)$ is explicit in terms of the measure of cylinders. Consider a cylinder in $\widehat{\Sigma}$, which, by the structure of $\widehat{\Sigma}$, has either the form

$$C_1 = [c_i^{a,b}, \dots, c_j^{a,b}], \quad (7)$$

where $1 \leq i \leq j \leq \tau(a, b)$, or

$$C_2 = [c_i^{a,p_1}, \dots, c_{\tau(a,p_1)}^{a,p_1}, w_{p_1,p_2} \cdots w_{p_{n-1},p_n}, c_1^{p_n,b}, \dots, c_j^{p_n,b}], \quad (8)$$

where $1 \leq i \leq \tau(a, p_1)$ and $1 \leq j \leq \tau(p_n, b)$. Note that $D_1 = \widehat{\sigma}^{-(i-1)}C_1 = [w_{a,b}]$ (excluding the case when $i = j = 1$, where $C_1 = [a]$) and $D_2 = \widehat{\sigma}^{-(i-1)}C_2 = [w_{a,p_1}, \dots, w_{p_n,b}]$. It follows from Kac's formula (6) for $f = 1_{D_i}$ that

$$\begin{aligned} \nu(C_1) &= \frac{1}{\int \tau d\varphi(\nu)} \varphi(\nu)([a, b]) \quad \text{and} \\ \nu(C_2) &= \frac{1}{\int \tau d\varphi(\nu)} \varphi(\nu)([a, p_1, p_2, \dots, p_n, b]). \end{aligned} \quad (9)$$

In particular, it also holds that

$$\nu([a]) = \sum_{b: M_{a,b}=1} \nu([a, c_1^{a,b}]) = \frac{1}{\int \tau d\varphi(\nu)} \sum_{b: M_{a,b}=1} \varphi(\nu)([a, b]) = \frac{1}{\int \tau d\varphi(\nu)} \varphi(\nu)([a]). \quad (10)$$

Equations (9) and (10) explicitly demonstrate that φ is a bijection. Indeed, for $\mu \in \mathcal{M}(\sigma; \tau)$, we define

$$\widehat{\mu}(C) := \frac{1}{\int \tau d\mu} \mu([a, p_1, \dots, p_n, b]),$$

where $C = C_i \subseteq \widehat{\Sigma}$ as in (7) or (8). We thus obtain a $\widehat{\sigma}$ -invariant probability measure on $\widehat{\Sigma}$ (it defines a measure by Kolmogorov extension theorem, and its invariance can be verified on cylinders). Note that the map $\mu \mapsto \widehat{\mu}$ is the inverse of φ .

Remark 5.4. It follows from (9) and (10) that for every cylinder $C \subseteq \widehat{\Sigma}$, there exists a cylinder $D \subseteq \Sigma$ such that $\widehat{\mu}(C) = 1/(\int \tau d\mu)\mu(D)$ for every measure $\mu \in \mathcal{M}(\sigma; \tau)$. Analogously, given a cylinder $D \subseteq \Sigma$, there exists a cylinder $C \subseteq \widehat{\Sigma}$ such that $\widehat{\mu}(C) = 1/(\int \tau d\mu)\mu(D)$ for every measure $\mu \in \mathcal{M}(\sigma; \tau)$.

In the next subsection, we will calculate the entropy at infinity of $(\widehat{\Sigma}, \widehat{\sigma})$ in relation to quantities defined on Σ . The next two lemmas will help to establish this connection. The first lemma is a direct consequence of Remark 5.4.

LEMMA 5.5. *Let $(\mu_n)_n$ be a sequence in $\mathcal{M}(\sigma; \tau)$.*

- (1) *Assume that $\limsup_{n \rightarrow \infty} \int \tau d\mu_n < \infty$. Then, $(\widehat{\mu}_n)_n$ converges on cylinders to the zero measure if and only if $(\mu_n)_n$ converges on cylinders to the zero measure.*
- (2) *If $\lim_{n \rightarrow \infty} \int \tau d\mu_n = \infty$, then $(\widehat{\mu}_n)_n$ converges on cylinders to the zero measure.*

LEMMA 5.6. *Let $(\mu_n)_n$ be a sequence in $\mathcal{M}(\sigma; \tau)$ that converges on cylinders to $\lambda_1 \mu$, where $\mu \in \mathcal{M}(\sigma; \tau)$ and $\lambda_1 \in [0, 1]$. Assume that $\lim_{n \rightarrow \infty} \int \tau d\mu_n = \lambda_2 \int \tau d\mu$. Then, $(\widehat{\mu}_n)_n$ converges in the cylinder topology of $\widehat{\Sigma}$ to $(\lambda_1/\lambda_2)\widehat{\mu}$.*

Proof. Let $C \subseteq \widehat{\Sigma}$ be a cylinder. By Remark 5.4, there exists a cylinder $D \subseteq \Sigma$ such that $\widehat{\mu}(C) = 1/(\int \tau d\mu)\mu(D)$ for every $\mu \in \mathcal{M}(\sigma; \tau)$. Then,

$$\lim_{n \rightarrow \infty} \widehat{\mu}_n(C) = \lim_{n \rightarrow \infty} \frac{1}{\int \tau d\mu_n} \mu_n(D) = \frac{\lambda_1}{\lambda_2} \frac{1}{\int \tau d\mu} \mu(D) = \frac{\lambda_1}{\lambda_2} \widehat{\mu}(C).$$

We conclude that $(\widehat{\mu}_n)_n$ converges on cylinders to $(\lambda_1/\lambda_2)\widehat{\mu}$. □

5.2. *Entropy at infinity of $(\widehat{\Sigma}, \widehat{\sigma})$.* Abramov's formula states that $h_{\widehat{\mu}}(\widehat{\sigma}) = 1/(\int \tau d\mu) h_{\mu}(\sigma)$ for every $\mu \in \mathcal{M}(\sigma; \tau)$ (see Remark 5.2). It follows that $h_{\text{top}}(\widehat{\sigma}) = \inf\{t \in \mathbb{R} : P(-t\tau) \leq 0\}$.

Remark 5.7. Since $\tau \geq 1$, the map $t \mapsto P(-t\tau)$ is strictly decreasing. In particular, if $P(-\tau) < \infty$, then $h_{\text{top}}(\widehat{\sigma}) < \infty$.

PROPOSITION 5.8. *The following hold.*

- (1) *If $\tau \in \Psi_1$, then $h_{\infty}(\widehat{\sigma}) = s_{\tau}$.*
- (2) *If $\tau \in \Psi_2$, then $h_{\infty}(\widehat{\sigma}) = s_{\infty}(-\tau)$.*

In other words, if $\tau \in \Psi$, then $h_{\infty}(\widehat{\sigma}) = h(\tau)$ (see Remark 3.12).

Proof.

Case 1. There exists a sequence $(\mu_n)_n$ in $\mathcal{M}(\sigma; \tau)$ that converges to the zero measure and such that $\lim_{n \rightarrow \infty} (h_{\mu_n}(\sigma) - s_\tau \int \tau d\mu_n) = 0$. Then, by Abramov's formula, $\lim_{n \rightarrow \infty} h_{\hat{\mu}_n} = s_\tau$ and $(\hat{\mu}_n)_n$ converges to the zero measure (see Lemma 5.5). It follows that $h_\infty(\hat{\sigma}) \geq s_\tau$.

Let $(\nu_n)_n$ be a sequence of measures in $\mathcal{M}(\hat{\sigma})$ that converges to the zero measure and such that $\lim_{n \rightarrow \infty} h_{\nu_n}(\hat{\sigma}) = h_\infty(\hat{\sigma})$ (the existence of such a sequence is given by Theorem 2.3). Let η_n be the measure in $\mathcal{M}(\sigma; \tau)$ such that $\hat{\eta}_n = \nu_n$. Then,

$$\lim_{n \rightarrow \infty} \left(\frac{h_{\eta_n}(\sigma) - h_\infty(\hat{\sigma}) \int \tau d\eta_n}{\int \tau d\eta_n} \right) = 0.$$

If $\limsup_{n \rightarrow \infty} \int \tau d\eta_n < \infty$, then $(\eta_n)_n$ converges on cylinders to the zero measure (see Lemma 5.5) and $\lim_{n \rightarrow \infty} (h_{\eta_n}(\sigma) - h_\infty(\hat{\sigma}) \int \tau d\eta_n) = 0$. In particular, $P_\infty(-h_\infty(\hat{\sigma})\tau) \geq 0$ and, therefore, $s_\tau \geq h_\infty(\hat{\sigma})$. If $\limsup_{n \rightarrow \infty} \int \tau d\eta_n = \infty$, then there exists a subsequence for which the integral diverges. Maybe after passing to a subsequence can we assume that $\lim_{n \rightarrow \infty} \int \tau d\eta_n = \infty$. Fix $\epsilon > 0$ and note that

$$\limsup_{n \rightarrow \infty} \frac{h_{\eta_n}(\sigma)}{\int \tau d\eta_n} \leq (s_\infty(-\tau) + \epsilon) + \limsup_{n \rightarrow \infty} \frac{P(-(s_\infty(-\tau) + \epsilon)\tau)}{\int \tau d\eta_n} = s_\infty(-\tau) + \epsilon. \quad (11)$$

Therefore, $h_\infty(\hat{\sigma}) \leq s_\infty(-\tau)$. This is not possible since we already proved that $h_\infty(\hat{\sigma}) \geq s_\tau$.

Case 2. Set $s = s_\infty(-\tau)$. We claim that $h_\infty(\hat{\sigma}) \geq s$. We assume that $s > 0$, otherwise there is nothing to prove. Fix $\epsilon > 0$ small. Since $P(-(s - \epsilon)\tau) = \infty$, there exists a sequence $(\mu_n)_n$ in $\mathcal{M}(\sigma; \tau)$ such that

$$\lim_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) - (s - \epsilon) \int \tau d\mu_n \right) = \infty. \quad (12)$$

In particular, the limit is greater than zero and, therefore, $\liminf_{n \rightarrow \infty} h_{\hat{\mu}_n}(\hat{\sigma}) \geq s - \epsilon$. It is enough to prove that $\limsup_{n \rightarrow \infty} \int \tau d\mu_n = \infty$, in which case, $(\hat{\mu}_n)_n$ has a subsequence that converges to the zero measure (see Lemma 5.5) and $\epsilon > 0$ was arbitrary. Suppose otherwise that $\limsup_{n \rightarrow \infty} \int \tau d\mu_n < \infty$ and, therefore, there exists M such that $\int \tau d\mu_n \leq M$ for all $n \in \mathbb{N}$. It follows that $h_{\mu_n}(\sigma) \leq M + P(-\tau)$, which contradicts (12).

