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MATRIX OPERATORS ON l_p TO l_q

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ABSTRACT. Workable necessary and sufficient conditions for a non-negative matrix to be a bounded operator from l_p to l_q when $1 < q \leq p < \infty$ are discussed. Alternative proofs are given for some known results, thereby filling a gap in the proof of the case $p = q$ of a result of Koskela's. The case $1 < q < p < \infty$ of Koskela's result is refined, and a weakened form of the Vere-Jones conjecture concerning matrix operators on l_p is shown to be false.

1. **Introduction.** Suppose throughout that $1 \leq p, q < \infty$, and write

$$p' := \frac{p}{p-1} \text{ when } p > 1, \quad \text{and } q' := \frac{q}{q-1} \text{ when } q > 1.$$

Suppose also that, unless otherwise stated, the indices of all sequences and matrices run through $\mathbb{N} := \{1, 2, \ldots\}$. Let l_p be the Banach space of all complex sequences $x := (x_j)$ with norm

$$\|x\|_p := \Big(\sum_{j=1}^{\infty} |x_j|^p\Big)^{1/p} < \infty.$$

Let $A := (a_{ij})$ be a real matrix. We say that $A \in (l_p, l_q)$ if for every $x := (x_j) \in l_p$, $y_i := \sum_{j=1}^{\infty} a_{ij}x_j$ is convergent for all $i \in \mathbb{N}$ and $Ax := y := (y_i) \in l_q$. We define

$$\|A\|_{p,q} := \sup_{\|x\|_p \leq 1} \|Ax\|_q,$$

so that $A \in (l_p, l_q)$ if and only if $\|A\|_{p,q} < \infty$, in which case $\|A\|_{p,q}$ is the p, q-norm of A. Following usual practice, we shall write $\|A\|_p$ for $\|A\|_{p,p}$. The matrix A is said to be *non-negative* (or *positive*) if $a_{ij} \geq 0$ (or $a_{ij} > 0$) for all $i, j \in \mathbb{N}$, and likewise a sequence $u := (u_j)$ is said to be non-negative (or positive) if $u_j \geq 0$ (or $u_j > 0$) for all $j \in \mathbb{N}$. To avoid trivial cases we assume in all that follows that no matrix A is identically zero.

The problem of obtaining workable necessary and sufficient conditions for $A \in (l_p, l_q)$ has been addressed by a number of authors. Ladyženskiĭ[7] proved the first part of the following result, and the complete result is essentially the case $p = q > 1$ of Koskela's Theorem 1 in [8].

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THEOREM A. *Let $p > 1$. Then a non-negative matrix $A := (a_{ij}) \in (l_p, l_p)$ if and only if there exist a positive number C and a positive sequence $u := (u_j)$ such that*

$$(1) \qquad \sum_{i=1}^{\infty} a_{ij} \Big(\sum_{k=1}^{\infty} a_{ik} u_k \Big)^{p-1} \le C u_j^{p-1}, \quad j = 1, 2, \ldots,$$

and then $\|A\|_p \le C^{1/p}$. Further, if the non-negative matrix $A \in (l_p, l_p)$, then there exists a positive sequence u for which (1) *holds with $C = (\|A\|_p)^p$.*

However, as noted in §3 below, Koskela's proof of the "only if" or necessity part of the above result is flawed. In this note we provide an alternative proof for the necessity part of Theorem A which also corrects the gap in Koskela's argument.

Koskela [8] (see also [1] and [10] for the sufficiency part) showed that the first part of Theorem A could be expressed in the more usable form:

THEOREM B. *Let $p > 1$. Then a non-negative matrix $A := (a_{ij}) \in (l_p, l_p)$ if and only if there exist positive numbers C_1 and C_2 and a positive sequence $u := (u_j)$ such that*

$$(2) \qquad \begin{cases} \sum_{j=1}^{\infty} a_{ij} u_j^{1/p} \le C_1 u_i^{1/p}, & i = 1, 2, \ldots, \\ \sum_{i=1}^{\infty} a_{ij} u_i^{1/p'} \le C_2 u_j^{1/p'}, & j = 1, 2, \ldots, \end{cases}$$

and then $\|A\|_p \le C_1^{1/p'} C_2^{1/p}$.

