22

Connection to Quantum Field Theory

In this chapter, we give an outlook on how to get a connection between the causal action principle and the dynamics of quantum fields. Since this direction of research is very recent and partly a work in progress, we do not enter any details but instead try to explain a few basic concepts and ideas. Our presentation is based on the recent research papers [57, 58, 84, 24]. Partly, our methods were already explored in the alternative approach in [44], which is more closely tied to the analysis of the continuum limit (as outlined in Chapter 21).

22.1 Convex Combinations of Measures and Dephasing Effects

Before beginning, we point out that in most examples of causal fermion systems considered in this book, the measure ρ was the push-forward of the volume measure on Minkowski space or a Lorentzian manifold. Thus, we first constructed a local correlation map (see (5.33))

$$F^\varepsilon \,:\, \mathscr{M} \to \mathcal{F}\,, \tag{22.1}$$

and then introduced the measure ρ on $\mathcal{F}$ by (see (5.34))

$$\rho = (F^\varepsilon)_* \mu_{\mathscr{M}}\,, \tag{22.2}$$

where $\mu_{\mathscr{M}}$ is the four-dimensional volume measure on $\mathscr{M}$. In all these examples, the measure ρ had the special property that it was supported on a smooth four-dimensional subset of $\mathcal{F}$ given by (for details, see Exercise 8.1)

$$M := \operatorname{supp} \rho = \overline{F^\varepsilon(\mathscr{M})}\,. \tag{22.3}$$

Also, when varying the measure in the derivation of the linearized field equations or in the study of interacting systems in the continuum limit, we always restricted attention to measures having this property (see (8.6) in Section 8.1 or Chapter 21). While this procedure seems a good starting point for the analysis of the causal action principle and gives good approximate solutions of the EL equations, we cannot expect that true minimizers are of this particular form.

With this in mind, our strategy is to allow for more general measures on $\mathcal{F}$ and to analyze the causal action principle for these general measures. As we will see, this analysis gives rise to close connections to quantum field theory. We proceed step by step and begin by explaining a construction that explains why going beyond push-forward measures of the form (22.2) makes it possible to further decrease the causal action. In other words, the following argument shows that minimizers of the causal action will *not* have the form of push-forward measures (22.2) but will have a more complicated structure. This argument is given in more detail in [45, §1.5.3]. Assume that we are given L measures $\rho_1, \ldots, \rho_L$ on $\mathcal{F}$. Then, their *convex combination* $\tilde{\rho}$, given by

$$\tilde{\rho} := \frac{1}{L} \sum_{\mathfrak{a}=1}^{L} \rho_{\mathfrak{a}}, \tag{22.4}$$

is again a positive measure on $\mathcal{F}$. Moreover, if the $\rho_{\mathfrak{a}}$ satisfy the linear constraints (i.e., the volume constraint (5.37) and the trace constraint (5.38)), then these constraints are again respected by $\tilde{\rho}$.

Next, we let ρ be a minimizing measure (describing, e.g., the vacuum). Choosing unitary transformations $U_1, \ldots, U_L$, we introduce the measures $\rho_{\mathfrak{a}}$ in (22.4) as

$$\rho_{\mathfrak{a}}(\Omega) := \rho\big(U^{-1} \Omega U\big) . \tag{22.5}$$

Thus, in other words, the measures $\rho_{\mathfrak{a}}$ are obtained from ρ by taking the unitary transformations by $U_{\mathfrak{a}}$. Since the causal action and the constraints are unitarily invariant, each of the measures $\rho_{\mathfrak{a}}$ is again minimized. Let us compute the action of the convex combination (22.4). First, by (5.36),

$$\mathcal{S}(\tilde{\rho}) = \frac{1}{L^2} \sum_{\mathfrak{a},\mathfrak{b}=1}^{L} \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}(x,y)\, \mathrm{d}\rho_{\mathfrak{a}}(x)\, \mathrm{d}\rho_{\mathfrak{b}}(y) . \tag{22.6}$$