We now claim that $h_\infty(\hat{\sigma}) \leq s$. Let $(\nu_n)_n$ be a sequence in $\mathcal{M}(\hat{\sigma})$ that converges to the zero measure and such that $\lim_{n \rightarrow \infty} h_{\nu_n}(\hat{\sigma}) = h_\infty(\hat{\sigma})$. Let η_n be the measure in $\mathcal{M}(\sigma; \tau)$ such that $\hat{\eta}_n = \nu_n$. Then,

$$\lim_{n \rightarrow \infty} \left(\frac{h_{\eta_n}(\sigma) - h_\infty(\hat{\sigma}) \int \tau d\eta_n}{\int \tau d\eta_n} \right) = 0.$$

If $\limsup_{n \rightarrow \infty} \int \tau d\eta_n < \infty$, then $\lim_{n \rightarrow \infty} (h_{\eta_n}(\sigma) - h_\infty(\hat{\sigma}) \int \tau d\eta_n) = 0$ and $(\eta_n)_n$ converges to the zero measure (see Lemma 5.5). In particular, $P_\infty(-h_\infty(\hat{\sigma})\tau) \geq 0$. If $h_\infty(\hat{\sigma}) > s$, then $\tau \in \Psi_1$, which contradicts our assumption. We therefore have that $h_\infty(\hat{\sigma}) \leq s$. Now, suppose that $\limsup_{n \rightarrow \infty} \int \tau d\nu_n = \infty$. Maybe after passing to a subsequence can we assume that $\lim_{n \rightarrow \infty} \int \tau d\nu_n = \infty$. Inequality (11) implies that $h_\infty(\hat{\sigma}) \leq s$. $\square$

5.3. Proof of Theorem 1.1. In this subsection, we will prove Theorem 1.1, a compactness result for sequences of invariant probability measures on countable Markov shifts that may not have the $\mathcal{F}$ -property. This result will be frequently used in later applications concerning the existence of equilibrium states and maximizing measures.

Proof of Theorem 1.1. Consider $M \in \mathbb{R}$ such that $\sup \phi < M$. Define $\psi = -(\phi - M)$. Note that $\psi \in \Psi$. Define $\tau : \Sigma \rightarrow \mathbb{R}$ by $\tau(x) = \sup_{y \in [x_1, x_2]} \lceil \psi(y) \rceil$, where $x = (x_1, x_2, \dots) \in \Sigma$. Since $\text{var}_2(\phi) = \text{var}_2(\psi)$ is finite, it follows that $\|\tau - \psi\|_0 < \infty$. In particular, $P(-\tau) < \infty$. Moreover, τ takes values in $\mathbb{N}$ and $\text{var}_2(\tau) = 0$.

Let $(\widehat{\Sigma}, \widehat{\sigma})$ be the CMS constructed at the beginning of this section for the pair (Σ, σ) and τ . Note that $(\widehat{\Sigma}, \widehat{\sigma})$ is transitive, because (Σ, σ) is transitive. Additionally, $h_{\text{top}}(\widehat{\sigma})$ is finite (see Remark 5.7). By Theorem 2.3, $\mathcal{M}_{\leq 1}(\widehat{\sigma})$ is compact with respect to the cylinder topology. As before, we denote by $\widehat{\nu}$ the measure in $\mathcal{M}(\widehat{\sigma})$ associated to $\nu \in \mathcal{M}(\sigma; \tau)$. Maybe after passing to a subsequence we can assume that $(\widehat{\mu}_n)_n$ converges on cylinders to $\lambda \widehat{\mu}$, where $\lambda \in [0, 1]$ and $\mu \in \mathcal{M}(\sigma; \tau)$. Maybe after passing to a further subsequence we can assume that $A := \lim_{n \rightarrow \infty} \int \tau d\mu_n$ is well defined and finite (the limsup is finite by assumption). Let C be a cylinder in Σ . By Remark 5.4, there exists a cylinder $D \subseteq \widehat{\Sigma}$ such that $\widehat{\nu}(D) = 1/(\int \tau d\nu)\nu(C)$ for every $\nu \in \mathcal{M}(\sigma; \tau)$. Then, $\lim_{n \rightarrow \infty} \mu_n(C) = \lambda A / (\int \tau d\mu) \mu(C)$. We conclude that $(\mu_n)_n$ converges on cylinders to $\mu_0 = \lambda A / (\int \tau d\mu) \mu$. Note that $\int \tau d\mu_0 < \infty$ and, therefore, $\int \phi d\mu_0 > -\infty$ (see Lemma 2.5). $\square$

The next result follows from Theorem 1.1 and will be used in the proof of Lemma 6.5.

COROLLARY 5.9. *Let $\tau \in \Psi$ and $m \in \mathbb{N}$. Let $(\mu_n)_n$ be a sequence in $\mathcal{M}(\sigma)$ and $\mu \in \mathcal{M}_{\leq 1}(\sigma)$. Assume that $\limsup_{n \rightarrow \infty} \int \tau d\mu_n < \infty$. Then, $(\mu_n)_n$ converges on cylinders to μ if and only if $\lim_{n \rightarrow \infty} \mu_n(C) = \mu(C)$ for every cylinder $C \subseteq \Sigma$ of length m .*

Proof. Suppose that $\lim_{n \rightarrow \infty} \mu_n(C) = \mu(C)$ for every cylinder $C \subseteq \Sigma$ of length m . It follows by Theorem 1.1 (applied to the potential $-\tau$) that every subsequence of $(\mu_n)_n$ has a subsubsequence that converges on cylinders to a countably additive measure $\nu \in \mathcal{M}_{\leq 1}(\sigma)$. Since $\nu(C) = \mu(C)$, for every cylinder of length m , it follows that $\nu = \mu$. We conclude that $(\mu_n)_n$ converges on cylinders to μ . The forward implication follows by definition. $\square$

6. Upper semi-continuity properties of the pressure map

In this section, we establish some upper semi-continuity results for the pressure map of uniformly continuous potentials. We will prove Theorem 1.3, which generalizes Theorem 2.7, and allows for the consideration of countable Markov shifts with infinite entropy and mild regularity assumptions on the potential.

6.1. A first upper semicontinuity result for the pressure.

PROPOSITION 6.1. *Let (Σ, σ) be a transitive CMS and $\tau : \Sigma \rightarrow \mathbb{N}$ be a potential such that $P(-\tau)$ is finite and $\text{var}_2(\tau) = 0$. Let $(\mu_n)_n$ be a sequence in $\mathcal{M}_{-\tau}(\sigma)$ that converges on cylinders to $\lambda \mu$, where $\lambda \in [0, 1]$ and $\mu \in \mathcal{M}_{-\tau}(\sigma)$. Assume that $\limsup_{n \rightarrow \infty} \int \tau d\mu_n < \infty$. Then,*

$$\limsup_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) - h(\tau) \int \tau \, d\mu_n \right) \leq \lambda \left(h_{\mu}(\sigma) - h(\tau) \int \tau \, d\mu \right). \quad (13)$$

Proof. It is sufficient to prove the proposition in the case where $(\int \tau \, d\mu_n)_n$ is convergent (otherwise, we can consider a subsequence for which the limsup in (13) is a limit and then pass to a further subsequence for which the integral converges). By Lemma 2.5, we have that $\int \tau \, d\mu < \infty$. Set λ_0 such that $\lim_{n \rightarrow \infty} \int \tau \, d\mu_n = \lambda_0 \int \tau \, d(\lambda\mu)$, which, by Lemma 2.5, also satisfies that $\lambda_0 \geq 1$.

Let $(\widehat{\Sigma}, \widehat{\sigma})$ be the CMS associated with (Σ, σ) and τ constructed in §5. By Remark 5.7, $h_{\text{top}}(\widehat{\sigma})$ is finite. It follows by Lemma 5.6 that $(\widehat{\mu}_n)_n$ converges on cylinders to $1/(\lambda_0)\widehat{\mu}$. We use Theorem 2.7 and Proposition 5.8 to conclude that

$$\limsup_{n \rightarrow \infty} \frac{h_{\mu_n}(\sigma)}{\int \tau \, d\mu_n} \leq \frac{1}{\lambda_0} \frac{h_{\mu}(\sigma)}{\int \tau \, d\mu} + \left(1 - \frac{1}{\lambda_0}\right) h(\tau).$$

Then,

$$\limsup_{n \rightarrow \infty} h_{\mu_n}(\sigma) \leq \lambda h_{\mu}(\sigma) + \left(\lim_{n \rightarrow \infty} \int \tau \, d\mu_n - \lambda \int \tau \, d\mu \right) h(\tau),$$

which is equivalent to inequality (13). $\square$

We will extend Proposition 6.1 to include potentials in Ψ . Let Ψ_n be the class of potentials in Ψ that are locally constant and depend only on the first n coordinates of Σ ; that is, their n -variation is zero. First, we will prove that Proposition 6.1 holds for potentials in Ψ_2 , removing the assumption that the potential takes values in $\mathbb{N}$. Next, we will prove that the proposition holds for potentials in Ψ_n and, finally, that it holds for potentials in Ψ . We begin with a lemma.

LEMMA 6.2. *Let $(\phi_n)_n, \phi$ be potentials in Ψ such that $\lim_{k \rightarrow \infty} \|\phi - \phi_k\|_0 = 0$. Then, $\lim_{n \rightarrow \infty} h(\phi_n) = h(\phi)$.*

Proof. If $\|\phi - \phi_k\| < \infty$, then $P(-t\phi) = \infty$ if and only if $P(-t\phi_k) = \infty$. It follows that $s_{\infty}(-\phi) = s_{\infty}(-\phi_k)$. By Lemma 3.4, we have also that $\lim_{k \rightarrow \infty} P_{\infty}(-t\phi_k) = P_{\infty}(-t\phi)$.

Let us first consider the case where $\phi \in \Psi_1$. Choose $\epsilon > 0$ small such that $P_{\infty}(-(s_{\phi} - \epsilon)\phi) > 0 > P_{\infty}(-(s_{\phi} + \epsilon)\phi)$. Then, $P_{\infty}(-(s_{\phi} - \epsilon)\phi_k) > 0 > P_{\infty}(-(s_{\phi} + \epsilon)\phi_k)$ for large enough k and, therefore, $|s_{\phi} - s_{\phi_k}| < \epsilon$. It follows that for large enough k , we have that $\phi_k \in \Psi_1$ and that $\lim_{k \rightarrow \infty} s_{\phi_k} = s_{\phi}$, which proves the claim in this case.

Suppose that $\phi \in \Psi_2$. For every $\epsilon > 0$, we have that $P_{\infty}(-(s_{\infty}(-\phi) + \epsilon)\phi) < 0$. Therefore, for large enough k , we have that $P_{\infty}(-(s_{\infty}(-\phi) + \epsilon)\phi_k) < 0$ and, thus, $s_{\phi_k} \leq s_{\infty}(-\phi) + \epsilon$. This implies that $\limsup_{n \rightarrow \infty} s_{\phi_k} \leq s_{\infty}(-\phi)$. Since $s_{\infty}(-\phi) = s_{\infty}(-\phi_k)$ and $h(\phi) = s_{\infty}(-\phi)$, the claim follows. $\square$

PROPOSITION 6.3. *Proposition 6.1 holds when $\tau \in \Psi_2$.*

Proof. Define $\tau_k(x) = 1/k \lceil k\tau(x) \rceil$ and observe that $\lim_{k \rightarrow 0} \|\tau - \tau_k\|_0 = 0$. Set $f_k = k\tau_k$ and note that $h(f_k) = 1/k h(\tau_k)$. The function f_k takes values in $\mathbb{N}$ and $\text{var}_2(f_k) = 0$. Proposition 6.1 holds for the functions f_k , which is equivalent to say that the proposition holds for τ_k . By Lemma 6.2, it follows that $\lim_{k \rightarrow \infty} h(\tau_k) = h(\tau)$. Additionally,

$|\int \tau_k dv - \int \tau dv| \leq \|\tau - \tau_k\|_0$ for every $v \in \mathcal{M}(\sigma)$. All of this together implies that Proposition 6.1 holds for $\tau \in \Psi_2$. $\square$

In §2.5, we constructed a CMS (Σ_m, σ_m) , which is a recoding of (Σ, σ) , and defined a canonical conjugacy $\pi_m : \Sigma_m \rightarrow \Sigma$. The map $(\pi_m)_* : \mathcal{M}_{\leq 1}(\sigma_m) \rightarrow \mathcal{M}_{\leq 1}(\sigma)$ is a bijection. For $\mu \in \mathcal{M}_{\leq 1}(\sigma)$, we defined $\tilde{\mu} = (\pi_m)_*^{-1} \mu \in \mathcal{M}_{\leq 1}(\sigma_m)$.