The sufficiency part of this result has proved particularly effective in establishing conditions for standard summability matrices to be in (l_p, l_p). (For Nörlund matrices see [3, 4, 6], and for Hausdorff matrices see [1,2,5].)

It was conjectured by Vere-Jones [10] that if $p > 1$ and the non-negative matrix $A \in (l_p, l_p)$, then there exists a positive sequence u for which (2) holds with $C_1 = C_2 = \|A\|_p$. Koskela [8] showed this conjecture to be false. A more compelling version of the conjecture would seem to be:

CONJECTURE V-J. If $p > 1$ and the non-negative matrix $A \in (l_p, l_p)$, then there exists a positive sequence u for which (2) holds with $C_1^{1/p'} C_2^{1/p} = \|A\|_p$.

We show in §5 that this conjecture is also false.

A useful adjunct to Theorem B is the following theorem which can sometimes be used (as in [2] and [3], for example) to show that $A \notin (l_p, l_p)$. It is essentially Lemma 2 of [2], and can be proved likewise. Its genesis is Theorem 4 of [1].

THEOREM C. *Let $A := (a_{ij})$ be a non-negative matrix, let (u_j) a bounded sequence of positive numbers such that $\sum u_j = \infty$, and let $\sigma_i := \sum_{j=1}^{\infty} a_{ij} (u_j / u_i)^{1/p}$ where $p > 1$. If $\sigma := \liminf_{i \to \infty} \sigma_i$, then $\|A\|_p \ge \sigma$. In particular, if $\sigma = \infty$, then $A \notin (l_p, l_p)$.*

In [8] Koskela gave essentially the following theorem concerning conditions for non-negative matrices to be in (l_p, l_q) with $p > q > 1$:

THEOREM D. *Let $p > q > 1$. Then a non-negative matrix $A := (a_{ij}) \in (l_p, l_q)$ if and only if there exist a positive constant C and a positive sequence $u := (u_j)$ with the following properties:*

(a) $\|u\|_p \leq 1$;

(b)

$$\sum_{i=1}^{\infty} a_{ij}\Big(\sum_{k=1}^{\infty} a_{ik}u_k\Big)^{q-1} \leq Cu_j^{p-1}, \quad j = 1, 2, \ldots; \tag{3}$$

and then $C^{1/q} \geq \|A\|_{p,q}$. Further, if the non-negative matrix $A \in (l_p, l_q)$, then there exists a non-negative sequence $u := (u_j)$ with $0 < \|u\|_p \leq 1$ for which (3) *holds with $C = (\|A\|_{p,q})^q$, and $u_j = 0$ only when $a_{ij} = 0$ for every $i \in \mathbb{N}$.*

Koskela stated the theorem with the sequence u non-negative throughout and with the following property in addition to (a) and (b):

(c) $u_j = 0$ *if and only if* $a_{ij} = 0$ *for every* $i \in \mathbb{N}$.

Since u_j can be given any positive value when the j-th column of A is identically zero without affecting the validity of (3), Koskela's original statement of his theorem is equivalent to the above.

In §3 of this note we provide an alternative proof for the necessity part of Theorem D. In addition, we show in §4 that when the non-negative matrix $A \in (l_p, l_q)$ with $p > q > 1$, and A has no zero columns, then there exists a positive sequence u with $\|u\|_p = 1$ which satisfies (3) with "$=$" in place of "$\leq$" and $C = (\|A\|_{p,q})^q$, but that there may be no such sequence when $p = q > 1$.

The following theorem shows directly that the necessity part of Theorem A is true with any $C > (\|A\|_p)^p$, and that the necessity part of Theorem D is true with any $C > (\|A\|_{p,q})^q$ even when $q > p$. The proof that we exhibit in §2 is an adaptation of that of Theorem 7.1.6 in [9]. We are indebted to Gordon Sinnamon for elaborating the details.

