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SOME OBSTACLES TO DUALITY IN TOPOLOGICAL ALGEBRA

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0. Introduction. Functors $\Phi : \mathscr{A} \rightarrow \mathscr{B}$, $\Gamma : \mathscr{B} \rightarrow \mathscr{A}$ form an *equivalence of categories* (see [**8**]) if $\Gamma(\Phi(A)) \cong A$ and $\Phi(\Gamma(B)) \cong B$ naturally for all objects A from $\mathscr{A}$ and B from $\mathscr{B}$. Letting $\mathscr{A}^*$ denote the opposite of $\mathscr{A}$, we say that $\mathscr{A}$ and $\mathscr{B}$ are *dual* if there is an equivalence between $\mathscr{A}^*$ and $\mathscr{B}$.

Let τ be a similarity type of finitary operation symbols. We let L_τ denote the first order language (with equality) using nonlogical symbols from τ, and consider the class $\mathscr{M}_\tau$ of all algebras of type τ as a category by declaring the morphisms to be all homomorphisms in the usual sense (i.e., those functions preserving the atomic sentences of L_τ). If $\mathscr{K}$ is a class in $\mathscr{M}_\tau$ (i.e., $\mathscr{K} \subseteq \mathscr{M}_\tau$ and $\mathscr{K}$ is closed under isomorphism), we view $\mathscr{K}$ as a full subcategory of $\mathscr{M}_\tau$, and we define the *order* of $\mathscr{K}$ to be the number of symbols occurring in τ.

If $\mathscr{S}$ is a class of topological spaces and $\mathscr{K}$ is a class of algebras, let $\mathscr{S} \cdot \mathscr{K}$ denote the category of "$\mathscr{S}$-topological $\mathscr{K}$-algebras" (i.e., the topologies are in $\mathscr{S}$, the algebras are in $\mathscr{K}$, and the operations are jointly continuous) plus continuous homomorphisms. A *dual pair*, for our purposes, is simply a pair $(\mathscr{S} \cdot \mathscr{K}, \mathscr{T} \cdot \mathscr{L})$ where the categories are dual to one another. (If S denotes the category of sets, we treat $S \cdot \mathscr{K}$ and $\mathscr{T} \cdot S$ as identical with $\mathscr{K}$ and $\mathscr{T}$ respectively.) Beautiful examples of dual pairs in topological algebra are well known (see (0.1) below), and it is our intention in this paper to underscore the special nature of some of these examples by laying down fairly general conditions on the classes $\mathscr{S}, \mathscr{T}, \mathscr{K}$, and $\mathscr{L}$ ensuring the nonexistence of a duality between $\mathscr{S} \cdot \mathscr{K}$ and $\mathscr{T} \cdot \mathscr{L}$.

In the following, certain categories of special interest will be given abbreviations.

(i) LCH = {locally compact Hausdorff spaces}, CH = {compact Hausdorff spaces}, and CCH, ZDCH, and EDCH denote respectively the connected, 0-dimensional, and extremally disconnected objects in CH.

(ii) AG = {abelian groups}, and TAG and TFAG denote respectively the torsion and torsion free objects in AG.

(iii) SL = {semilattices}.

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(iv) BA = {Boolean algebras}, and CBA denotes the complete objects in BA.

(v) RCF = {rings of continuous real-valued functions with domains in CH}, and IFRCF denotes the idempotent-free objects in RCF.

0.1. *Examples*. The following are dual pairs.

(i) ([**9**]): (LCH·AG, LCH·AG), (CH·AG, AG), (CCH·AG, TFAG), (ZDCH·AG, TAG).

(ii) ([**10**]): (ZDCH·SL, SL).

(iii) ([**14**]): (ZDCH, BA), (EDCH, CBA).

(iv) (A.O. Gel'fand, A. N. Kolmogorov): (CH, RCF), (CCH, IFRCF).

Note that in all of the above pairs $(\mathscr{S}\cdot\mathscr{K}, \mathscr{T}\cdot\mathscr{L})$, the classes $\mathscr{K}$ are equational classes; and in all but one example, $\mathscr{T} = \mathscr{S}$. In [**3**] we used the ultraproduct construction in a categorical setting (see also [**1**], [**4**], [**6**], [**12**]) to prove that there can be no dual pairs where $\mathscr{S} = \mathscr{T} = S$ and $\mathscr{K}$, $\mathscr{L}$ are nontrivial elementary SP-classes (see Section 1 for terminology). Here we continue to use ultraproducts and other techniques (e.g. the existence of enough free objects) to examine those of the above dualities in which $\mathscr{L}$ is not a nice elementary class and show that $\mathscr{L}$ cannot be replaced by another class $\mathscr{L}'$ which in some sense "improves matters". The following is a summary of our results.

0.2. THEOREM. (i) ZDCH·AG *is not dual to either an elementary* P-*class or a class with representable underlying set functor ("representable class" for short).*

(ii) EDCH *is not dual to either an elementary* P-*class or an* SP-*class of order* $<c$ (= *the cardinality of the real line*).

(iii) CCH *is not dual to a* P-*class.*

(iv) CH *is not dual to either a representable elementary* P-*class* (*e.g. an elementary* SP-*class, a universal Horn class*) *or a representable class of order* $<c$.

(v) *If* CH *is dual to an elementary* P-*class then* CCH *is dual to an elementary class.*

In connection with (0.2 (iv)) above, a question which has stubbornly remained open is whether CH is dual to an elementary P-class. Trying to deal with this problem has led to results of independent interest (see [**4**]). It is interesting to note that, although CH is not dual to an equational class in the usual (finitary) sense, its dual category is monadic, hence "like" an equational class from a more abstract categorical viewpoint.