Remark 6.4. Let $(\mu_n)_n$ be a sequence in $\mathcal{M}(\sigma)$ such that $\limsup_{n \rightarrow \infty} \int \tau d\mu_n < \infty$, where $\tau \in \Psi$. Then, $(\mu_n)_n$ converges on cylinders to $\mu \in \mathcal{M}_{\leq 1}(\sigma)$ if and only if $(\tilde{\mu}_n)_n$ converges on cylinders to $\tilde{\mu}$ (see Remark 2.12 and Corollary 5.9).

In the next lemma, we have added superscripts Σ and Σ_m to the pressure to indicate the CMS we are considering. Let $\tau_m = \tau \circ \pi_m$.

LEMMA 6.5. $h(\tau) = h(\tau_m)$.

Proof. Observe that $h_\mu(\sigma) = h_{\tilde{\mu}}(\sigma_m)$ and $\int \tau d\mu = \int \tau_m d\tilde{\mu}$ for every $\mu \in \mathcal{M}(\sigma)$. It follows that $P^{\Sigma_m}(-t\tau_m) = P^\Sigma(-t\tau)$ and, therefore, $s_\infty(-\tau_m) = s_\infty(-\tau)$.

We will prove that if $t > s_\infty(-\tau)$, then $P_\infty^{\Sigma_m}(-t\tau_m) = P_\infty^\Sigma(-t\tau)$. It follows from this fact that $\tau \in \Psi_1$ if and only if $\tau_m \in \Psi_1$, and that $s_\tau = s_{\tau_m}$, which finishes the proof of the lemma, as we can conclude that $h(\tau) = h(\tau_m)$.

If there exists a sequence $(\mu_n)_n$ in $\mathcal{M}(\sigma)$ that converges on cylinders to the zero measure and such that $\lim_{n \rightarrow \infty} (h_{\mu_n}(\sigma) - t \int \tau d\mu_n) = P_\infty(-t\tau)$, then $(\tilde{\mu}_n)_n$ converges to the zero measure (see Remark 2.12) and $\lim_{n \rightarrow \infty} (h_{\tilde{\mu}_n}(\sigma_m) - t \int \tau_m d\tilde{\mu}_n) = P_\infty(-t\tau)$. Therefore, we have $P_\infty^{\Sigma_m}(-t\tau_m) \geq P_\infty^\Sigma(-t\tau)$. If no such sequence exists, then we still have the inequality $P_\infty^{\Sigma_m}(-t\tau_m) \geq P_\infty^\Sigma(-t\tau)$. Note that if $P_\infty^{\Sigma_m}(-t\tau_m) = -\infty$, then $P_\infty^\Sigma(-t\tau) = -\infty$. We will now suppose that $P_\infty^{\Sigma_m}(-t\tau_m) > -\infty$.

Let $(\tilde{\mu}_n)_n$ be a sequence in $\mathcal{M}_{-\tau_m}(\sigma_m)$ that converges on cylinders to the zero measure and such that $P_\infty^{\Sigma_m}(-t\tau_m) = \lim_{n \rightarrow \infty} (h_{\tilde{\mu}_n}(\sigma_m) - t \int \tau_m d\tilde{\mu}_n)$. Since $t > s_\infty(-\tau_m)$, we have that $\limsup_{n \rightarrow \infty} \int \tau_m d\tilde{\mu}_n < \infty$ (see Lemma 3.10); equivalently, $\limsup_{n \rightarrow \infty} \int \tau d\mu_n < \infty$. By Remark 6.4, $(\mu_n)_n$ converges to the zero measure and, therefore, $P_\infty^{\Sigma_m}(-t\tau_m) \leq P_\infty^\Sigma(-t\tau)$. We conclude that $P_\infty^{\Sigma_m}(-t\tau_m) = P_\infty^\Sigma(-t\tau)$ for every $t > s_\infty(-\tau)$. $\square$

PROPOSITION 6.6. Proposition 6.1 holds when $\tau \in \Psi_m$ for $m \geq 3$.

Proof. Let $(\mu_n)_n$ be a sequence in $\mathcal{M}_{-\tau}(\sigma)$ that converges on cylinders to $\lambda\mu$, where $\mu \in \mathcal{M}_{-\tau}(\sigma)$ and $\lambda \in [0, 1]$. By assumption, $\limsup_{n \rightarrow \infty} \int \tau d\mu_n < \infty$. Consider (Σ_m, σ_m) and note that $\tau_m \in \Psi_1$, as it only depends on the first coordinate of Σ_m . We can apply Proposition 6.3 to the pair (Σ_m, σ_m) and τ_m . Since $(\tilde{\mu}_n)_n$ converges on cylinders to $\lambda\tilde{\mu}$ (see Remark 2.12), we have that

$$\limsup_{n \rightarrow \infty} \left(h_{\tilde{\mu}_n}(\sigma_m) - h(\tau_m) \int \tau_m d\tilde{\mu}_n \right) \leq \lambda \left(h_{\tilde{\mu}}(\sigma_m) - h(\tau_m) \int \tau_m d\tilde{\mu} \right).$$

Finally, recall that $h_v(\sigma) = h_{\tilde{v}}(\sigma_m)$ and $\int \tau dv = \int \tau_m d\tilde{v}$ for every $v \in \mathcal{M}(\sigma)$, and that $h(\tau) = h(\tau_m)$ (see Lemma 6.5). We conclude that inequality (13) holds for τ . $\square$

THEOREM 6.7. *Let (Σ, σ) be a transitive CMS and $\tau \in \Psi$. Let $(\mu_n)_n$ be a sequence in $\mathcal{M}_{-\tau}(\sigma)$ that converges on cylinders to $\lambda\mu$, where $\mu \in \mathcal{M}_{-\tau}(\sigma)$ and $\lambda \in [0, 1]$. Suppose that $\limsup_{n \rightarrow \infty} \int \tau d\mu_n < \infty$. Then,*

$$\limsup_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) - h(\tau) \int \tau d\mu_n \right) \leq \lambda \left(h_{\mu}(\sigma) - h(\tau) \int \tau d\mu \right).$$

Proof. Since $\tau \in \Psi$, we have that $\text{var}_2(\tau) < \infty$ and that $\lim_{n \rightarrow \infty} \text{var}_n(\tau) = 0$. Fix $m \geq 2$ and define $\tau_m(x) = \sup\{\tau(y) : y \in [x_1, \dots, x_m]\}$, where $x = (x_1, x_2, \dots)$. Note that $\|\tau - \tau_m\|_0 \leq \text{var}_m(\tau)$ and that $\text{var}_2(\tau_m) < \infty$. Moreover, $\tau_m \in \Psi_m$ and by Proposition 6.6, we have that

$$\limsup_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) - h(\tau_m) \int \tau_m d\mu_n \right) \leq \lambda \left(h_{\mu}(\sigma) - h(\tau_m) \int \tau_m d\mu \right).$$

Since $|\int \tau_m d\nu - \int \tau d\nu| \leq \|\tau_m - \tau\|_0$ for every $\nu \in \mathcal{M}(\sigma)$, and $\|\tau_m - \tau\|_0 \rightarrow 0$, we obtain the desired inequality (see Lemma 6.2). $\square$

6.2. Proof of Theorem 1.3. This subsection is devoted to the proof of Theorem 1.3. First, we prove that the pressure map is upper semi-continuous on the full shift over a countable alphabet for a suitable class of potentials. For a general CMS, our strategy is to induce on a cylinder of length one and use the results obtained for the full shift.

LEMMA 6.8. *Let $(\mathbb{N}^{\mathbb{N}}, \sigma)$ be the full shift. Let $\phi \in C_{uc}(\mathbb{N}^{\mathbb{N}})$ be a potential such that $\text{var}_1(\phi)$ and $P(-\phi)$ are finite. Then,*

$$\lim_{n \rightarrow \infty} \inf_{x \in [n]} \phi(x) = \infty. \quad (14)$$

Moreover, if a sequence $(\nu_n)_n$ in $\mathcal{M}(\sigma)$ converges on cylinders to the zero measure, then $\lim_{n \rightarrow \infty} \int \phi d\nu_n = \infty$.

Proof. Let $a : \mathbb{N} \rightarrow \mathbb{R}$ be given by $a(n) = \inf_{x \in [n]} \phi(x)$. Define $\phi_1 : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{R}$, where $\phi_1(x) = a(k)$ if $x \in [k]$. Note that $\|\phi - \phi_1\|_0 \leq \text{var}_1(\phi)$. In particular, $P(-\phi_1) = \log(\sum_{n=1}^{\infty} e^{-a(n)})$ is finite (see Example 2.9). We conclude that $\lim_{n \rightarrow \infty} a(n) = \infty$.

Let us now suppose that $(\nu_n)_n$ converges on cylinders to the zero measure. We argue by contradiction and suppose that $\liminf_{n \rightarrow \infty} \int \phi d\nu_n < \infty$. In this case, the inequality

$$1 - \nu_n \left(\bigcup_{k < n} [k] \right) = \nu_n \left(\bigcup_{k \geq n} [k] \right) \leq \frac{\int \phi d\nu_n}{\inf\{\phi(x) : x \in \bigcup_{k \geq n} [k]\}} = \frac{\int \phi d\nu_n}{\inf_{k \geq n} a(k)}$$

implies that $(\nu_n)_n$ does not converge on cylinders to the zero measure. $\square$

Remark 6.9. If $\phi : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{R}$ is a potential such that both $P(-\phi)$ and $\text{var}_1(\phi)$ are finite, then it follows by Lemma 6.8 that ϕ is bounded below. Additionally, suppose $\phi \in \Psi$ and $t > s_{\infty}(-\phi)$. In this case, $P_{\infty}(-t\phi) = -\infty$ (see Lemmas 3.10 and 6.8). In particular, $\phi \in \Psi_2$ and, therefore, $h(\phi) = s_{\infty}(-\phi)$.