THEOREM 1. *Suppose that $p, q > 1$, that the non-negative matrix $A := (a_{ij}) \in (l_p, l_q)$, and that $C > (\|A\|_{p,q})^q$. Then there exists a positive sequence $u := (u_j)$ such that $\|u\|_p \leq 1$ and* (3) *is true.*

2. **Preliminary results and proof of Theorem 1.** In order to prove Theorem 1 we first adopt some notations and prove two lemmas. We define B_p^+ to be the set of non-negative sequences u with $\|u\|_p \leq 1$; and $E_r u := (u_j^r)$ for $u := (u_j) \geq 0$.

LEMMA 1. *Let $p \geq 1$. Suppose that S is a continuous, order preserving map from B_p^+ to B_p^+ and that $0 < t < 1$. Then there exists a positive $u \in B_p^+$ such that $tSu < u$.*

PROOF. Choose $u^{(1)} \in B_p^+$ such that $u^{(1)} > 0$ and $\|u^{(1)}\|_p = 1 - t$. For $n \in \mathbb{N}$, define $u^{(n+1)} := u^{(1)} + tSu^{(n)} > 0$. Note that if $u^{(n)} \in B_p^+$ for any $n \in \mathbb{N}$, then $Su^{(n)} \in B_p^+$, so $\|u^{(n+1)}\|_p \leq \|u^{(1)}\|_p + t\|Su^{(n)}\|_p \leq 1 - t + t = 1$, and hence $u^{(n+1)} \in B_p^+$. Also $u^{(2)} - u^{(1)} = tSu^{(1)} \geq 0$, and if $u^{(n+1)} - u^{(n)} \geq 0$ for any $n \in \mathbb{N}$, then $u^{(n+2)} - u^{(n+1)} = t(Su^{(n+1)} - Su^{(n)}) \geq 0$ since S is order preserving. It follows that the sequence of sequences $(u^{(n)})$ is term-wise

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non-decreasing in B_p^+. By the monotonic convergence theorem, u, the term-wise limit of $u^{(n)}$, is also the l_p-limit of $(u^{(n)})$. Hence $u \in B_p^+$, and since S is continuous, we have that $u = u^{(1)} + tSu > tSu \geq 0$ as required. ∎

LEMMA 2. *Let $p \geq 1$ and $r > 0$. Then E_r is a continuous, order preserving map from B_{pr}^+ to B_p^+.*

PROOF. Only the continuity of E_r is not immediately evident.

Let $x := (x_j)$ and $y := (y_j)$ be sequences in B_{pr}^+.

If $r \leq 1$, then $|x_j^r - y_j^r| \leq |x_j - y_j|^r$ by basic calculus, and so

$$\|E_r x - E_r y\|_p = \Big(\sum_{j=1}^{\infty} |x_j^r - y_j^r|^p\Big)^{1/p} \leq \Big(\sum_{j=1}^{\infty} |x_j - y_j|^{pr}\Big)^{1/p} = (\|x - y\|_{pr})^r.$$

If $r > 1$, then $|x_j^r - y_j^r| \leq r(x_j + y_j)^{r-1}|x_j - y_j|$ by the mean value theorem, and so, using Hölder's and Minkowski's inequalities, we get the estimate

$$\begin{aligned}\|E_r x - E_r y\|_p &= \Big(\sum_{j=1}^{\infty} |x_j^r - y_j^r|^p\Big)^{1/p} \leq \Big(\sum_{j=1}^{\infty} r^p (x_j + y_j)^{p(r-1)} |x_j - y_j|^p\Big)^{1/p} \\ &\leq r\Big(\sum_{j=1}^{\infty} (x_j + y_j)^{pr}\Big)^{(r-1)/pr} \Big(\sum_{j=1}^{\infty} |x_j - y_j|^{pr}\Big)^{1/pr} \\ &\leq r(\|x\|_{pr} + \|y\|_{pr})^{r-1} \|x - y\|_{pr}.\end{aligned}$$

