NEAR-RING HOMOMORPHISMS

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1. Introduction. Blackett [4] introduced the concepts of near-ring homomorphism and near-ring ideal. Beidleman [1] established the fundamental homomorphism theorem and the isomorphism theorems for (left) near-rings obeying the condition that $0.a = 0$ for every a in the near-ring. Several others, for example [3], [5], and [7], have taken up the study of ideals. This paper takes up the study of homomorphisms of (left) near-rings not subject to the condition $0.a = 0$. It is shown that such homomorphisms can be decomposed into homomorphisms of two special sub-near-rings. Conversely, conditions are sought under which homomorphisms of the two sub-near-rings may be mated to produce a homomorphism of the near-ring.

We require the following theorem which is proved in [2].

Near-ring Decomposition Theorem. Let R be a near-ring Each $r \varepsilon R$ has a unique decomposition in each of the forms

$$0r + (-0r + r) \text{ and } (r - 0r) + 0r.$$

Thus
$$R = R_z + R_c = R_c + R_z,$$

where $R_c = \{p \mid p \varepsilon R, 0p = 0\}$ and $R_z = \{0r \mid r \varepsilon R\}$. Therefore every near-ring may be expressed as a sum of its maximal sub-C-ring and its maximal sub-Z-ring.

2. Decomposition.

PROPOSITION 1. Let α be a near-ring homomorphism of R into N. Then the image of R_c is a sub-near-ring of

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N_c <u>and the image of</u> R_z <u>is a sub-near-ring of</u> N_z.

<u>Proof</u>. It is known that the image of a near-ring is a near-ring. Let $r \varepsilon R_c$. Then the first assertion follows from

$$0(\alpha r) = (\alpha 0)(\alpha r) = \alpha(0r) = \alpha 0 = 0.$$

For the second assertion, recall that $p \varepsilon R_z$ if and only if there exists a $q \varepsilon R$ such that $0q = p$, and consider

$$\alpha p = \alpha(0q) = (\alpha 0)(\alpha q) = 0(\alpha q).$$

In light of the near-ring decomposition theorem, we may say that a homomorphism from R to N is completely determined by its restrictions to R_c and R_z. So we may associate with each near-ring homomorphism of R a unique homomorphism pair; the first member of the pair being a near-ring homomorphism of R_c into N_c and the second member being a near-ring homomorphism of R_z into N_z. Now, the problem is to determine which such homomorphism pairs give rise to homomorphisms of R into N. First, let us emphasize the manner in which a homomorphism pair might "give rise" to a homomorphism.

Let α_1 and α_2 be, respectively, near-ring homomorphisms of R_c into N_c and R_z into N_z. If these are to be the restrictions of some near-ring homomorphism α, then α must be such that

$$\alpha r = \alpha(c + z) = \alpha c + \alpha z = \alpha_1 c + \alpha_2 z, \quad c \varepsilon R_c \text{ and } z \varepsilon R_z.$$

Moreover, the mapping is defined unambiguously. For if $c + z = z + d$, $d \varepsilon R_c$, then

$$\alpha(z + d) = \alpha(z + d - z + z) = \alpha(c + z).$$

When we speak of the map (and possible homomorphism) α on R to N arising from the pair α_1 and α_2, we use the notation $\alpha = [\alpha_1, \alpha_2]$ and mean the map defined by

$\alpha r = \alpha_1 c + \alpha_2 z$, where $r = c + z$, $c \in R_c$ and $z \in R_z$.

3. <u>An example</u>. The following example shows that a homomorphism pair need not give rise to a homomorphism.

Let $(R, +)$ be the group of order 8 with generating relations

$$4a = 2b = 0, \quad b + a = 3a + b.$$

For $x, y \in R$ define multiplication by

$$x.y = \begin{cases} 0, & y \in \{0, 2a, 3a + b, a + b\} \\ 2a + b, & y \in \{a, 3a, b, 2a + b\}. \end{cases}$$

Then $(R, +, .)$ is a near-ring with $R_c = \{0, 2a, 3a + b, a + b\}$ and $R_z = \{0, 2a + b\}$. On R_c consider the automorphism α_1 which fixes 0 and $a + b$ and permutes $3a + b$ and $2a$. On R_z take the identity map as α_2. If $\alpha = [\alpha_1, \alpha_2]$ is a near-ring homomorphism, we have

$$\alpha b = \alpha(2a + (2a + b)) = \alpha_1(2a) + \alpha_2(2a + b)$$

$$= (3a + b) + (2a + b) = a$$

and $$\alpha b = \alpha((2a + b) + 2a) = \alpha_2(2a + b) + \alpha_1(2a)$$

$$= (2a + b) + (3a + b) = 3a.$$

Thus $[\alpha_1, \alpha_2]$ is not a near-ring homomorphism.

4. <u>Conditions on the pair</u>. The α of the example failed to be a homomorphism because it did not behave properly on an element of the form $z + c$. The following proposition shows that this is, indeed, a critical question.