PROPOSITION 6.10. Let $(\mathbb{N}^{\mathbb{N}}, \sigma)$ be the full shift and $\phi \in C_{uc}(\mathbb{N}^{\mathbb{N}})$ be a potential such that $\text{var}_1(\phi)$ and $P(-\phi)$ are finite. Let $(\mu_n)_n$ be a sequence in $\mathcal{M}_{-\phi}(\sigma)$ that converges weak* to $\mu \in \mathcal{M}_{-\phi}(\sigma)$. Suppose that $\limsup_{n \rightarrow \infty} \int \phi d\mu_n < \infty$. Then,

$$\limsup_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) - \int \phi d\mu_n \right) \leq h_{\mu}(\sigma) - \int \phi d\mu.$$

Proof. By Remark 6.9, the potential ϕ is bounded below. After adding a positive constant, we can assume that ϕ is bounded away from zero and, therefore, $\phi \in \Psi$. By Remark 6.9, we also have that $\phi \in \Psi_2$ and, therefore, $h(\phi) = s_{\infty}(-\phi)$. It follows by Theorem 6.7 that

$$\limsup_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) - s_{\infty}(-\phi) \int \phi d\mu_n \right) \leq h_{\mu}(\sigma) - s_{\infty}(-\phi) \int \phi d\mu.$$

Lemma 2.5 implies that

$$\limsup_{n \rightarrow \infty} -(1 - s_{\infty}(-\phi)) \int \phi d\mu_n \leq -(1 - s_{\infty}(-\phi)) \int \phi d\mu.$$

The result follows from adding these inequalities together. $\square$

Remark 6.11. Under the assumptions in Proposition 6.10, assume further that $s_{\infty}(-\phi) < 1$. If $\lim_{n \rightarrow \infty} (h_{\mu_n}(\sigma) - \int \phi d\mu_n) = h_{\mu}(\sigma) - \int \phi d\mu$, then $\lim_{n \rightarrow \infty} \int \phi d\mu_n = \int \phi d\mu$ and $\lim_{n \rightarrow \infty} h_{\mu_n}(\sigma) = h_{\mu}(\sigma)$.

The next proposition is the key ingredient in the proof of Theorem 1.3.

PROPOSITION 6.12. Let (Σ, σ) be a mixing CMS, and ϕ a weakly Hölder potential such that $P(-\phi)$ and $\text{var}_2(\phi)$ are finite. Let $(\mu_n)_n$ be a sequence in $\mathcal{M}_{-\phi}(\sigma)$ that converges on cylinders to $\lambda\mu$, where $\lambda \in [0, 1]$ and $\mu \in \mathcal{M}_{-\phi}(\sigma)$. Suppose that $\limsup_{n \rightarrow \infty} \int \phi d\mu_n < \infty$. Then,

$$\limsup_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) - \int \phi d\mu_n \right) \leq \lambda \left(h_{\mu}(\sigma) - \int \phi d\mu \right) + (1 - \lambda)P(-\phi).$$

Proof. The case $\lambda = 0$ follows from the variational principle. Suppose that $\lambda > 0$. We will further suppose that $P(-\phi) = 0$ (otherwise, we can subtract $P(-\phi)$ from both sides and consider the potential $\phi' = \phi + P(-\phi)$).

Fix $a \in S$ such that $\mu([a]) > 0$. We will induce on the 1-cylinder $[a]$. As described in §4, the induced system $(\Sigma(a), \sigma_a)$ is topologically conjugate to the full shift and cylinders in $\Sigma(a)$ are certain cylinders in Σ . Let $\tau : \Sigma(a) \rightarrow \mathbb{N}$ be the first return time map and $\bar{\phi} : \Sigma(a) \rightarrow \mathbb{R}$ the induced potential, which is defined by $\bar{\phi}(x) = \sum_{i=0}^{\tau(x)-1} \phi(\sigma^i x)$. Let $\mathcal{M}^{[a]}(\sigma) = \{\mu \in \mathcal{M}(\sigma) : \mu([a]) > 0\}$. For $\nu \in \mathcal{M}^{[a]}(\sigma)$, define $\bar{\nu}(\cdot) = 1/(\nu([a]))\nu(\cdot \cap [a])$, which is a measure supported on $\Sigma(a)$. By Kac's formula, we have that $\int \tau d\bar{\nu} = 1/(\nu([a]))$ and $\int \phi d\nu = 1/(\int \tau d\bar{\nu}) \int \bar{\phi} d\bar{\nu}$ for every $\nu \in \mathcal{M}^{[a]}(\sigma)$. Moreover, the map $\nu \mapsto \bar{\nu}$ is a bijection between $\mathcal{M}^{[a]}(\sigma)$ and $\mathcal{M}_{\tau}(\sigma_a)$.

Since $\lim_{n \rightarrow \infty} \mu_n([a]) = \lambda\mu([a])$, we can assume that $\mu_n([a]) > 0$ for all $n \in \mathbb{N}$. Then, $\mu, (\mu_n)_n$ are in $\mathcal{M}^{[a]}(\sigma)$. Since $(\mu_n)_n$ converges on cylinders to $\lambda\mu$, we have that

$$\lim_{n \rightarrow \infty} \int \tau \, d\bar{\mu}_n = \lim_{n \rightarrow \infty} \frac{1}{\mu_n([a])} = \frac{1}{\lambda\mu([a])} = \frac{1}{\lambda} \int \tau \, d\bar{\mu}. \quad (15)$$

Moreover, since cylinders in $\Sigma(a)$ are cylinders in Σ , it follows that

$$\lim_{n \rightarrow \infty} \bar{\mu}_n(C) = \lim_{n \rightarrow \infty} \frac{1}{\mu_n([a])} \mu_n(C) = \frac{1}{\mu([a])} \mu(C) = \bar{\mu}(C)$$

for every cylinder $C \subseteq \Sigma(a)$. In particular, $(\bar{\mu}_n)_n$ converges weak* to $\bar{\mu}$ (see Remark 2.2). Set $M := \sup_n \int \phi \, d\mu_n$. It follows from Kac's formula that $\int \bar{\phi} \, d\bar{\mu}_n \leq M \int \tau \, d\bar{\mu}_n$ and, therefore, $\limsup_{n \rightarrow \infty} \int \bar{\phi} \, d\bar{\mu}_n < \infty$. Note that $\text{var}_1(\bar{\phi}) \leq \text{var}_2(\phi) < \infty$.

We will consider two cases.

Case 1: ($-\phi$ is recurrent). In this case, we have that $P(-\bar{\phi}) = 0$ (see Remark 4.1). It follows by Proposition 6.10 that

$$\limsup_{n \rightarrow \infty} \left(h_{\bar{\mu}_n}(\sigma_a) - \int \bar{\phi} \, d\bar{\mu}_n \right) \leq h_{\bar{\mu}}(\sigma_a) - \int \bar{\phi} \, d\bar{\mu}.$$

Combining this with (15), we obtain that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) - \int \phi \, d\mu_n \right) &= \limsup_{n \rightarrow \infty} \left(\frac{h_{\bar{\mu}_n}(\sigma_a) - \int \bar{\phi} \, d\bar{\mu}_n}{\int \tau \, d\bar{\mu}_n} \right) \leq \lambda \left(\frac{h_{\bar{\mu}}(\sigma_a) - \int \bar{\phi} \, d\bar{\mu}}{\int \tau \, d\bar{\mu}} \right) \\ &= \lambda \left(h_{\mu}(\sigma) - \int \phi \, d\mu \right). \end{aligned}$$

Case 2: ($-\phi$ is transient). Set $t_0 := \Delta_a(-\phi) < 0$, where $\Delta_a(-\phi)$ is the a -discriminant of $-\phi$ (see (5)). By the discriminant theorem (see Theorem 4.2), $-\phi + t1_{[a]}$ is transient for every $t \in [0, -t_0)$. It follows that $P(-\phi + t1_{[a]}) = 0$ for every $t \in [0, -t_0]$ (see [CS, Lemma 4.1]). Since $\Delta_a(-\phi - t_01_{[a]}) = 0$, the discriminant theorem implies that $-\phi_0 := -\phi - t_01_{[a]}$ is recurrent. Note that $P(-\phi_0) = 0$ and that $-\phi_0$ satisfies the assumption in Case 1, so we obtain that

$$\limsup_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) - \int \phi \, d\mu_n - t_0\mu_n([a]) \right) \leq \lambda \left(h_{\mu}(\sigma) - \int \phi \, d\mu - t_0\mu([a]) \right).$$

Since $\lim_{n \rightarrow \infty} \mu_n([a]) = \lambda\mu([a])$, we conclude that

$$\limsup_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) - \int \phi \, d\mu_n \right) \leq \lambda \left(h_{\mu}(\sigma) - \int \phi \, d\mu \right). \quad (16)$$

In particular, inequality (16) holds in both cases. $\square$

PROPOSITION 6.13. *Let (Σ, σ) be a transitive CMS and $\phi \in C_{uc}(\Sigma)$ a potential such that $P(-\phi)$ and $\text{var}_2(\phi)$ are finite. Let $(\mu_n)_n$ be a sequence in $\mathcal{M}_{-\phi}(\sigma)$ that converges on cylinders to $\lambda\mu$, where $\lambda \in [0, 1]$ and $\mu \in \mathcal{M}_{-\phi}(\sigma)$. Suppose that $\limsup_{n \rightarrow \infty} \int \phi \, d\mu_n < \infty$. Then,*

$$\limsup_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) - \int \phi \, d\mu_n \right) \leq \lambda \left(h_{\mu}(\sigma) - \int \phi \, d\mu \right) + (1 - \lambda)P_{\infty}(-\phi).$$

Proof. We will assume that (Σ, σ) is mixing and explain how to derive the transitive case from the mixing case in the remark below. Fix $m \geq 2$ and define $\phi_m(x) = \sup\{\phi(y) : y \in [x_1, \dots, x_m]\}$, where $x = (x_1, x_2, \dots)$. Note that $\|\phi - \phi_m\|_0 \leq \text{var}_m(\phi)$ and that $\text{var}_2(\phi_m) \leq \text{var}_2(\phi)$. Moreover, the potential ϕ_m is locally constant (in particular, weakly Hölder) and $\limsup_{n \rightarrow \infty} \int \phi_m d\mu_n < \infty$. Let $V = \sum_{k \in \mathbb{N}} (1/k) 1_{[k]}$. Note that $\phi_m + tV$ satisfies the hypothesis in Proposition 6.12 and, therefore,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) - \int (\phi_m + tV) d\mu_n \right) \\ \leq \lambda \left(h_\mu(\sigma) - \int (\phi_m + tV) d\mu \right) + (1 - \lambda)P(-\phi_m - tV). \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \int V d\mu_n = \lambda \int V d\mu$, we obtain that

$$\limsup_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) - \int \phi_m d\mu_n \right) \leq \lambda \left(h_\mu(\sigma) - \int \phi_m d\mu \right) + (1 - \lambda)P(-\phi_m - tV).$$

It follows from Proposition 3.2 that

$$\limsup_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) - \int \phi_m d\mu_n \right) \leq \lambda \left(h_\mu(\sigma) - \int \phi_m d\mu \right) + (1 - \lambda)P_\infty(-\phi_m).$$

Finally, use that $\|\phi - \phi_m\|_0 \rightarrow 0$ to conclude the desired inequality. $\square$

Remark 6.14. Let (Σ, σ) be a transitive CMS. In Remark 2.10, we described the spectral decomposition of a transitive CMS; in this paragraph, we will follow the notation introduced in Remark 2.10. Recall that $\Sigma = \bigcup_{i=1}^p \Sigma_i$, where p is the period of Σ , $\sigma(\Sigma_i) = \Sigma_{i+1}$, $\Sigma_{p+1} = \Sigma_1$ and (Σ_i, σ^p) is mixing. We induce on $\Sigma_1 \subseteq \Sigma$, where the first return time map is constant with values equal to p . It worth noticing that (Σ, σ) can be identified with the CMS constructed from the pair (Σ_1, σ^p) and $\tau : \Sigma_1 \rightarrow \mathbb{N}$, where τ is constant equal to p . Note that $h_{\bar{\mu}}(\sigma^p) + \int \bar{\phi} d\bar{\mu} = p(h_\mu(\sigma) + \int \phi d\mu)$, where $\bar{\mu}$ is the measure in $\mathcal{M}(\Sigma_1)$ corresponding to $\mu \in \mathcal{M}(\sigma)$, and that a sequence $(\mu_n)_n$ in $\mathcal{M}(\sigma)$ converges on cylinders to $\lambda\mu \in \mathcal{M}_{\leq 1}(\sigma)$ if and only if $(\bar{\mu}_n)_n$ converges on cylinders to $\lambda\bar{\mu} \in \mathcal{M}_{\leq 1}(\sigma^p)$. It follows that $P_\infty(\bar{\phi}) = pP_\infty(\phi)$. Since (Σ_1, σ^p) is mixing, we can apply Proposition 6.13 to the sequence $(\bar{\mu}_n)_n$ and deduce the desired conclusion for the sequence $(\mu_n)_n$.

