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Free products and residual nilpotency

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Let G be the free product of groups G_α, $[G_\alpha]$ the cartesian subgroup of G and $k[G_\alpha]$ the intersection of $[G_\alpha]$ with the k-th term of the lower central series for G. Then the $k[G_\alpha]$ form a descending chain of subgroups of $[G_\alpha]$ and it is shown that if the intersection of all the subgroups in this chain is trivial then G and hence each G_α, is residually nilpotent. This answers a question of S. Moran.

Let G_α, $\alpha \in \Lambda$ be a set of non-trivial groups where the cardinality of Λ is at least 2 and let $G = \underset{\alpha\in\Lambda}{*} G_\alpha$ be the free product of the G_α. If $[G_\alpha]$ is the *cartesian subgroup* of G, that is the kernel of the map from G to the direct product of the G_α, we define subgroups $k[G_\alpha]$ of $[G_\alpha]$ for each integer $k \geqq 1$ as follows;

$$1[G_\alpha] = [G_\alpha] \ , \ k[G_\alpha] = [G \ , \ (k-1)[G_\alpha]] \ .$$

These subgroups were first introduced by Golovin and it follows from his results that

$$\gamma_k(G) \cap [G\ ] = k[G_\alpha] \tag{1}$$

where $\gamma_k(G)$ is the k-th term of the lower central series of G (see Moran [1], p. 559).

If we put $\omega[G_\alpha] = \bigcap_{k\geqq 1} k[G_\alpha]$

and $\omega G = \bigcap_{k\geqq 1} \gamma_k(G)$

then Moran ([1], p. 561, Theorem 8.6) proves the following theorem.

THEOREM 1. *If* G_α, $\alpha \in \Lambda$ *is a set of residually nilpotent groups such that either no* G_α *possesses generalized periodic elements or, for*

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some prime p *no* G_α *possesses elements of infinite* p*-height then* $\omega[G_\alpha] = 1$.

Moran states that he does not know if the condition of residual nilpotence for the G_α is necessary. Here we show that this is so.

THEOREM 2. *If* G *is the free product of a set of groups* G_α , $\alpha \in \Lambda$ *such that* $\omega[G_\alpha] = 1$ *then* $\omega G = 1$.

COROLLARY. *In the statement of Theorem 1 the condition that each* G_α *is residually nilpotent is necessary.*

Proof of the Corollary. If $\omega[G_\alpha] = 1$ then Theorem 2 implies that $\omega G = 1$ which is just the statement that G is residually nilpotent. Now any subgroup of a residually nilpotent group is also residually nilpotent so that each G_α , as a subgroup of G , is also residually nilpotent.

Proof of Theorem 2. Let $K = \omega G$; then $K \subseteq \gamma_k(G)$ for each k so that

$$\begin{aligned} K \cap [G_\alpha] &\subseteq \gamma_k(G) \cap [G_\alpha] \\ &\subseteq k[G_\alpha] \quad \text{from (1)} . \end{aligned}$$

This implies

$$K \cap [G_\alpha] \subseteq \omega[G_\alpha] = 1 ,$$

so that K and $[G_\alpha]$ are two normal subgroups of G with trivial intersection and therefore the normal subgroup M of G generated by K and $[G_\alpha]$ is their direct product.

Now the Kurosh Subgroup Theorem (MacLane [4]) implies that M is itself a free product of a free group and certain conjugates of subgroups of the factors G_α . However from Baer and Levi [2] a free product can not be expressed non-trivially as a direct product; so either M is infinite cyclic or, as M is normal in G , M is a subgroup of some factor G_α .

As $[G_\alpha] \subseteq M$ and $[G_\alpha]$ is a subgroup of G which intersects each factor G_α trivially, M must be infinite cyclic. Now as an infinite cyclic group is directly indecomposable and $[G_\alpha]$ is a free group of rank at least one (Dey [3]), $K = \omega G$ must be trivial as required.

References

[1] S. Moran, "Associative operations on groups II", *Proc. London Math. Soc. (3)* 8 (1958), 548-568.

[2] Reinhold Baer und Friedrich Levi, "Freie Produkte und ihre Untergruppen", *Compositio Math.* 3 (1936), 391-398.

[3] Ian M.S. Dey, "Relations between the free and direct products of groups", *Math. Zeitschr.* 80 (1962), 121-147.

[4] Saunders MacLane, "A proof of the subgroup theorem for free products", *Mathematika* 5 (1958), 13-19.

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