It follows that E_r is a continuous map from B_{pr}^+ to B_r^+. ∎

PROOF OF THEOREM 1. Since $\|A\|_{p,q} > 0$, we can divide A by $\|A\|_{p,q}$ and thereby reduce the problem to the case $\|A\|_{p,q} = 1$. Note that the transpose matrix A^* satisfies $\|A^*\|_{q',p'} = 1$. Further, A is a continuous, order preserving map from B_p^+ to B_q^+ and A^* is an order preserving map from $B_{q'}^+$ to $B_{p'}^+$. Therefore

$$S := E_{p'/p} A^* E_{q/q'} A$$

is a continuous, order preserving map from B_p^+ to B_p^+. Let $0 < t < 1$. Then, by Lemma 1, there is a positive $u := (u_j) \in B_p^+$ such that $tSu < u$, that is

$$t\Big(\sum_{i=1}^{\infty} a_{ij} \Big(\sum_{k=1}^{\infty} a_{ik} u_k\Big)^{q/q'}\Big)^{p'/p} < u_j, \quad j = 1, 2, \ldots,$$

and therefore

$$\sum_{i=1}^{\infty} a_{ij} \Big(\sum_{k=1}^{\infty} a_{ik} u_k\Big)^{q-1} < t^{1-p} u_j^{p-1}, \quad j = 1, 2, \ldots.$$ ∎

3. **Alternative proofs of Theorems A and D.** While the case $p = q$ of Koskela's Theorem 1 in [8] (our Theorem A above) is certainly correct, the original proof in [8] of the necessity part of it was flawed for the following reason:

When A is positive, then a positive sequence (u_j) satisfying (1) can be found by the method used in [8]. For a general non-negative matrix A, Koskela suggested first applying the result to $A + \epsilon B$, where $\epsilon > 0$ and B is a fixed positive matrix in (l_p, l_p), to obtain a positive sequence $(u_j^{(\epsilon)})$ satisfying the appropriate version of (1), and then using a simple continuity argument. He gave no indication, however, of how to prevent $\liminf_{\epsilon \to 0+} u_j^{(\epsilon)}$ from being infinite or 0.

The proof of the sufficiency part of Theorem 1 in [8] involves only straightforward applications of Hölder's inequality, and serves to establish the sufficiency parts of Theorems A and D.

In §3 we use Theorem 1 to provide alternative proofs of the necessity parts of Theorems A and D, which show, inter alia, how to avoid the sort of difficulty mentioned above in the case of Theorem A. To deal with this difficulty we introduce an equivalence relation "$\sim$", and prove an additional lemma.

Given a non-negative matrix $A = (a_{ij})$, let $\mathbb{N}_+$ be the set of positive integers j such that $a_{ij} > 0$ for some $i \in \mathbb{N}$. We define an equivalence relation "$\sim$" on $\mathbb{N}_+$ as follows: For $j, k \in \mathbb{N}_+$, we say that $j \sim k$ either if $j = k$, or if there is a chain of distinct positive integers $j_1, j_2, \ldots, j_{r-1}, j_r$ such that $j = j_1$, $k = j_r$ and, for each $\nu \in \{1, 2, \ldots, r-1\}$, there is an $i_\nu \in \mathbb{N}$ such that $a_{i_\nu j_\nu} > 0$ and $a_{i_\nu j_{\nu+1}} > 0$.

LEMMA 3. *Let $p > 1$. Suppose that $A := (a_{ij})$ is a non-negative matrix, that $C > 0$, and that $u := (u_j)$ is a positive sequence satisfying*

$$\sum_{i=1}^{\infty} a_{ij} \Big(\sum_{k=1}^{\infty} a_{ik} u_k \Big)^{p-1} \leq C u_j^{p-1}, \quad j = 1, 2, \ldots. \tag{4}$$

Then

(i) for any fixed $m \in \mathbb{N}_+$ and $a > 0$, (4) continues to hold if u is replaced by $v := (v_j)$ where

$$v_j := \begin{cases} u_j & \text{if } j \not\sim m, \\ a u_j & \text{if } j \sim m; \end{cases}$$