If $\mathfrak{a} = \mathfrak{b}$, we obtain the action of the measure $\rho_{\mathfrak{a}}$, which, due to unitary invariance, is equal to the action of ρ. We thus obtain

$$\mathcal{S}(\tilde{\rho}) = \frac{\mathcal{S}(\rho)}{L} + \frac{1}{L^2} \sum_{\mathfrak{a}\neq\mathfrak{b}} \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}(x,y)\, \mathrm{d}\rho_{\mathfrak{a}}(x)\, \mathrm{d}\rho_{\mathfrak{b}}(y) . \tag{22.7}$$

Let us consider the contributions for $\mathfrak{a} \neq \mathfrak{b}$ in more detail. In order to simplify the explanations, it is convenient to assume that the measures $\rho_{\mathfrak{a}}$ have mutually disjoint supports (this can typically be arranged by a suitable choice of the unitary transformations $U_{\mathfrak{a}}$). Then, the spacetime $\tilde{M} := \operatorname{supp} \tilde{\rho}$ can be decomposed into L "sub-spacetimes" $M_{\mathfrak{a}} := \operatorname{supp} \rho_{\mathfrak{a}}$,

$$\tilde{M} = M_1 \cup \cdots \cup M_L \qquad \text{and} \qquad M_{\mathfrak{a}} \cap M_{\mathfrak{b}} = \varnothing \quad \text{if } \mathfrak{a} \neq \mathfrak{b} . \tag{22.8}$$

The Lagrangian of the last summand in (22.7) is computed from the fermionic projector $P_{\mathfrak{a},\mathfrak{b}}(x,y)$, where $x \in M_{\mathfrak{a}}$ and $y \in M_{\mathfrak{b}}$ are in different sub-spacetimes. Similar to (5.58), it can be expressed in terms of the physical wave functions by (for details, see [45, Lemma 1.5.2])

$$P_{\mathfrak{a},\mathfrak{b}}(x,y) = -\sum_{i,j} |\psi^{e_i}(x) \succ (U_{\mathfrak{a}}\, U_{\mathfrak{b}}^*)^i_j \prec \psi^{e_j}(y)| . \tag{22.9}$$

The point is that this fermionic projector involves the operator product $U_{\mathfrak{a}} U_{\mathfrak{b}}^*$. By choosing the unitary operators $U_{\mathfrak{a}}$ and $U_{\mathfrak{b}}$ suitably, one can arrange that this operator product involves many phase factors. Moreover, one can arrange that, when carrying out the sums in (22.9), these phases cancel each other due to destructive interference. In this way, the kernel $P(x,y)$ can be made small if x and y lie in different sub-spacetimes. As a consequence, the last summand in (22.7) can be arranged to be very small. Taking into account the factor $1/N$ in the first summand in (22.7), also the causal action of $\tilde{\rho}$ becomes small. Clearly, this argument applies only if the number L of sub-spacetimes is not too large because otherwise it becomes more and more difficult to arrange destructive interference for all summands of the sum in (22.7) (estimating the optimal number L of subsystems is a difficult problem, which we do not enter here). Also, we cannot expect that the simple ansatz (22.4) will already give a minimizer. But at least, our argument explains why it is too naive to think of a minimizing measure as being the push-forward measure of a volume measure under a smooth local correlation map. Instead, a minimizing measure could be composed of a large number of sub-spacetimes.

In the abovementioned consideration, it is crucial that the kernels $P_{\mathfrak{a},\mathfrak{b}}(x,y)$ for $\mathfrak{a} \neq \mathfrak{b}$ are very small due to decoherence effects. It is a subtle point how small these kernels are. If they are so small that we may assume that they vanish, then this means that the sub-spacetimes do not interact with each other. Therefore, one can take the point of view that, in order to describe all physical phenomena, it suffices to restrict attention to one sub-spacetime. The appearance of many sub-spacetimes that are completely decoherent is an intriguing mathematical effect, which may have interesting philosophical implications, but it is of no relevance as far as physical predictions are concerned. For this reason, here we shall not discuss these decoherent sub-spacetimes further. Also, we leave the question open whether they really occur for minimizing measures. Instead, we take the point of view that, in case our minimizing measure consists of several decoherent sub-spacetimes, we restrict it to one sub-spacetime and denote the resulting measure by ρ.

