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A NOTE ON TASKINEN'S COUNTEREXAMPLES ON THE PROBLEM OF TOPOLOGIES OF GROTHENDIECK

by JOSE BONET and ANTONIO GALBIS

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By the work of Taskinen (see [**4**, **5**]), we know that there is a Fréchet space E such that $L_b(E, l_2)$ is not a (DF)-space. Moreover there is a Fréchet–Montel space F such that $F'_b \hat{\otimes}_\varepsilon F'_b$ is not (DF). In this second example, the duality theorem of Buchwalter (cf. [**2**, §45.3]) can be applied to obtain that $F'_b \hat{\otimes}_\varepsilon F'_b \cong (F \hat{\otimes}_\pi F)'_{co}$ and hence $F'_b \hat{\otimes}_\varepsilon F'_b$ is a (gDF)-space (cf. [**1**, Ch. 12 or **3**, Ch. 8]). The (gDF)-spaces were introduced by several authors to extend the (DF)-spaces of Grothendieck and to provide an adequate frame to consider strict topologies.

It seemed to be open whether, for Fréchet spaces E and F, the spaces $L_b(E, F'_b)$ and $E'_b \hat{\otimes}_\varepsilon F'_b$ were (gDF). In this short note we observe that the constructions of Taskinen in [**4**] can be adapted to give a negative answer to these two questions.

Our notations are standard and can be seen in [**1**, **2**, **3**].

Let X be a Fréchet space with a basis of absolutely convex 0-neighbourhoods (U_n) with $2U_{n+1} \subseteq U_n$. We denote by $B(X)$ the set of all bounded subsets of X. Let A be a saturated subset of $B(X)$ whose union covers X. We denote by τ_A the topology on X' of uniform convergence on the elements of A. Using the characterizations of (gDF)-spaces and a standard argument by polarization, one gets

Lemma 1. *Let X be a Fréchet space. Then (X', τ_A) is (gDF) if and only if for every squence of absolutely convex subsets $C_n \in A$, $n \in \mathbb{N}$, there is $C \in A$ such that $\cup(C_n \cap U_n: n \in \mathbb{N}) \subseteq C$.*

Let E and F be Fréchet spaces with decreasing sequences of 0-neighbourhoods (W_n) and (V_n) such that $(n^{-1}W_n)$ and $(n^{-1}V_n)$ are basis of 0-neighbourhoods in E and F respectively. Then $L_b(E, F'_b)$ is topologically isomorphic to the dual $(E \hat{\otimes}_\pi F)'$ endowed with the topology of uniform convergence on the sets $\overline{\Gamma(A \otimes B)}$, $A \in B(E)$, $B \in B(F)$. Then we have:

Corollary 2. *$L_b(E, F'_b)$ is (gDF) if and only if for every $(B_n) \subseteq B(E)$, $(C_n) \subseteq B(F)$, there are $A \in B(E)$, $B \in B(F)$ such that $\cup(\overline{\Gamma(B_n \otimes C_n)} \cap \overline{\Gamma(2^{-n} W_n \otimes V_n)}: n \in \mathbb{N}) \subseteq \overline{\Gamma(A \otimes B)}$, all the closures taken in $E \hat{\otimes}_\pi$ F.*

Observe that by symmetry the condition above is also equivalent to $L_b(F, E'_b)$ being (gDF).

We will use the notations of [**4**, §4] and we will take $M_n = M'_n = l_2^n$. Hence we can suppose that the projection constants $\rho(n)$ from G_n onto M_n satisfy

$$\sqrt{\frac{2n}{\pi}} < \rho(n) < 2\sqrt{n} \quad \text{(see [4, p. 22]).}$$

Example 3. *There is a Fréchet space E such that $L_b(E,l_2)$ is not (gDF).*

This will be a consequence of Corollary 2, once we prove that for $F=l_2$, V the unit ball of F, and E the Fréchet space of Taskinen [4, §4.4], we have that the set

$$\cup(\overline{\Gamma(2^{-n}W_n\otimes V)} \cap \overline{\Gamma(W^n_{n+1}\otimes 2^{4n}V)}: n\in\mathbb{N})$$

is not contained in $\overline{\Gamma(A\otimes B)}$ for all $A\in B(E)$, $B\in B(F)$, the closures taken in $E\hat{\otimes}_\pi F$ (observe that $W^n_{n+1}\in B(E)$).