(ii) for fixed $j, k \in \mathbb{N}_+$ with $j \sim k$ and $j \neq k$, there is a positive integer r and positive constants K_1, K_2 such that

$$K_1 C^{-r(p'-1)} u_k \leq u_j \leq K_2 C^{r(p'-1)} u_k.$$

PROOF. (i) Let $\mathbb{N}_m := \{k \in \mathbb{N}_+ \mid k \sim m\}$. Then

$$\sum_{i=1}^{\infty} a_{ij} \Big(\sum_{k=1}^{\infty} a_{ik} v_k \Big)^{p-1} = \sum_{i=1}^{\infty} \Big(\sum_{k \in \mathbb{N}_m} a_{ij}^{p'-1} a_{ik} a u_k + \sum_{k \notin \mathbb{N}_m} a_{ij}^{p'-1} a_{ik} u_k \Big)^{p-1},$$

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from which the desired result follows since

$$\sum_{k\notin \mathbb{N}_m} a_{ij}^{p'-1} a_{ik} u_k = 0 \text{ when } j \sim m, \text{ and } \sum_{k\in \mathbb{N}_m} a_{ij}^{p'-1} a_{ik} a u_k = 0 \text{ when } j \not\sim m.$$

(ii) Let $j = j_1, j_2, \ldots, j_{r-1}, j_r = k$ be the chain of integers and $i_1, i_2, \ldots, i_{r-1}, i_r$ the corresponding indices of the definition of $j \sim k$. It follows from (4) that, for $\nu = 1, 2, \ldots, r-1$,

$$u_{j_\nu}^{p-1} \geq C^{-1} \sum_{i=1}^{\infty} a_{ij} \Big(\sum_{k=1}^{\infty} a_{ik} u_k \Big)^{p-1} \geq C^{-1} a_{i_\nu j_\nu} a_{i_\nu j_{\nu+1}}^{p-1} u_{j_{\nu+1}}^{p-1} > 0.$$

Combining these inequalities we see that $u_j \geq K_1 C^{-r(p'-1)} u_k$ for some positive constant K_1. Likewise there is a positive constant K_2 such that $u_k \geq K_2^{-1} C^{-r(p'-1)} u_j$. ■

PROOFS OF THE NECESSITY PARTS OF THEOREMS A AND D. Suppose that $p \geq q > 1$ and that the non-negative matrix $A := (a_{ij}) \in (l_p, l_q)$. Let $C_n := (\|A\|_{p,q})^p + 1/n$ for $n \in \mathbb{N}$. Then, by Theorem 1, there is a positive sequence $u^{(n)} := (u_j^{(n)})$ such that $\|u^{(n)}\|_p \leq 1$ and

$$\sum_{i=1}^{\infty} a_{ij} \Big(\sum_{k=1}^{\infty} a_{ik} u_k^{(n)} \Big)^{q-1} \leq C_n (u_j^{(n)})^{p-1}, \quad j = 1, 2, \ldots. \tag{5}$$

CASE 1. Let $p > q > 1$. Define $u := (u_j)$ where $u_j := \liminf_{n\to\infty} u_j^{(n)}$. Then $\|u\|_p \leq 1$ and

$$\sum_{i=1}^{\infty} a_{ij} \Big(\sum_{k=1}^{\infty} a_{ik} u_k \Big)^{q-1} \leq (\|A\|_{p,q})^p u_j^{p-1}, \quad j = 1, 2, \ldots. \tag{6}$$

Hence, for every $i \in \mathbb{N}$, $a_{ij}^q \leq (\|A\|_{p,q})^p u_j^{p-q}$. It follows that $u_j > 0$ whenever $j \in \mathbb{N}_+$. The above process could yield $u_j = 0$, but only when $j \in \mathbb{N} \setminus \mathbb{N}_+$, that is, when the j-th column of A is identically 0. This establishes the necessity part of Theorem D.