In order to understand the dynamics of a causal fermion system, it is more interesting to consider convex combinations of measures that are *not* completely decoherent. In order to explain the resulting effects in a simple example, suppose we choose electromagnetic potentials $A_1, \ldots, A_L$ in Minkowski space (which do not need to satisfy Maxwell's equations). Constructing the regularized kernels $P^\varepsilon_{\mathfrak{a}}(x,y)$ (as explained in Chapters 18 and 21), one gets corresponding causal fermion systems described by measures $\rho_{\mathfrak{a}}$. Abstractly, these measures can be written as explained in the context of the linearized field equations (see (8.18) in Section 8.1) as

$$\tilde{\rho} = \sum_{\mathfrak{a}=1}^L (F_{\mathfrak{a}})_* \big(f_{\mathfrak{a}}\, \rho \big) , \tag{22.10}$$

where $F_{\mathfrak{a}}$ is the corresponding local correlation map, and $f_{\mathfrak{a}}$ is a weight function. Since these measures are obtained from each other by small perturbations, it seems a good idea to depict the corresponding supports $M_{\mathfrak{a}} := \operatorname{supp} \rho_{\mathfrak{a}}$ as being close together (see Figure 8.1(b)). The convex combination of these measures (22.4) is referred to as a measure with *fragmentation* (see [49, Sections 1 and 5] or [51, Section 5]). The reason why we consider convex combinations (rather than general linear combinations) is that we need to preserve the positivity of the measure. In

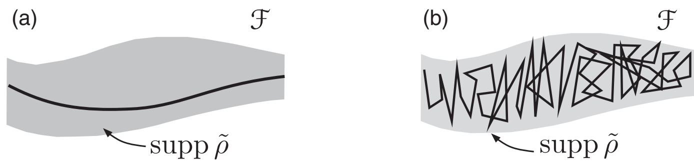

Figure 22.1 A measure obtained by fragmentation (a) and by holographic mixing with fluctuations (b).

the limit when N gets large, the fragmented measure $\tilde{\rho}$ goes over to a measure with enlarged support (see Figure 8.1(c)). Integrating with respect to this measure also involves an integration over the "internal degrees of freedom" corresponding to the directions that are transverse to $M := \operatorname{supp} \rho$ (see Figure 22.1(a)). This integral with respect to $\tilde{\rho}$ bears similarity to the path integral formulation of quantum field theory if one identifies the abovementioned "internal degrees of freedom" with field configurations.

22.2 The Mechanism of Holographic Mixing

For the mathematical description of the interacting measure $\tilde{\rho}$, working with fragmented measures as introduced in Section 22.1 does not seem to be the best method. One difficulty is that it is a priori not clear how large the number L of fragments is to be chosen. Moreover, mechanisms where L changes dynamically are difficult to implement, at least perturbatively. For these reasons, it seems preferable to describe $\tilde{\rho}$ with a different method, referred to as *holographic mixing*. At first sight, this method seems very different from fragmentation. However, as we will explain at the very end of this section, fragmentation also allows for the description of fragmented measures, at least if the construction is carried out in sufficiently large generality. We now explain the general idea and a few related constructions.

Let $(\mathcal{H}, \mathcal{F}, \rho)$ be a causal fermion system (e.g., describing the Minkowski vacuum). The *wave evaluation operator* Ψ introduced in (5.59) is a mapping that to every vector in $\mathcal{H}$ associates the corresponding physical wave function (for more details, see, e.g., [45, §1.1.4])

$$\Psi \,:\, \mathcal{H} \rightarrow C^0(M, SM)\,, \qquad u \mapsto \psi^u\,, \tag{22.11}$$

where the physical wave function ψ^u is again given by (5.56). Evaluating at a fixed spacetime point gives the mapping

$$\Psi(x) \,:\, \mathcal{H} \rightarrow S_xM\,, \qquad u \mapsto \psi^u(x)\,. \tag{22.12}$$