LEMMA 6.15. *Let (Σ, σ) be a transitive CMS and $\phi \in C_{uc}(\Sigma)$ a potential such that $P(-\phi)$ and $\text{var}_2(\phi)$ are finite. Let $(\mu_n)_n$ be a sequence in $\mathcal{M}_{-\phi}(\sigma)$ that converges on cylinders to $\lambda\mu$, where $\lambda \in [0, 1]$ and $\mu \in \mathcal{M}_{-\phi}(\sigma)$. Suppose that $s_\infty(-\phi) < 1$. Then,*

$$\limsup_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) - \int \phi d\mu_n \right) \leq \lambda \left(h_\mu(\sigma) - \int \phi d\mu \right) + (1 - \lambda)P_\infty(-\phi).$$

Proof. Maybe after passing to a subsequence that maximizes the left-hand side can we assume that $(\int \phi d\mu_n)_n$ converges to a real number or to ∞ . If $\lim_{n \rightarrow \infty} \int \phi d\mu_n < \infty$, the result follows from Proposition 6.13. If $\lim_{n \rightarrow \infty} \int \phi d\mu_n = \infty$, then $\lim_{n \rightarrow \infty} h_{\mu_n}(\sigma) - \int \phi d\mu_n = -\infty$ (see Lemma 3.9), in which case, the inequality trivially holds. $\square$

Proof of Theorem 1.3. The proof of the inequality follows by Proposition 6.13 and Lemma 6.15 for the potential $-\phi$. To see that the inequality is sharp, consider a sequence $(v_n)_n$ in $\mathcal{M}_\phi(\sigma)$ that converges to the zero measure and such that $\lim_{n \rightarrow \infty} (h_{\mu_n}(\sigma) + \int \phi d\mu_n) = P_\infty(\phi)$. Set $\eta_n = \lambda\mu + (1 - \lambda)v_n$. Note that $(\eta_n)_n$ converges to $\lambda\mu$ and that it realizes the equality. $\square$

COROLLARY 6.16. *Let (Σ, σ) be a transitive CMS with finite entropy and $\phi \in C_{b,uc}(\Sigma)$. Let $(\mu_n)_n$ be a sequence in $\mathcal{M}(\sigma)$ that converges on cylinders to $\lambda\mu$, where $\lambda \in [0, 1]$ and $\mu \in \mathcal{M}(\sigma)$. Then, $\limsup_{n \rightarrow \infty} (h_{\mu_n}(\sigma) + \int \phi d\mu_n) \leq \lambda(h_\mu(\sigma) + \int \phi d\mu) + (1 - \lambda)P_\infty(\phi)$.*

7. Existence of equilibrium states and maximizing measures

In this section, we derive applications of Theorems 1.1 and 1.3 to the problem of existence and non-existence of equilibrium states and maximizing measures. Recall that we have defined $\mathcal{H} = \{\phi \in C_{uc}(\Sigma) : \sup(\phi), \text{var}_2(\phi), P(\phi) < \infty\}$.

7.1. Existence and non-existence of equilibrium states. We will now prove Theorem 1.4, which provides a criterion for the existence of equilibrium states and describes what occurs when no equilibrium state exists.

Proof of Theorem 1.4. (1) Observe that if $s_\infty(\phi) < 1$ and $\liminf_{n \rightarrow \infty} \int \phi d\mu_n = -\infty$, then $\liminf_{n \rightarrow \infty} (h_{\mu_n}(\sigma) + \int \phi d\mu_n) = -\infty$ (see Lemma 3.9), which is not possible. We can therefore assume that $\liminf_{n \rightarrow \infty} \int \phi d\mu_n > -\infty$. It follows by Theorem 1.1 that $(\mu_n)_n$ has a subsequence that converges on cylinders to $\lambda\mu \in \mathcal{M}_{\leq 1}(\sigma)$, where $\lambda \in [0, 1]$ and $\mu \in \mathcal{M}_\phi(\sigma)$. By Theorem 1.3,

$$\limsup_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) + \int \phi d\mu_n \right) \leq \lambda \left(h_\mu(\sigma) + \int \phi d\mu \right) + (1 - \lambda)P_\infty(\phi).$$

Therefore, $P(\phi) = \lambda(h_\mu(\sigma) + \int \phi d\mu) + (1 - \lambda)P_\infty(\phi)$. Finally, note that $P_\infty(\phi) < P(\phi)$ implies that $\lambda = 1$ and, therefore, μ is an equilibrium state.

(2) If there is no subsequence for which the integral goes to $-\infty$, then we have that $\liminf_{n \rightarrow \infty} \int \phi d\mu_n > -\infty$. In this case, Theorem 1.1 implies that $(\mu_n)_n$ has a subsequence that converges on cylinders to a measure $\lambda\mu \in \mathcal{M}_{\leq 1}(\sigma)$, where $\lambda \in [0, 1]$ and $\mu \in \mathcal{M}_\phi(\sigma)$. By Theorem 1.3, we have that

$$\limsup_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) + \int \phi d\mu_n \right) \leq \lambda \left(h_\mu(\sigma) + \int \phi d\mu \right) + (1 - \lambda)P_\infty(\phi).$$

Therefore, $P(\phi) = \lambda(h_\mu(\sigma) + \int \phi d\mu) + (1 - \lambda)P_\infty(\phi)$. Since $h_\mu(\sigma) + \int \phi d\mu < P(\phi)$, we conclude that $\lambda = 0$. $\square$

Remark 7.1. Let $\phi \in \mathcal{H}$ and define $f(t) = P(t\phi)$. If $t > s_\infty(\phi)$ and $t\phi$ is SPR, then $t\phi$ has an equilibrium state (note that $s_\infty(t\phi) < 1$). If ϕ is SPR and $s_\infty(\phi) = 1$, then it admits an equilibrium state if and only if the slopes of the supporting lines for the graph of f at $(t, f(t))$ remain bounded as $t \rightarrow 1^+$.

Remark 7.2. Let $\phi \in \mathcal{H}$ be a potential that does not have an equilibrium state. Let $(\mu_n)_n$ be a sequence in $\mathcal{M}_\phi(\sigma)$ such that $P(\phi) = \lim_{n \rightarrow \infty} (h_{\mu_n}(\sigma) + \int \phi d\mu_n)$. Then, the following hold.

- (1) If (Σ, σ) has finite topological entropy, then $(\mu_n)_n$ converges to the zero measure.
- (2) If (Σ, σ) is the full shift on a countable alphabet and $\text{var}_1(\phi) < \infty$, then $\lim_{n \rightarrow \infty} \int \phi d\mu_n = -\infty$ (see Lemma 6.8).
- (3) If $P_\infty(\phi) = -\infty$, then $(\mu_n)_n$ does not have any subsequence converging to the zero measure (otherwise, $P_\infty(\phi) = P(\phi)$). Therefore, $\lim_{n \rightarrow \infty} \int \phi d\mu_n = -\infty$.

Example 7.3. Let (Σ, σ) be a null recurrent finite entropy CMS. As described in §4, the induced transformation over a 1-cylinder $[a]$ is conjugate to the full shift $(\mathbb{N}^\mathbb{N}, \sigma_f)$. Let $\tau : \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}$ be the first return time map and set $f(t) = P(-t\tau)$. Note that τ depends only on the first coordinate, that is, $\text{var}_1(\tau) = 0$. Since (Σ, σ) is null recurrent, we have that $f(t) = \infty$ if $t < h_{\text{top}}(\sigma)$ and $f(h_{\text{top}}(\sigma)) = 0$. The potential $\phi := -h_{\text{top}}(\sigma)\tau$ is positive recurrent (see [Sa4, Corollary 2]; see also Proposition A.2). Let μ_ϕ be the RPF measure of ϕ . Since (Σ, σ) has no measure of maximal entropy, we have that $\int \tau d\mu_\phi = \infty$ or $\int \phi d\mu_\phi = -\infty$; equivalently, ϕ does not have an equilibrium state in the classical sense. For $t > 1$, the potential $t\phi$ has a unique equilibrium state, which we denote by μ_t . Then,

$$0 = P(\phi) = \lim_{t \rightarrow 1^+} \left(h_{\mu_t}(\sigma) + \int \phi d\mu_t \right) \quad \text{and} \quad \lim_{t \rightarrow 1^+} \int \phi d\mu_t = -\infty.$$

In this example, we observe that the finiteness condition on the integral of the sequence of measures cannot be removed from the assumptions in Theorem 1.4(1) to guarantee the existence of an equilibrium state.

7.2. Ergodic optimization. For $\phi \in C(\Sigma)$, we define

$$\beta(\phi) = \sup_{\mu \in \mathcal{M}(\sigma)} \int \phi d\mu.$$

We say that $\mu \in \mathcal{M}(\sigma)$ is a maximizing measure of ϕ if $\beta(\phi) = \int \phi d\mu$. As mentioned in §1, we consider a quantity analogous to the pressure at infinity. Define

$$\beta_\infty(\phi) = \sup_{(\mu_n)_n \rightarrow 0} \limsup_{n \rightarrow \infty} \int \phi d\mu_n,$$

where the supremum runs over sequences in $\mathcal{M}(\sigma)$ that converge on cylinders to the zero measure. If there is no such sequence, we define $\beta_\infty(\phi) = -\infty$. Note that $P_\infty(\phi) \geq \beta_\infty(\phi)$.

Continuous potentials on subshifts of finite type admit maximizing measures. This is not generally the case for countable Markov shifts and we are interested in establishing a criterion for the existence of a maximizing measure in this setting. It is fairly easy to construct potentials without maximizing measures, as we can see in the following example.

Example 7.4. Let (Σ, σ) be a CMS for which there exists a sequence $(\mu_n)_n$ in $\mathcal{M}(\sigma)$ that converges on cylinders to the zero measure (for instance, if $h_{\text{top}}(\sigma)$ is finite; see also Theorem 4.5). Let $\phi = 1 - \sum_{k=1}^{\infty} (1/k) 1_{[k]}$. Note that $\lim_{n \rightarrow \infty} \int \phi d\mu_n = 1$ and,

therefore, $\beta(\phi) \geq 1$. Since $\phi < 1$, we conclude that $\beta(\phi) = 1$ and that ϕ does not have any maximizing measure. Furthermore, note that $\beta_\infty(\phi) = 1$.