To prove this it is enough to see that for every $t_m\geqq 1$, $m\in\mathbb{N}$, the set

$$\cup(\Gamma(2^{-n}W_n\otimes V)\cap\Gamma(2^{4n}W^n_{n+1}\otimes V): n\in\mathbb{N})$$

is not contained in the closure of $\Gamma((\cap(t_mW_m: m\in\mathbb{N}))\otimes V)$ in $E\otimes_\pi F$. First observe that the key Lemma 4.3 of [4] yields for $s(n):=\rho(n)/4$ that $\Gamma(U\otimes V)\cap\Gamma(2^{5n}\tilde{U}\otimes V)\not\subseteq \Gamma((s(2^{5n})U\cap 4s\tilde{U})\otimes V)$ for all $n\in\mathbb{N}$ and $s>0$. Then

$$\Gamma(2^{-n}U\otimes V)\cap\Gamma(2^{4n}\tilde{U}\otimes V)\not\subseteq\Gamma(((s(2^{5n})/2^n)U\cap 2^{-n+2}s\tilde{U})\otimes V)$$

for all $n\in\mathbb{N}$ and $s>0$. Since $\lim\rho(n)/4n^{1/5}=\infty$, there is $n_0\in\mathbb{N}$ such that $s(2^{5n_0})/2^{n_0}\geqq 2t_1$. Then $\Gamma(2^{-n_0}U\otimes V)\cap\Gamma(2^{4n_0}\tilde{U}\otimes V)$ is not contained in $2\Gamma((t_1U\cap t_{n_0+1}\tilde{U})\otimes V)$. Since

$$(\Gamma((\cap(t_mW_m: m\in\mathbb{N}))\otimes V)+\Gamma(W_{n_0+1}\otimes V))\cap(E_{n_0}\otimes F)\subseteq 2\Gamma((t_1U\cap t_{n_0+1}\tilde{U})\otimes V)$$

([4; proof of 4.5]), $\Gamma(W_{n_0+1}\otimes V)$ is a 0-neighbourhood in $E\otimes_\pi F$, and

$$[\Gamma[2^{-n_0}W_{n_0}\otimes V)\cap\Gamma(2^{4n_0}W^{n_0}_{n_0+1}\otimes V)]\cap(E_{n_0}\cap F)=\Gamma(2^{-n_0}U\otimes V)\cap\Gamma(2^{4n_0}\tilde{U}\otimes V)$$

the conclusion follows.

Example 4. *There is a Fréchet–Schwartz space F with the bounded approximation property and a Fréchet space E such that $F'_b\hat{\otimes}_\varepsilon E'_b\cong L_b(F,E'_b)$ is not (gDF).*

F is the Fréchet–Schwartz and E is the Fréchet space constructed as in [4, §4.7]. In this case the proof is a little more involved. Using now the second key Lemma 4.8 in [4] one can prove that

$$\cup(\Gamma(2^{-n}W_n\otimes V_n)\cap(\cap(\Gamma(2^{4n}W^n_p\otimes V_p): p\geqq n+1)): n\in\mathbb{N})$$

is not contained in the closure of $\Gamma(A\otimes B)$ in $E\otimes_\pi F$ for $A\in B(E)$ and $B\in B(F)$.

Now since $W^n_p=\tilde{U}$ for $p\geqq n+1$, we can apply the positive result [4, §3.1] for $(H,\tilde{h})$

and F to obtain for each $n \in \mathbb{N}$, $C_n \in B(F)$ such that

$$\cap(\Gamma(2^{4n} W_p^n \otimes V_p) : p \geqq n+1) \subseteq \overline{\Gamma(W_{n+1}^n \otimes C_n)}$$

the closure taken in $E \hat{\otimes}_\pi F$. Consequently, for $(W_{n+1}^n) \subseteq B(E)$, $(C_n) \subseteq B(F)$ the set

$$\cup((\overline{\Gamma(2^{-n} W_n \otimes V_n)} \cap \overline{\Gamma(W_{n+1}^n \otimes C_n)}) : n \in \mathbb{N})$$

is not contained in $\overline{\Gamma(A \otimes B)}$, the closure taken in $E \hat{\otimes}_\pi F$. The conclusion follows from Corollary 2. □

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Departamento de Matemáticas
E.T.S. Arquitectura
Universidad Politécnica de Valencia
46022 Valencia
Spain

Departamento de Análisis
Facultad de Matemáticas
Universidad de Valencia
46071 Burjasot (Valencia)
Spain