Working with the wave evaluation operator makes it possible to write the kernel of the fermionic projector (5.58) in the short form (for a detailed proof, see [45, Lemma 1.1.3])

$$P(x,y) = -\Psi(x)\, \Psi(y)^*\,. \tag{22.13}$$

The general procedure of holographic mixing is to replace the wave evaluation operator with a linear combination of wave evaluation operators $\Psi_{\mathfrak{a}}$,

$$\tilde{\Psi} := \sum_{\mathfrak{a}=1}^{L} \tilde{\Psi}_{\mathfrak{a}} \,, \tag{22.14}$$

which in turn are all obtained by perturbing Ψ (as will be described in more detail, in (22.18)). Now we form the corresponding local correlation map,

$$\tilde{F} \,:\, M \to \mathcal{F} \,, \qquad \tilde{F}(x) := -\tilde{\Psi}(x)^* \, \tilde{\Psi}(x) \,, \tag{22.15}$$

and take the corresponding push-forward measure,

$$\tilde{\rho} := \tilde{F}_* \rho \,. \tag{22.16}$$

In this way, we have constructed a new measure ρ that incorporates the perturbations described all the wave evaluation operators $\tilde{\Psi}_1, \ldots, \tilde{\Psi}_L$. However, in contrast to the convex combination of measures (22.10), the support of the measure (22.16), in general, does not decompose into several fragments. In fact, if the mapping $\tilde{F}$ is continuous, injective and closed, the support of $\tilde{\rho}$ will again be homeomorphic to M. In other words, the topological structure of spacetime remains unchanged by the abovementioned procedure.

More concretely, the perturbed wave evaluation operators $\Psi_{\mathfrak{a}}$ can be obtained as follows. Suppose that the causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ was constructed similarly to that given in Section 5.5 from a system of Dirac wave functions satisfying, for example, the Dirac equation

$$\big(\mathcal{D} - m\big)\psi = 0 \,. \tag{22.17}$$

Then, one can perturb the system by considering the Dirac equation in the presence of classical potentials $\mathcal{B}_1, \ldots, \mathcal{B}_L$,

$$\big(\mathcal{D} + \mathcal{B}_{\mathfrak{a}} - m\big)\tilde{\psi}_{\mathfrak{a}} = 0 \,. \tag{22.18}$$

The corresponding wave evaluation operators $\tilde{\Psi}_{\mathfrak{a}}$ are built up of all these Dirac solutions. In this way, the resulting wave evaluation operator (22.14) involves all the classical potentials $\mathcal{B}_{\mathfrak{a}}$. Qualitatively speaking, the resulting spacetime $\tilde{M}$ can be thought of as being in a "superposition" of all these potentials. But this analogy does not carry over to a more technical level.

As already mentioned after (22.16), taking the push-forward with respect to a mapping F does not change the topological structure of spacetime. Even more, if F is smooth and varies only on macroscopic scales, then all microscopic structures of spacetime remain unchanged. This does not account for the picture of a measure $\tilde{\rho}$, which accounts for additional "internal degrees of freedom" as shown in Figure 8.1(c) and Figure 22.1(a). In order to allow the description of such measures, one needs to consider mappings F that are *not smooth* but instead "fluctuate" on a microscopic scale (as is shown symbolically in Figure 22.1(b)). If we allow for such fluctuations even on the Planck scale, then the procedure (22.14) does allow for the description of all measures described previously with fragmentation (22.10). This consideration explains why the wave evaluation operators $\tilde{\Psi}_{\mathfrak{a}}$

should be constructed not only by introducing classical potentials (22.18) but, in addition, by introducing small-scale fluctuations. This can be realized as follows. We choose operators $A_\mathfrak{a}$ on $\mathcal{H}$, which add up to the identity,

$$\sum_{\mathfrak{a}=1}^{N} A_\mathfrak{a} = \mathbb{1} \,, \tag{22.19}$$

and then decompose the local correlation operator by multiplying from the right by $A_\mathfrak{a}$,

$$\Psi_\mathfrak{a} := \Psi A_\mathfrak{a} \,. \tag{22.20}$$

In the second step, the physical wave functions in $\Psi_\mathfrak{a}$ are perturbed by classical potentials $A_\mathfrak{a}$, again by considering the Dirac equation (22.18). In the last step, we again take the sum of the wave evaluation operators (22.14) and form the push-forward measure (22.16). This procedure is referred to as *holographic mixing*.