We will prove a result that describes the limiting behaviour of the integral of a potential when escape of mass is allowed, which is analogous to Theorem 1.3 and improves Lemma 2.5. For this, we will use the fact that β_∞ is the asymptotic slope of the map $t \mapsto P_\infty(t\phi)$.

LEMMA 7.5. *Let $\phi \in \mathcal{H}$. Then,*

$$\beta_\infty(\phi) = \lim_{t \rightarrow \infty} \frac{1}{t} P_\infty(t\phi).$$

Proof. First, note that $P_\infty(t\phi) \geq \beta_\infty(t\phi) = t\beta_\infty(\phi)$ and, therefore, $\liminf_{t \rightarrow \infty} (1/t) P_\infty(t\phi) \geq \beta_\infty(\phi)$. Set $A = \limsup_{t \rightarrow \infty} (1/t) P_\infty(t\phi)$. Observe that $P_\infty(t\phi) \leq tP_\infty(\phi)$ for all $t \geq 1$, and therefore, $A \leq P_\infty(\phi) < \infty$. We claim that $A \leq \beta_\infty(\phi)$. We will assume that $P_\infty(t\phi) > -\infty$ for all $t > 0$, otherwise $A = \beta_\infty(\phi) = -\infty$.

Let $(t_n)_n$ be a sequence going to infinity such that $\lim_{n \rightarrow \infty} (1/t_n) P_\infty(t_n\phi) = A$. Denote by ν_0 the zero measure. Let $(\nu_n)_n$ be a sequence in $\mathcal{M}(\sigma)$ such that $\rho(\nu_n, \nu_0) \leq 1/n$ (see (1)) and $(h_{\nu_n}(\sigma) + t_n \int \phi d\nu_n) \in [P_\infty(t_n\phi) - (1/n), P_\infty(t_n\phi) + (1/n)]$. We conclude that

$$\lim_{n \rightarrow \infty} \left(\frac{1}{t_n} h_{\nu_n}(\sigma) + \int \phi d\nu_n \right) = A$$

and that $(\nu_n)_n$ converges to the zero measure. Note that

$$\frac{t_n - 1}{t_n} h_{\nu_n}(\sigma) + \left(\frac{1}{t_n} h_{\nu_n}(\sigma) + \int \phi d\nu_n \right) \leq P(\phi)$$

and, therefore, $\limsup_{n \rightarrow \infty} h_{\nu_n}(\sigma)$ is finite. We obtain that $\lim_{n \rightarrow \infty} (1/t_n) h_{\nu_n}(\sigma) = 0$. Finally, note that $A = \lim_{n \rightarrow \infty} \int \phi d\nu_n \leq \beta_\infty(\phi)$, which concludes the proof of the lemma. $\square$

Remark 7.6. It is known that if $\phi \in \mathcal{H}$, then $\beta(\phi) = \lim_{t \rightarrow \infty} (1/t) P(t\phi)$ (the proof is analogous to the proof of Lemma 7.5).

THEOREM 7.7. *Let (Σ, σ) be a transitive CMS and $\phi \in \mathcal{H}$. Let $(\mu_n)_n$ be a sequence in $\mathcal{M}(\sigma)$ that converges on cylinders to $\lambda\mu$, where $\lambda \in [0, 1]$ and $\mu \in \mathcal{M}(\sigma)$. Then,*

$$+ \limsup_{n \rightarrow \infty} \int \phi d\mu_n \leq \lambda \int \phi d\mu + (1 - \lambda)\beta_\infty(\phi).$$

Moreover, the inequality is sharp.

Proof. Maybe after passing to a subsequence can we assume that $\lim_{n \rightarrow \infty} \int \phi d\mu_n$ is well defined. By Lemma 2.5, we have that $\int \phi d\mu > -\infty$. By Theorem 1.3, applied to the potential $t\phi$, for $t \geq 1$, we obtain

$$\limsup_{n \rightarrow \infty} \left(\frac{1}{t} h_{\mu_n}(\sigma) + \int \phi d\mu_n \right) \leq \lambda \left(\frac{1}{t} h_\mu(\sigma) + \int \phi d\mu \right) + (1 - \lambda) \frac{1}{t} P_\infty(t\phi).$$

Sending t to infinity, we obtain the desired inequality (see Lemma 7.5). To see that the inequality is sharp, consider a sequence $(\nu_n)_n$ converging to the zero measure and such that $\lim_{n \rightarrow \infty} \int \phi d\nu_n = \beta(\phi)$, and define $\eta_n = \lambda\mu + (1 - \lambda)\nu_n$. The sequence $(\eta_n)_n$ converges to $\lambda\mu$ and achieves equality in the inequality above. $\square$

The next result is analogous to Theorem 1.4 and the proof is essentially the same, considering Theorem 7.7 instead of Theorem 1.3. We leave the details to the reader.

THEOREM 7.8. *Let (Σ, σ) be a transitive CMS and $\phi \in \mathcal{H}$. Let $(\mu_n)_n$ be a sequence in $\mathcal{M}(\sigma)$ such that $\lim_{n \rightarrow \infty} \int \phi d\mu_n = \beta(\phi)$. Then, the following hold.*

- (1) *If $\beta_\infty(\phi) < \beta(\phi)$, then $(\mu_n)_n$ has a subsequence that converges weak* to a maximizing measure of ϕ . In particular, ϕ admits a maximizing measure.*
- (2) *If ϕ does not have a maximizing measure, then $(\mu_n)_n$ converges on cylinders to the zero measure.*

Note that Theorem 1.5 is a direct consequence of Theorem 7.8. We finish this subsection with a result about zero temperature limits for a suitable class of potentials.

PROPOSITION 7.9. *Let (Σ, σ) be a transitive CMS and $\phi \in \mathcal{H}$ a potential with summable variations such that $\beta_\infty(\phi) < \beta(\phi)$. Then, there exists $t_0 \in [0, \infty)$ such that $t\phi$ has a unique equilibrium state μ_t for every $t \in (t_0, \infty)$. Moreover, if $(t_n)_n$ goes to ∞ , then $(\mu_{t_n})_n$ has a subsequence that converges weak* to a maximizing measure of ϕ .*

Proof. Since $\beta_\infty(\phi) = \lim_{t \rightarrow \infty} (1/t)P_\infty(t\phi)$ and $\beta_\infty(\phi) < \beta(\phi)$, it follows that $P_\infty(t\phi) < t\beta(\phi) \leq P(t\phi)$ for large enough t . Therefore, there exists $t_0 \geq 1$ such that if $t > t_0$, then $t\phi$ is SPR. Since $s_\infty(t\phi) = (1/t)s_\infty(\phi) < 1$ for all $t > t_0$, we conclude that $t\phi$ has an equilibrium state (see Theorem 1.4). Since $t\phi$ has summable variations, the equilibrium state is unique. Note that $P'(t\phi) = \int \phi d\mu_t$ for every $t > t_0$ (see [Sa3]). By convexity of the map $t \mapsto P(t\phi)$, $\lim_{t \rightarrow \infty} P'(t\phi) = \lim_{t \rightarrow \infty} (1/t)P(t\phi) = \beta(\phi)$ (see Remark 7.6). Then, $\lim_{t \rightarrow \infty} \int \phi d\mu_t = \beta(\phi)$ and the result follows from Theorem 7.8. $\square$

8. Pressure at infinity of suspension flows

Let (Σ, σ) be a transitive CMS and τ a continuous potential bounded away from zero, that is, there exists $c = c(\tau) > 0$ such that $\tau(x) \geq c$ for every $x \in \Sigma$. Define

$$Y = \{(x, t) \in \Sigma \times \mathbb{R} : 0 \leq t \leq \tau(x)\} / \sim,$$

where we identify the points $(x, \tau(x))$ and $(\sigma(x), 0)$ for all $x \in \Sigma$. The suspension flow over σ with roof function τ is the semi-flow $\Theta = (\theta_t)_{t \geq 0}$ on Y defined by

$$\theta_t(x, s) = (x, s + t) \quad \text{whenever } s + t \in [0, \tau(x)].$$

Denote by $\mathcal{M}(\Theta)$ the space of Θ -invariant probability measures on Y , and $\mathcal{M}_{\leq 1}(\Theta)$ the space Θ -invariant sub-probability measures on Y . Let $\mathcal{M}(\sigma; \tau) = \{\mu \in \mathcal{M}(\sigma) : \int \tau d\mu < \infty\}$. It follows from a classical result of Ambrose and Kakutani [AK] that the map $\varphi : \mathcal{M}(\sigma; \tau) \rightarrow \mathcal{M}(\Theta)$, defined by $\varphi(\mu) = (\mu \times \text{Leb}) / (\int \tau d\mu)$, where Leb is the one-dimensional Lebesgue measure, is a bijection. In particular, a measure $\nu \in \mathcal{M}_{\leq 1}(\Theta)$

different from the zero measure can be uniquely written as $\nu = \lambda(\mu \times \text{Leb})/(\int \tau \, d\mu)$ for some $\mu \in \mathcal{M}(\sigma; \tau)$ and $\lambda \in (0, 1]$.

Recall that $\Psi = \{\psi \in C_{uc}(\Sigma) : \inf \psi > 0, \text{var}_2(\psi) < \infty, P(-\psi) < \infty\}$. For our applications, we will be mainly interested in the case where $\tau \in \Psi$.

8.1. Topology of convergence on cylinders. In [IV, §6], the authors introduced a topology on the space $\mathcal{M}_{\leq 1}(\Theta)$ that is analogous to the cylinder topology on $\mathcal{M}_{\leq 1}(\sigma)$.

Definition 8.1. Let $(\nu_n)_n$ and ν be measures in $\mathcal{M}_{\leq 1}(\Theta)$. We say that $(\nu_n)_n$ converges on cylinders to ν if

$$\lim_{n \rightarrow \infty} \nu_n(C \times [0, c]) = \nu(C \times [0, c])$$

for every cylinder $C \subseteq \Sigma$, and $c = c(\tau)$.

This notion of convergence induces the cylinder topology on $\mathcal{M}_{\leq 1}(\Theta)$. Let $(C_i)_{i \in \mathbb{N}}$ be an enumeration of the cylinders of Σ . The function $\rho_\tau : \mathcal{M}_{\leq 1}(\Theta) \times \mathcal{M}_{\leq 1}(\Theta) \rightarrow \mathbb{R}_{\geq 0}$, given by

$$\rho_\tau(\nu_1, \nu_2) = \sum_{i \in \mathbb{N}} \frac{1}{2^i} |\nu_1(C_i \times [0, c]) - \nu_2(C_i \times [0, c])|,$$

is a metric on $\mathcal{M}_{\leq 1}(\Theta)$, compatible with the cylinder topology (see [IV, Lemma 6.6]). A sequence $(\nu_n)_n$ in $\mathcal{M}(\Theta)$ converges on cylinders to $\nu \in \mathcal{M}(\Theta)$ if and only if $(\nu_n)_n$ converges in the weak* topology to ν (see [IV, Lemma 6.7]). In other words, the cylinder topology on $\mathcal{M}_{\leq 1}(\Theta)$ induces the weak* topology on $\mathcal{M}(\Theta)$.