CASE 2. Let $p = q > 1$. Let $\mathbb{N}'$ be the set of first elements in the equivalence classes associated with the equivalence relation "$\sim$" on $\mathbb{N}_+$. For each $k \in \mathbb{N}'$ and $j \sim k$ divide $u_j^{(n)}$ by $u_k^{(n)}$ which, by Lemma 3(i), we can do without affecting the validity of (6). Thus we now have $u_k^{(n)} = 1$ for all $k \in \mathbb{N}'$. Also, by Lemma 3(ii), we have, for fixed distinct $j, k \in \mathbb{N}_+$ with $k \in \mathbb{N}'$ and $j \sim k$, that there is a positive integer r and positive constants K_1, K_2 such that

$$K_1 C_n^{-r(p'-1)} \leq u_j^{(n)} \leq K_2 C_n^{r(p'-1)}.$$

Define

$$u_j := \begin{cases} \liminf_{n\to\infty} u_j^{(n)} & \text{for } j \in \mathbb{N}_+, \\ 1 & \text{for } j \in \mathbb{N} \setminus \mathbb{N}_+. \end{cases}$$

Then $\infty > u_j > 0$ for all $j \in \mathbb{N}$, and $u := (u_j)$ is a positive sequence satisfying (1) with $C = (\|A\|_{p,p})^p$. Note that, for $j \in \mathbb{N} \setminus \mathbb{N}_+$, we could have defined u_j to be any positive number. This completes the proof of the necessity part of Theorem A. ■

4. **Refinement of Theorem D.**

THEOREM 2. *Let $1 < q < p < \infty$, and suppose that the non-negative matrix $A := (a_{ij}) \in (l_p, l_q)$. Then there exists a non-negative sequence $u := (u_j)$ having the following properties:*

(a) $\|u\|_p = 1$;

(b) $u_j = 0$ *only if* $a_{ij} = 0$ *for every* $i \in \mathbb{N}$;

(c) for each $j \in \mathbb{N}$,

$$\sum_{i=1}^{\infty} a_{ij} \Big(\sum_{k=1}^{\infty} a_{ik} u_k \Big)^{q-1} = \|A\|_{p,q}^{q} u_j^{p-1}. \tag{7}$$

PROOF. Let $C := (\|A\|_{p,q})^q$, and define $fu := (f_j u)$ where

$$f_j u := \sum_{i=1}^{\infty} a_{ij} \Big(\sum_{k=1}^{\infty} a_{ik} u_k \Big)^{q-1} \text{ for } j \in \mathbb{N}.$$

By Theorem D, there exists a non-negative sequence $u := (u_j)$ such that $0 < \|u\|_p \leq 1$ and $fu \leq Cu^{p-1}$, C being the smallest possible constant for which such an inequality can hold. If $\|u\|_p < 1$, then pick a constant $t > 1$ satisfying $t\|u\|_p \leq 1$ and let $v := tu$. We then get $fv \leq t^{q-1} Cu^{p-1} = t^{q-p} Cv^{p-1}$, which is a contradiction because $t^{q-p} < 1$. This proves that $\|u\|_p = 1$.

Now assume that there exists $j_0 \in \mathbb{N}$ such that $f_{j_0} u < Cu_{j_0}^{p-1}$, then on replacing u_{j_0} by λu_{j_0} for a λ less than and close enough to 1, we get a new u with p-norm less than 1 for which $fu \leq Cu^{p-1}$. But this is impossible by what we proved in the previous paragraph. Hence we must have $f_j u = Cu_j^{p-1}$ for every $j \in \mathbb{N}$. ∎

That the above theorem does not hold for $p = q$ is shown by the following example involving a matrix A with no zero columns:

Let $p = q > 1$, and let $A := (a_{ij})$ with $a_{11} :=2$, $a_{ii} := 1$ for $i = 2, 3, \ldots$, and all other $a_{ij} := 0$. Then, for $j = 1, 2, \ldots$,

$$\sum_{i=1}^{\infty} a_{ij} \Big(\sum_{k=1}^{\infty} a_{ik} u_k \Big)^{p-1} = \lambda_j u_j^{p-1}$$

where $\lambda_1 = 2^p$ and $\lambda_j = 1$ for $j = 2, 3, \ldots$. So it is impossible to have a positive sequence $u := (u_j)$ satisfying (7).