The resulting wave evaluation operator $\tilde{\Psi}$ involves both the operators $A_\mathfrak{a}$ and the potentials $\mathcal{B}_\mathfrak{a}$. As explained in (22.9) in the context of fragmentation, the operators $A_\mathfrak{a}$ enter the kernel of the fermionic projector,

$$P(x,y) = -\sum_{\mathfrak{a},\mathfrak{b}=1}^{N} |\psi^{e_i}(x) \succ (A_\mathfrak{a}\, A_\mathfrak{b}^*)^i_j \prec \psi^{e_j}(y)| \,. \tag{22.21}$$

In this way, one can build phase factors into this kernel, possibly giving rise to destructive interference. In other words, the wave evaluation operator $\tilde{\Psi}$ is a sum of many, partly decoherent components. The name "holographic mixing" is inspired by the similarity to a hologram in which several pictures are stored, each of which becomes visible only when looking at the hologram in the corresponding coherent light.

The abovementioned ideas and constructions are implemented in the recent paper [24] in an enhanced way. The main improvement compared to the abovementioned description is to build in *current conservation*. Indeed, forming the wave evaluation as the sum of terms (22.14), each being a solution of a different Dirac equation (22.18) has the disadvantage that the conservation of the Dirac current (which holds for each wave function ψ_a) no longer holds for the sum. This is not satisfying, also because we know from our general setup that, even in the setting of general quantum spacetimes, there should be a conserved inner product (namely the commutator inner product introduced in Section 9.4). In order to resolve this shortcoming, it is preferable to work with a single Dirac equation of the form

$$\big(\mathcal{D} + \mathcal{B} - m\big)\tilde{\Psi} = 0 \,. \tag{22.22}$$

This is indeed possible if the operator $\mathcal{B}$ is chosen as an integral operator with an integral kernel of the form

$$\mathcal{B}(x,y) = \sum_{\mathfrak{a}=1}^{N} \mathcal{B}_\mathfrak{a}\Big(\frac{x+y}{2}\Big)\, L_\mathfrak{a}(y-x) \,, \tag{22.23}$$

where $\mathcal{B}_\mathfrak{a}$ are again classical potentials and $L_\mathfrak{a}$ are certain symmetric kernels. In this description, there is a conserved current and a corresponding conserved inner product on the Dirac solutions, which has a similar structure as the commutator inner product (9.50). We refer the interested reader for detailed explanations to [24]. We finally remark that the nonlocal operator $\mathcal{B}$ of the form (22.23) composed of many potentials $\mathcal{B}_\mathfrak{a}$ was also derived in [52] by a thorough analysis of the linearized field equations for causal fermion systems describing Minkowski space. The existence theory for the Dirac equation involving integral operators is studied in Finster et al. [148].

22.3 A Distinguished Quantum State

The constructions outlined in the previous sections make it possible to construct general measures $\tilde{\rho}$, which go beyond measures describing a classical spacetime with classical bosonic fields. The EL equations for these measures can be understood as equations describing the dynamics in these generalized spacetimes. With this in mind, the remaining question is how to interpret the resulting measure $\tilde{\rho}$. Can it be understood in terms of an interaction via quantum fields? Or, in more physical terms, what does the measure $\tilde{\rho}$ tell us about measurements performed in the corresponding spacetime? In order to address these questions in a systematic way, in [58], a *distinguished quantum state* is constructed. It describes how the interacting measure $\tilde{\rho}$ looks like if measurements are performed at a given time using the objects of a causal fermion describing the vacuum. This "measurement" can also be understood more generally as a "comparison" of the measures $\tilde{\rho}$ and ρ at time t. In technical terms, the quantum state, denoted by ω^t, is a positive linear functional on the algebra of fields $\mathscr{A}$ of the noninteracting spacetime,

$$\omega : \mathscr{A} \to \mathbb{C} \qquad \text{with} \qquad \omega(A^* A) \geq 0 \quad \text{for all } A \in \mathscr{A} \,. \tag{22.24}$$

Here, we use the language of algebraic quantum field theory (as introduced, e.g., in the textbooks [4, 19, 132]), which seems most suitable for describing quantum fields in the needed generality. This notion of a quantum state is illustrated in Exercise 22.1.