By Kac's formula, we have that $\nu(C \times [0, c]) = (c\mu(C))/(\int \tau \, d\mu)$, whenever $\nu \in \mathcal{M}(\Theta)$ and $\nu = \varphi(\mu)$. In particular, a sequence $(\nu_n)_n$ in $\mathcal{M}(\Theta)$ converges on cylinders to $\lambda\nu$, where $\lambda \in [0, 1]$ and $\nu \in \mathcal{M}(\Theta)$, if and only if

$$\lim_{n \rightarrow \infty} \frac{\mu_n(C)}{\int \tau \, d\mu_n} = \lambda \frac{\mu(C)}{\int \tau \, d\mu} \quad (17)$$

for every cylinder $C \subseteq \Sigma$, where $\varphi(\mu_n) = \nu_n$ and $\varphi(\mu) = \nu$ (see [IV, Remark 6.5]). A more detailed analysis on the relation given by (17) gives us the following useful fact.

LEMMA 8.2. *Let $(\mu_n)_n, \mu$ be measures in $\mathcal{M}(\sigma; \tau)$ and set $\nu_n = \varphi^{-1}(\mu_n)$, $\nu = \varphi^{-1}(\mu)$. Then, the following statements are equivalent.*

- (1) $(\nu_n)_n$ converges on cylinders to the zero measure.
- (2) Every subsequence of $(\mu_n)_n$ has a subsubsequence $(\mu_{n_k})_k$ that converges on cylinders to the zero measure or such that $\lim_{k \rightarrow \infty} \int \tau \, d\mu_{n_k} = \infty$.

Similarly, the following statements are equivalent.

- (1) $(\nu_n)_n$ converges on cylinders to $\lambda\nu$, where $\lambda \in (0, 1]$.
- (2) Every subsequence of $(\mu_n)_n$ has a subsubsequence $(\mu_{n_k})_k$ that converges on cylinders to $\lambda_1\mu$ and $\lim_{k \rightarrow \infty} \int \tau \, d\mu_{n_k} = \lambda_2 \int \tau \, d\mu$ for some $\lambda_1 \in (0, 1]$ and $\lambda_2 \in [1, \infty)$ satisfying $\lambda = \lambda_1/\lambda_2$.

Remark 8.3. It follows by Lemma 8.2 that $(v_n)_n$ converges weak* to ν if and only if $(\mu_n)_n$ converges weak* to μ and $\lim_{n \rightarrow \infty} \int \tau d\mu_n = \int \tau d\mu$, where $\nu_n = \varphi^{-1}(\mu_n)$, $\nu = \varphi^{-1}(\mu)$.

We now provide a proof of Theorem 1.6, which is analogous to Theorem 2.3 in the finite entropy case.

Proof of Theorem 1.6. Let $(v_n)_n$ be a sequence in $\mathcal{M}(\Theta)$ and set $\mu_n = \varphi^{-1}(v_n)$. We will prove that $(v_n)_n$ has a convergent subsequence. Maybe after passing to a subsequence can we assume that $(\int \tau d\mu_n)_n$ converges to ∞ or a real number. Note that if $\lim_{n \rightarrow \infty} \int \tau d\mu_n = \infty$, then by Lemma 8.2, $(v_n)_n$ converges on cylinders to the zero measure. Let us now suppose that $\lim_{n \rightarrow \infty} \int \tau d\mu_n < \infty$. It follows by Theorem 1.1 (applied to $-\tau$) that there exists a subsequence $(\mu_{n_k})_k$ that converges on cylinders to $\lambda\mu$, where $\lambda \in [0, 1]$ and $\mu \in \mathcal{M}(\sigma; \tau)$. By Lemma 8.2, the sequence $(v_{n_k})_k$ converges on cylinders to a measure in $\mathcal{M}_{\leq 1}(\Theta)$. In general, if $(v_n)_n$ is any sequence in $\mathcal{M}_{\leq 1}(\Theta)$, we consider a subsequence where the mass converges and apply the discussion above.

To prove the density of $\mathcal{M}(\Theta)$ in $\mathcal{M}_{\leq 1}(\Theta)$, it is enough to prove that there exists a sequence $(v_n)_n$ in $\mathcal{M}(\Theta)$ that converges to the zero measure. Indeed, if $\lambda\mu \in \mathcal{M}_{\leq 1}(\Theta)$, then we can consider the sequence $(\lambda v + (1 - \lambda)v_n)_n$ that converges to λv . If (Σ, σ) has finite entropy, then there exists a sequence $(\mu_n)_n$ that converges to the zero measure (see [IV, Lemma 4.6]). If (Σ, σ) has infinite entropy, consider a sequence $(\mu_n)_n$ in $\mathcal{M}(\sigma)$ such that $\lim_{n \rightarrow \infty} h_{\mu_n}(\sigma) = \infty$; since $h_{\mu_n}(\sigma) - \int \tau d\mu_n \leq P(-\tau)$, we conclude that $\lim_{n \rightarrow \infty} \int \tau d\mu_n = \infty$. In both cases, it follows by Lemma 8.2 that $(\varphi(\mu_n))_n$ converges to the zero measure. $\square$

8.2. Entropy and pressure at infinity for suspension flows. The entropy of a measure $\nu \in \mathcal{M}(\Theta)$ is denoted by $h_\nu(\Theta)$ (this is defined as the entropy of ν with respect to the time one map θ_1). By Abramov's formula, we have that $h_\nu(\Theta) = (h_\mu(\sigma)) / (\int \tau d\mu)$, where $\varphi(\mu) = \nu$. The entropy of the suspension flow (Y, Θ) is defined by $h_{\text{top}}(\Theta) = \sup_{\nu \in \mathcal{M}(\Theta)} h_\nu(\Theta)$. Similarly, for a potential $\phi : Y \rightarrow \mathbb{R}$, we define its pressure by the formula

$$P^\Theta(\phi) = \sup_{\nu \in \mathcal{M}(\Theta)} \left(h_\nu(\Theta) + \int \phi d\nu \right).$$

We added the superscript Θ to emphasize that the base dynamical system is the suspension flow (Y, Θ) . We define the pressure at infinity of ϕ by the formula

$$P_\infty^\Theta(\phi) = \sup_{(v_n)_n \rightarrow 0} \limsup_{n \rightarrow \infty} \left(h_{v_n}(\Theta) + \int \phi dv_n \right),$$

where the supremum runs over all sequences $(v_n)_n$ in $\mathcal{M}(\Theta)$ that converge on cylinders to the zero measure. If there is no such sequence, we set $P_\infty^\Theta(\phi) = -\infty$. The entropy at infinity of (Y, Θ) , which we denote by $h_\infty(\Theta)$, is defined as the pressure at infinity of the zero potential.

For a potential $\phi : Y \rightarrow \mathbb{R}$, we define $\Delta_\phi : \Sigma \rightarrow \mathbb{R}$ by the formula $\Delta_\phi(x) = \int_0^{\tau(x)} \phi(\theta_t x) dt$. It follows by Kac's formula that $\int \phi dv = (\int \Delta_\phi d\mu) / (\int \tau d\mu)$, where $\mu = \varphi^{-1}(\nu)$. Therefore,

$$P^\Theta(\phi) = \inf\{t : P(\Delta_\phi - t\tau) \leq 0\}. \quad (18)$$

We will prove that a similar formula holds for the pressure at infinity.

LEMMA 8.4. $P_\infty^\Theta(\phi) = \inf\{t : P_\infty^{\text{top}}(\Delta_\phi - t\tau) \leq 0\}$.

Proof. Since τ is bounded away from zero, the function $f(t) = P_\infty^{\text{top}}(\Delta_\phi - t\tau)$ is decreasing. Define $s = \inf\{t : P_\infty^{\text{top}}(\Delta_\phi - t\tau) < \infty\}$ and set $a = \inf\{t : P_\infty^{\text{top}}(\Delta_\phi - t\tau) \leq 0\}$. Note that $f(t) = \infty$ if $t < s$ and $f(t) < \infty$ if $t > s$.

If there is no sequence in $\mathcal{M}(\Theta)$ that converges to the zero measure, then there is no sequence in $\mathcal{M}(\sigma)$ that converges to the zero measure (see Lemma 8.2). In this case, we have that $P_\infty^\Theta(\phi) = -\infty$ and that $P_\infty^{\text{top}}(\Delta_\phi - t\tau) = -\infty$ for all t (see Remark 3.7), and thus the formula holds. Let us now assume that there exists a sequence in $\mathcal{M}(\Theta)$ that converges to the zero measure. In particular, $P_\infty(\phi)$ can be realized by a sequence of measures converging to the zero measure.

Fix $t > a$. Let $(v_n)_n$ be a sequence in $\mathcal{M}(\Theta)$ that converges to the zero measure and such that $P_\infty^\Theta(\phi) = \lim_{n \rightarrow \infty} (h_{v_n}(\Theta) + \int \phi dv_n)$. Set $\mu_n \in \mathcal{M}(\sigma)$ such that $\varphi(\mu_n) = v_n$. Maybe after passing to a subsequence we can assume that $(\int \tau d\mu_n)_n$ converges to a real number or ∞ . In the first case, we have that $(\mu_n)_n$ converges to the zero measure (see Lemma 8.2) and, therefore, $\limsup_{n \rightarrow \infty} (h_{\mu_n}(\sigma) + \int (\Delta_\phi - t\tau) d\mu_n) < 0$, which implies that $\lim_{n \rightarrow \infty} (h_{v_n}(\Theta) + \int \phi dv_n) < t$. In the second case, consider $\epsilon > 0$ and note that

$$h_{\mu_n}(\sigma) + \int (\Delta_\phi - t\tau) d\mu_n \leq P(\Delta_\phi - (t + \epsilon)\tau) + \epsilon \int \tau d\mu_n,$$

which implies that $\lim_{n \rightarrow \infty} (h_{v_n}(\Theta) + \int \phi dv_n) \leq t + \epsilon$ (observe that $P(\Delta_\phi - (t + \epsilon)\tau) < \infty$, because $P_\infty^{\text{top}}(\Delta_\phi - (t + \epsilon)\tau) < \infty$, see Remark 3.7(1)). In both cases, we conclude that $P_\infty^\Theta(\phi) < t$ for every $t > a$ and, therefore, $P_\infty^\Theta(\phi) \leq a$.

It remains to prove that $P_\infty^\Theta(\phi) \geq a$. We consider two cases.

Case 1: ($s < a$). In this case, we have that $P_\infty^{\text{top}}(\Delta_\phi - a\tau) = 0$. Therefore, there exists a sequence $(\mu_n)_n$ that converges on cylinders to the zero measure such that $\lim_{n \rightarrow \infty} (h_{\mu_n}(\sigma) + \int (\Delta_\phi - a\tau) d\mu_n) = 0$ and, thus, $\lim_{n \rightarrow \infty} (h_{v_n}(\Theta) + \int \phi dv_n) = a$, where $v_n = \varphi(\mu_n)$. The sequence $(v_n)_n$ converges to zero and, therefore, $P_\infty^\Theta(\phi) \geq a$.