5. **Weakened form of the Vere-Jones conjecture.**

THEOREM 3. *Let $p > 1$. If Conjecture V-J holds for $A + E$, where $A := (a_{ij})$ is a non-negative matrix in (l_p, l_p) and either*

(i) E is the infinite identity matrix, or

(ii) $E := (e_{ij})$ with $e_{ii} := 1$ for $i = 1, 2, \ldots, n$ and all other $e_{ij} := 0$, and $a_{ij} := 0$ for

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all $i,j > n$, then

$$\|A+E\|_p = \|A\|_p + 1.$$

PROOF. Let $C := \|A + E\|_p$. Applying Theorem B to $A + E$, we see that there is a positive sequence $u := (u_j)$ such that

$$\sum_{j=1}^{\infty} a_{ij} u_j^{1/p} \le (C_1 - 1) u_i^{1/p}, \quad i = 1, 2, \ldots,$$

$$\sum_{i=1}^{\infty} a_{ij} u_i^{1/p'} \le (C_2 - 1) u_j^{1/p'}, \quad j = 1, 2, \ldots,$$

with $C = C_1^{1/p'} C_2^{1/p}$. By Theorem B and Hölder's inequality, we get

$$\|A\|_p \le (C_1 - 1)^{1/p'} (C_2 - 1)^{1/p} \le C - 1 = \|A+E\|_p - 1 \le \|A\|_p. \qquad \blacksquare$$

To show that Conjecture V-J is not true in general, we require, in addition to Theorem 3, the following proposition concerning $n \times n$ matrices which is due to Koskela [8]. It should be noted that the p-norm of an $n \times n$ matrix $A := (a_{ij})$ with respect to the l_p space of n-tuples is the same as the p-norm of the infinite form of that matrix obtained by setting $a_{ij} := 0$ for all $i,j > n$.

PROPOSITION. *Let $p > 1$, let I denote the unit $n \times n$ matrix, and let A be a non-negative $n \times n$ matrix. Then*

$$\|A+I\|_p = \|A\|_p + 1$$

if and only if $\|A\|_p = \lambda_A$, the greatest non-negative eigenvalue of A.

Since there are non-negative $n \times n$ matrices A with greatest non-negative eigenvalues $\lambda_A < \|A\|_p$, the failure, in general, of Conjecture V-J follows from Theorem 3(ii) and the proposition. A simple example of such a matrix is given by

$$A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

for which $\lambda_A = 0 < \|A\|_p = 1$.

REFERENCES

1. D. Borwein and A. Jakimovski, *Matrix operators on l_p*, Rocky Mountain J. Math. **9**(1979), 463–476.

2. D. Borwein, *Generalized Hausdorff matrices as bounded operators on l_p*, Math. Z. **183**(1983), 483–487.

3. D. Borwein and F. P. Cass, *Nörlund matrices as bounded operators on l_p*, Arch. Math. **42**(1984), 464–469.

4. D. Borwein, *Nörlund operators on l_p*, Canad. Math. Bull. (1) **36**(1993), 8–14.

5. D. Borwein and X. Gao, *Generalized Hausdorff and weighted mean matrices as operators on l_p*, J. Math. Anal. Appl. **178**(1993), 517–528.

6. F. P. Cass and W. Kratz, *Nörlund and weighted mean matrices as bounded operators on l_p*, Rocky Mountain J. Math. **20**(1990), 59–74.

7. L. A. Ladyženskiĭ, *Über ein Lemma von Schur*, Latviĭsk Mat. Ežegodnik **9**(1971), 139–150.

8. M. Koskela, *A characterization of non-negative matrix operators on l_p to l_q with $\infty > p \ge q > 1$*, Pacific J. Math. **75**(1978), 165–169.

9. P. Szeptycki, *Dissertationes Mathematicae*, Warsaw, Notes on Integral Transformations, 1984.
10. D. Vere-Jones, *Ergodic properties of non-negative matrices-II,* Pacific J. Math. **26**(1968), 601-620.

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