We now outline the construction of the quantum state as given in [58]. We are given two causal fermion systems $(\tilde{\mathcal{H}}, \tilde{\mathcal{F}}, \tilde{\rho})$ and $(\mathcal{H}, \mathcal{F}, \rho)$ describing the interacting system and the vacuum, respectively. Our goal is to "compare" these causal fermion systems at a given time. In order to specify the time, we choose sets $\tilde{\Omega} \subset \tilde{M} := \operatorname{supp} \tilde{\rho}$ and $\Omega \subset M := \operatorname{supp} \rho$, which can be thought of as the past of this time in the respective spacetime. We want to relate the two causal fermions systems with the help of the nonlinear surface layer integral (9.62) introduced in Section 9.6. However, we need to take into account that the causal fermion systems are defined on two different Hilbert spaces $\tilde{\mathcal{H}}$ and $\mathcal{H}$. Therefore, in order to make sense of the nonlinear surface layer integral, we need to identify the Hilbert spaces $\mathcal{H}$ and $\tilde{\mathcal{H}}$ by a unitary transformation denoted by V,

$$V : \mathcal{H} \to \tilde{\mathcal{H}} \qquad \text{unitary} \,. \tag{22.25}$$

Then, the operators in $\tilde{F}$ can be identified with operators in $\mathcal{F}$ by the unitary transformation,

$$\mathcal{F} = V^{-1}\,\tilde{\mathcal{F}}\,V\,. \tag{22.26}$$

An important point to keep in mind is that this identification is not canonical, but it leaves the freedom to transform the operator V according to

$$V \to V\mathcal{U} \qquad \text{with} \qquad \mathcal{U} \in \mathrm{L}(\mathcal{H}) \text{ unitary}\,. \tag{22.27}$$

The freedom in choosing $\mathcal{U}$ must be taken into account in the nonlinear surface layer integral, which now takes the form

$$\begin{aligned} &\gamma^{\tilde{\Omega},\Omega}(\tilde{\rho}, \mathcal{U}\rho) \\ &\quad= \int_{\tilde{\Omega}} \mathrm{d}\tilde{\rho}(x) \int_{M\setminus\Omega} \mathrm{d}\rho(y)\; \mathcal{L}(x, \mathcal{U}y\mathcal{U}^{-1}) \\ &\qquad - \int_{\Omega} \mathrm{d}\rho(x) \int_{\tilde{M}\setminus\tilde{\Omega}} \mathrm{d}\tilde{\rho}(y)\; \mathcal{L}(\mathcal{U}y\mathcal{U}^{-1}, y)\,. \end{aligned} \tag{22.28}$$

The method for dealing with the freedom in choosing $\mathcal{U}$ is to integrate over the unitary group. Moreover, it is preferable to consider the exponential of the nonlinear surface layer integral. This leads us to introduce the *partition function* $Z^{\tilde{\Omega},\Omega}$ by

$$Z^{\tilde{\Omega},\Omega}\big(\beta, \tilde{\rho}\big) = \fint_{\mathcal{G}} \exp\Big(\beta\,\gamma^{\tilde{\Omega},\Omega}\big(\tilde{\rho}, \mathcal{U}\rho\big)\Big)\, \mathrm{d}\mu_{\mathcal{G}}(\mathcal{U})\,, \tag{22.29}$$

where $\mu_{\mathcal{G}}$ is the normalized Haar measure on the unitary group (in order for this Haar measure to be well defined, one needs to assume that the Hilbert space $\mathcal{H}$ is finite-dimensional, or else one must exhaust $\mathcal{H}$ by finite-dimensional subspaces).