Case 2: ($s = a$). Fix $\epsilon > 0$. Since $P_\infty^{\text{top}}(\Delta_\phi - (s - \epsilon)\tau) = \infty$, it follows that $P(\Delta_\phi - (s - \epsilon)\tau) = \infty$. There exists a sequence $(\mu_n)_n$ in $\mathcal{M}(\sigma)$ such that

$$\lim_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) + \int (\Delta_\phi - (s - \epsilon)\tau) d\mu_n \right) = \infty.$$

Then, $\limsup_{n \rightarrow \infty} (h_{v_n}(\Theta) + \int \phi dv_n) \geq s - \epsilon$, where $v_n = \varphi(\mu_n)$. Note that

$$h_{\mu_n}(\sigma) + \int (\Delta_\phi - (s - \epsilon)\tau) d\mu_n \leq P(\Delta_\phi - (s + \epsilon)\tau) + 2\epsilon \int \tau d\mu_n$$

and, therefore, $\lim_{n \rightarrow \infty} \int \tau \, d\mu_n = \infty$. We conclude that $(v_n)_n$ converges to zero and then $P_\infty^\Theta(\phi) \geq s - \epsilon$. Since $\epsilon > 0$ was arbitrary, it follows that $P_\infty^\Theta(\phi) \geq s$. $\square$

Remark 8.5. Let $\tau \in \Psi$. Note that $h_{\text{top}}(\Theta) < \infty$ (see (18)). In particular, if $\phi \in C_b(Y)$, then $P^\Theta(\phi) < \infty$. Furthermore, the entropy at infinity of (Y, Θ) is given by $h_\infty(\Theta) = h(\tau)$ (see Definition 3.11 and Lemma 8.4).

We now provide a proof of Theorem 1.7, which is similar to Theorem 1.3 and analogous to Corollary 6.16, where we considered a finite entropy CMS and a bounded potential.

Proof of Theorem 1.7. Maybe after passing to a subsequence we can assume that $(h_{v_n}(\Theta) + \int \phi \, dv_n)_n$ is convergent and converges to the limsup. Set $\mu_n = \varphi^{-1}(v_n)$ and $\mu = \varphi^{-1}(v)$. Assume that $\lambda \in (0, 1]$, otherwise, the result follows by definition of the pressure at infinity. By Lemma 8.2, maybe after passing to a subsequence, we can assume that $(\mu_n)_n$ converges on cylinders to $\lambda_1 \mu$ and $\lim_{k \rightarrow \infty} \int \tau \, d\mu_k = \lambda_2 \int \tau \, d\mu$, where $\lambda_1 \in (0, 1]$, $\lambda_2 \in [1, \infty)$ and $\lambda = \lambda_1/\lambda_2$.

By Lemma 8.4, we have that if $t > P_\infty^\Theta(\phi)$, then $P_\infty^{\text{top}}(\Delta_\phi - t\tau) < 0$. Note that $\Delta_\phi - t\tau$ is uniformly continuous with finite second variation and finite pressure (see Lemma 3.5). In particular, we can apply Theorem 1.3 and conclude that

$$\limsup_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) - \int (\Delta_\phi - t\tau) \, d\mu_n \right) \leq \lambda_1 \left(h_\mu(\sigma) - \int (\Delta_\phi - t\tau) \, d\mu \right),$$

and therefore,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left(h_{v_n}(\Theta) - \int (\phi - t) \, dv_n \right) &= \limsup_{n \rightarrow \infty} \left(\frac{h_{\mu_n}(\sigma) - \int (\Delta_\phi - t\tau) \, d\mu_n}{\int \tau \, d\mu_n} \right) \\ &\leq \frac{\lambda_1}{\lambda_2} \left(\frac{h_\mu(\sigma) - \int (\Delta_\phi - t\tau) \, d\mu}{\int \tau \, d\mu} \right) \\ &= \lambda (h_v(\Theta) - \int (\phi - t) \, dv), \end{aligned}$$

which is equivalent to

$$\limsup_{n \rightarrow \infty} \left(h_{v_n}(\Theta) + \int \phi \, dv_n \right) \leq \lambda \left(h_v(\Theta) + \int \phi \, dv \right) + (1 - \lambda)t.$$

Since $t > P_\infty^\Theta(\phi)$ was arbitrary, we obtain the desired inequality.

We now prove that the inequality is sharp. Let $(\eta_n)_n$ be a sequence in $\mathcal{M}(\Theta)$ that converges to the zero measure and such that $\lim_{n \rightarrow \infty} (h_{\eta_n}(\Theta) + \int \phi \, d\eta_n) = P_\infty^\Theta(\phi)$. The existence of such a sequence follows by Theorem 1.6. Set $v_n = \lambda v + (1 - \lambda)\eta_n$. Note that $(v_n)_n$ converges to λv and it achieves equality in the pressure inequality. $\square$

Analogous to the case of CMS, we say that $\phi : Y \rightarrow \mathbb{R}$ is SPR if $P^\Theta(\phi) > P_\infty^\Theta(\phi)$. The next result follows from combining Theorems 1.6 and 1.7. The proof is analogous to the proof of Theorem 1.4 and we leave the details to the reader.

THEOREM 8.6. *Let $\tau \in \Psi$ and $\phi \in \mathcal{H}_Y$. Let $(v_n)_n$ be a sequence in $\mathcal{M}(\Theta)$ such that $P^\Theta(\phi) = \lim_{n \rightarrow \infty} (h_{v_n}(\Theta) + \int \phi \, dv_n)$. Then, the following hold.*

- (1) If ϕ is SPR, then $(v_n)_n$ has a subsequence that converges in the weak* topology to an equilibrium state of ϕ . In particular, ϕ admits equilibrium states.
- (2) If ϕ does not have an equilibrium state, then $(v_n)_n$ converges to the zero measures. In this case, $P_\infty^\Theta(\phi) = P^\Theta(\phi)$.

Acknowledgements. The author was supported by FONDECYT Iniciación 11220409 and FONDECYT Regular 1250928.

A. Appendix. Tightness in the space of invariant probability measures

Let (Σ, σ) be a transitive countable Markov shift. Let $(\mu_n)_n$ be a sequence in $\mathcal{M}(\sigma)$. The condition

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mu_n \left(\bigcup_{s \geq k} [s] \right) = 0, \quad (\text{A.1})$$

says that $(\mu_n)_n$ does not lose mass in the cylinder topology. In Lemma A.1, we prove that this condition is equivalent to saying that $(\mu_n)_n$ is tight. Since Σ may not be locally compact, it is convenient to replace tightness with something more suitable to our setup.

LEMMA A.1. *Let $(\mu_n)_n$ be a sequence in $\mathcal{M}(\sigma)$ such that (A.1) holds. Then, there exists a subsequence of $(\mu_n)_n$ that converges in the weak* topology.*

Proof. We follow closely the idea of the proof of [IV, Proposition 4.11]. By countability of the set of cylinders, we can find a subsequence $(\mu_{n_\ell})_\ell$ such that $\lim_{\ell \rightarrow \infty} \mu_{n_\ell}(C) =: F(C)$ is well defined for every cylinder $C \subseteq \Sigma$. We claim that F extends to a countably additive measure on Σ . As explained in [IV, Proposition 4.11], it is enough to prove that

$$\lim_{k \rightarrow \infty} \lim_{\ell \rightarrow \infty} \mu_{n_\ell} \left(\bigcup_{s \geq k} [a_1, \dots, a_m, s] \right) = 0 \quad (\text{A.2})$$

for every cylinder $[a_1, \dots, a_m]$. Since $\mu_n([s]) = \mu_n(\sigma^{-m}[s]) \geq \mu_n([a_1, \dots, a_m, s])$, it follows that (A.1) implies that (A.2). We conclude that F extends to a countably additive measure μ and that $(\mu_{n_\ell})_\ell$ converges on cylinders to $\mu \in \mathcal{M}_{\leq 1}(\sigma)$ (see [IV, Proposition 4.11 and Lemma 4.1]). Observe that (A.1) implies $\lim_{k \rightarrow \infty} (1 - \mu(\bigcup_{s < k} [s])) = 0$ and, therefore, μ is a probability measure. Since $(\mu_{n_\ell})_\ell$ converges on cylinders to a probability measure, we conclude that the sequence converges in the weak* topology. $\square$

We apply Lemma A.1 to prove that, for the full shift, the escape of mass can be completely ruled out in certain cases.

PROPOSITION A.2. *Let $(\mathbb{N}^\mathbb{N}, \sigma)$ be the full shift and $\psi \in C_{uc}(\Sigma)$ a potential such that $\text{var}_1(\psi)$ and $P(\psi)$ are finite. Let $(\mu_n)_n$ be a sequence in $\mathcal{M}(\sigma)$ such that*

$$P(\psi) = \lim_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) + \int \psi \, d\mu_n \right).$$

Then, $(\mu_n)_n$ has a subsequence that converges in the weak topology.*

Proof. Maybe after passing to a subsequence can we assume that $\lim_{n \rightarrow \infty} \mu_n(C)$ is well defined for every cylinder $C \subseteq \Sigma$. By Lemma A.1, it is enough to prove that

$$\lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \mu_n \left(\bigcup_{s \geq k} [s] \right) = 0.$$

We argue by contradiction and suppose that this is not the case. Since the sequence $(\lim_{n \rightarrow \infty} \mu_n(\bigcup_{s \geq k} [s]))_k$ is decreasing, there exists $\epsilon > 0$ such that

$$\lim_{n \rightarrow \infty} \mu_n \left(\bigcup_{s \geq k} [s] \right) \geq \epsilon$$

for every $k \in \mathbb{N}$. Set $A_k = \bigcup_{s \geq k} [s]$. In particular,

$$P(\psi + t1_{A_k}) \geq \limsup_{n \rightarrow \infty} \left(h_{\mu_n}(\sigma) + \int \psi d\mu_n + t\mu_n(A_k) \right) \geq P(\psi) + t\epsilon$$

for all $t \in [0, +\infty)$ and $k \in \mathbb{N}$. Consider $\psi_0(x) = \sup\{\psi(y) : y \in [x_1]\}$, where $x = (x_1, \dots) \in \mathbb{N}^{\mathbb{N}}$. Note that since $\text{var}_1(\psi)$ is finite, $\|\psi - \psi_0\|_0 =: M < \infty$. Also note that $\text{var}_1(\psi_0) = 0$. The inequality above implies that

$$P(\psi_0 + t1_{A_k}) \geq P(\psi_0) + t\epsilon - 2M$$

for all $t \in [0, +\infty)$ and $k \in \mathbb{N}$. Note that $P(\psi_0) \leq P(\psi) + M < \infty$. By Example 2.9, we have that $P(\psi_0) = \log(\sum_{i \in \mathbb{N}} e^{\psi_0(i)})$ and, therefore, the series $\sum_{i \in \mathbb{N}} e^{\psi_0(i)}$ converges. Set $t_0 \in [0, +\infty)$ such that $(t_0\epsilon - 2M) > 2$, and $m \in \mathbb{N}$ such that $\sum_{i \geq m} e^{\psi_0(i)} < e^{P(\psi_0) - t_0}$. It follows that

$$\begin{aligned} P(\psi_0) + 2 &< P(\psi_0) + (t_0\epsilon - 2M) < P(\psi_0 + t_01_{A_m}) \\ &= \log \left(\sum_{i < m} e^{\psi_0(i)} + \sum_{i \geq m} e^{\psi_0(i) + t_0} \right) \\ &\leq \log \left(\sum_{i < m} e^{\psi_0(i)} + e^{P(\psi_0)} \right) \\ &\leq \log(2e^{P(\psi_0)}) \\ &= P(\psi_0) + \log 2, \end{aligned}$$

which is a contradiction. We conclude that the sequence $(\mu_n)_n$ satisfies (A.1) and by Lemma A.1, we conclude that $(\mu_n)_n$ has a subsequence that converges in the weak* topology to an invariant probability measure. $\square$

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