PROPOSITION 2. *A necessary and sufficient condition that* $\alpha = [\alpha_1, \alpha_2]$ *preserves addition is that*

$$\alpha(z + c) = \alpha_2 z + \alpha_1 c, \quad z \in R_z \quad \text{and} \quad c \in R_c.$$

Proof. Of course, if α preserves addition the implication is trivial.

Assume the stated equality. Recalling that $(R_c, +)$ is normal in $(R, +)$, we see that

$$\begin{aligned}
\alpha r_1 + \alpha r_2 &= \alpha_1 c_1 + \alpha_2 z_1 + \alpha_1 c_2 + \alpha_2 z_2 \\
&= \alpha_1 c_1 + \alpha(z_1 + c_2) + \alpha_2 z_2 \\
&= \alpha_1 c_1 + \alpha((z_1 + c_2 - z_1) + z_1) + \alpha_2 z_2 \\
&= \alpha_1 c_1 + \alpha_1(z_1 + c_2 - z_1) + \alpha_2 z_1 + \alpha_2 z_2 \\
&= \alpha_1(c_1 + z_1 + c_2 - z_1) + \alpha_2(z_1 + z_2) \\
&= \alpha(c_1 + z_1 + c_2 + z_2) = \alpha(r_1 + r_2).
\end{aligned}$$

Thus, preservation of addition follows from the given special case.

PROPOSITION 3. A necessary and sufficient condition that $\alpha = [\alpha_1, \alpha_2]$ preserves multiplication, given that α preserves addition, is that

$$\alpha(rd) = (\alpha r)(\alpha_1 d), \quad r \in R, \ d \in R_c.$$

Proof. Assume the equality holds and compare

$$\begin{aligned}
\alpha(r_1 r_2) &= \alpha((c_1 + z_1) c_2 + z_2) \\
&= \alpha((c_1 + z_1)c_2) + \alpha_2 z_2
\end{aligned}$$

with $$(\alpha r_1)(\alpha r_2) = (\alpha_1 c_1 + \alpha_2 z_1)(\alpha_1 c_2) + \alpha_2 z_2 .$$

Then, by the equality, $\alpha(r_1 r_2) = (\alpha r_1)(\alpha r_2)$.

5. Another example. In order to show that the above conditions may be fulfilled, we indicate a type of near-ring for which every homomorphism pair gives rise to a homomorphism.

Start with any C-ring R_c and any Z-ring R_z. Let (R, +) be the direct sum of $(R_c, +)$ and $(R_z, +)$. Then $(R_z, +)$ is normal in (R, +) and $c + z = z + c$ for every $c \in R_c$ and $z \in R_z$. Define multiplication in R by

$$r_1 . r_2 = (c_1 + z_1)(c_2 + z_2) = c_1 c_2 + z_2$$

It is easily checked that multiplication in R is associative and left-distributive over addition. For some N, let α_1 and α_2 have the usual meaning. It is easy to see that $\alpha = [\alpha_1, \alpha_2]$ is a near-ring homomorphism of R into N.

6. Homomorphisms on C- and Z- rings. In view of the decomposition of near-ring homomorphisms, it is appropriate to look at homomorphisms between C-rings and homomorphisms between Z-rings.

The case for Z-rings is easily treated. Let Z_1 and Z_2 be two Z-rings and let β be a homomorphism on $(Z_1, +)$ to $(Z_2, +)$. Note that

$$\beta(w. z) = \beta z = \beta w . \beta z, w, z \in Z_1 .$$

Thus any homomorphism on the additive group is a near-ring homomorphism. The following proposition shows that no such sweeping answer is possible for C-rings.

PROPOSITION 4. *Let (G, +) and (H, +) be groups and let α be a group homomorphism from G onto H. It is possible to define multiplications on G and H so that (G, +, .) and (H, +, .) are near-rings and α is a near-ring homomorphism.*

Proof. Let K be the (group) kernel of α and identify H with G/K. A multiplication on G is needed so that K becomes a near-ring ideal. Recall from [4] that a subset N of a

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near-ring R is an ideal if and only if

1) $(N, +)$ is a normal subgroup of $(R, +)$,

2) $RN \subseteq N$,

3) $(r_1 + n)r_2 - r_1 r_2 \in N$, r_1 and $r_2 \in R$, $n \in N$.

K, of course, satisfies 1) regardless of the multiplication used.

On G define a multiplication so that

$$x . y = \begin{cases} 0 & x \in K \text{ and } y \in G, \\ y & x \in G-K \text{ and } y \in G. \end{cases}$$

With this multiplication $(G, +, .)$ is a near-ring (see [6]). Also, condition 2) is immediate for K. Next,

$$(g_1 + k)g_2 - g_1 g_2 = 0 - 0 = 0 \in K, \qquad g_1 \in K,$$

and

$$= g_2 - g_2 = 0 \in K, \qquad g_1 \in G - K$$

Hence K is an ideal under this multiplication and G/K is a near-ring.

Lastly, to specifically show that α preserves multiplication, consider (keeping in mind that $(G, +, .)$ is a C-ring)

$$\alpha(xy) = \alpha 0 = \alpha x . \alpha y, \quad x \in K \text{ and } y \in G,$$

$$= \alpha y = \alpha x . \alpha y, \quad x \in G-K \text{ and } y \in G.$$

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