In analogy to the path integral formulation of quantum field theory, the quantum state is obtained by introducing insertions into the integrand of the partition function, that is, symbolically,

$$\omega(\cdots) = \frac{1}{Z^{\tilde{\Omega},\Omega}\big(\beta, \tilde{\rho}\big)} \fint_{\mathcal{G}} (\cdots) \exp\Big(\beta\,\gamma^{\tilde{\Omega},\Omega}\big(\tilde{\rho}, \mathcal{U}\rho\big)\Big)\, \mathrm{d}\mu_{\mathcal{G}}(\mathcal{U})\,. \tag{22.30}$$

These insertions have the structure of surface layer integrals involving linearized solutions in the vacuum spacetime. Likewise, the argument of the state on the left-hand side is formed of operators that are parametrized by the same linearized solutions that enter the insertions on the right-hand side. More precisely, they are operators of the field algebra $\mathscr{A}$, being defined as the $*$-algebra generated by the linearized solutions, subject to the canonical commutation and anti-commutation relations. The commutation relations involve the causal fundamental solution of the linearized solutions, which can be constructed with energy methods as outlined in Chapter 14 (for details, see [22]). Likewise, for the anti-commutation relations, we use the causal fundamental solutions of the dynamical wave equation mentioned at the end of Section 9.4 in (9.49) (for more details, see [82]). The positivity property of the state is ensured by the specific form of the insertions. We refer the interested reader to [58]. We remark that, as is worked out in [84], the abovementioned quantum state allows for the description of general entanglement. Moreover, the dynamics of the quantum state is studied in [24].

We finally note that the definition of the partition function (22.29) and of the insertions in the definition of the state (22.30) bears a similarity with the path integral formulation of quantum theory (see, e.g., [110, 93]). However, this similarity does not seem to go beyond formal analogies. In particular, one should keep in mind that, in contrast to the integral over field configurations in the path integral formulation, in (22.29), one integrates over the unitary transformations arising from the freedom in identifying the Hilbert spaces $\mathcal{H}$ and $\tilde{\mathcal{H}}$ (see (22.27)). This is a major conceptual difference that, at least at present, prevents us from getting a tighter connection to path integrals and the functional integral approach.

22.4 Exercises

Exercise 22.1 The purpose of this exercise is to get familiar with the notion of a *quantum state* as defined by (22.24). In quantum mechanics, the system is usually described by a unit vector ψ in a Hilbert space $(\mathcal{H}, \langle .|. \rangle)$. An observable corresponds to a symmetric operator $A \in \mathrm{L}(\mathcal{H})$ on this Hilbert space (for simplicity, we here restrict attention to bounded operators). The expectation value of a measurement is given by the expectation $\langle \psi | A | \psi \rangle$.

(a) Show that the linear operator $W \in \mathrm{L}(\mathcal{H})$ defined by

$$W\phi = \langle \phi | \psi \rangle \, \psi \qquad \text{or, in bra/ket notation,} \qquad W = |\psi\rangle\langle\psi| , \tag{22.31}$$

is a projection operator (i.e., it is symmetric and idempotent). Show that the expectation value of a measurement can be written as

$$\langle \psi | A | \psi \rangle = \mathrm{tr}_{\mathcal{H}} \big(WA \big) . \tag{22.32}$$

(b) Show that the mapping

$$\omega \;:\; A \mapsto \mathrm{tr}_{\mathcal{H}} \big(WA \big) \tag{22.33}$$

is a quantum state in the sense (22.24) (here for the algebra $\mathscr{A}$, we take the *algebra of observables*, i.e., the set of all operators obtained from all observables by taking products and linear combinations).

(c) Let ψ_1 and ψ_2 be two distinct, nonzero vectors of $\mathcal{H}$. Show that, choosing

$$W := |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| , \tag{22.34}$$

the mapping (22.33) again defines a quantum state in the sense (22.24). Show that this quantum state cannot be written in the form (22.31). One refers to (22.31) as a *pure state*, whereas (22.34) is a *mixed state*.

(d) Is the quantum state in (c) properly normalized in the sense that $\omega(\mathbb{1}) = 1$? If not, how can this normalization be arranged?