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1. Preliminaries. We specify classes $K \subseteq \mathcal{M}_\tau$ using closure conditions as well as "syntactic" conditions, to wit: $\mathcal{K}$ is a P-*class* (resp. S-*class*, H-*class*, SP-*class*, etc.) if $\mathcal{K}$ is closed under the formation of arbitrary direct products (resp. subalgebras, homomorphic images, both subalgebras and products, etc.); $\mathcal{K}$ is an *elementary class* (resp. *Horn class*, *universal Horn class*, *equational class*, etc.) if $\mathcal{K}$ = the models of a set of L_τ-sentences (resp. Horn sentences, universal Horn sentences, equations, etc. (see [**5**] for definitions of syntactic notions)). We are thus able to talk about elementary P-classes, Horn S-classes, and the like. Among the well-known relations involving these sorts of classes are: all Horn classes are elementary P-classes; all universal Horn classes are SP-classes; and the equational classes are precisely the HSP-classes.

One important feature of SP-classes is that their underlying set functors are representable (i.e., free algebras over singleton sets exist); in fact, their underlying set functors have left adjoints (i.e., free algebras over arbitrary sets exist). We will not use the full power of this latter fact, however.

The key feature of elementary P-classes, from our standpoint, is that they admit an "ultraproduct construction" (see also [**1**], [**3**], [**4**], [**6**], [**12**]) which behaves nicely. In particular, let $\mathcal{A}$ be a category with products and let $\langle A_i : i \in I \rangle$ be an indexed family of $\mathcal{A}$-objects, with D an ultrafilter of subsets of I. Then the *D-ultraproduct* $\prod_D^{\mathcal{A}} A_i$ is the direct limit, when it exists, of the functor $\Phi : D \to \mathcal{A}$; where, for $J \in D$, $\Phi(J) = \prod_{i \in J}^{\mathcal{A}} A_i$; and, whenever $J \supseteq K \in D$, $\Phi(J, K)$ is the natural "projection" from $\prod_{i \in J}^{\mathcal{A}} A_i$ to $\prod_{i \in K}^{\mathcal{A}} A_i$. When $A_i = A$ for each $i \in I$, we let $A_{\mathcal{A}}{}^{(D)}$ denote $\prod_D^{\mathcal{A}} A_i$. $A_{\mathcal{A}}{}^{(D)}$ is called the *D-ultrapower* of A, and there is always a naturally defined "D-diagonal" morphism $\Delta_D : A \to A_{\mathcal{A}}{}^{(D)}$. The following is well known.

1.1. THEOREM. *Let τ be a finitary similarity type. Then ultraproducts in $\mathcal{M}_\tau$ are always defined and equal to the usual model theoretic ones (à la Łoś, et al). If $\mathcal{K} \subseteq \mathcal{M}_\tau$ is an elementary* P-*class then the ultraproduct in $\mathcal{K}$ is the one inherited from $\mathcal{M}_\tau$ ($\prod_D^{\mathcal{K}} A_i = \prod_D A_i$). Furthermore, the diagonal morphism $\Delta_D : A \to A^{(D)}$ is always an elementary embedding (hence in particular a monomorphism), and is an isomorphism for all ultrafilters D if and only if (the underlying set of) A is finite (notation: $|A|$ will denote the cardinality of the underlying set of A).*

1.2. *Remark.* In [**6**], a notion of finiteness in a category $\mathcal{A}$ is defined using ultrapowers: $A \in \text{Ob}(\mathcal{A})$ is $\mathcal{A}$*-ultrafinite* if $\Delta_D : A \to A_{\mathcal{A}}{}^{(D)}$ is an isomorphism for all ultrafilters D. They give examples of concrete categories $\mathcal{A}$ in which infinite objects are $\mathcal{A}$-ultrafinite, and others in which finite objects are not $\mathscr{A}$-ultrafinite. We will return to this concept (and its dual notion) later on.

2. Subcategories of CH•AG. A well known consequence of the Pontryagin-van Kampen duality is that if G is a compact Hausdorff abelian group and if A is the (discrete) group of characters of G then G is connected if and only if A is torsion free and G is 0-dimensional if and only if A is a torsion group. Now TFAG is not an equational class in the type of groups; but it is a universal Horn class, which is almost as good. (We do not know whether there is a dual pair (CCH·AG, $\mathscr{L}$) where $\mathscr{L}$ is an equational class, as there do not seem to be any known categorical tools which distinguish universal Horn classes from equational ones.)

The situation involving ZDCH·AG is in marked contrast to that involving CCH·AG (as well as ZDCH·SL and ZDCH). We will have established (0.2 (i)) once we prove the following two theorems.

2.1. THEOREM. TAG *is not equivalent to an elementary* P-*class.*

Proof. First of all, TAG does indeed have ultraproducts: simply take the torsion subgroup of the usual ultraproduct. Moreover, it is proved in [**6**] that a torsion group A is TAG-ultrafinite if, and only if, for all positive integers n, $A_n = \{a \in A : na = 0\}$ is finite. So let $A = \mathbf{Q}/\mathbf{Z}$, the rationals modulo its subgroup of integers. Then clearly A is TAG-ultrafinite. Now if $\Phi : \text{TAG} \rightarrow \mathscr{L}$ were an equivalence between TAG and an elementary P-class $\mathscr{L}$, then $\Phi(A)$ would have to be $\mathscr{L}$-ultrafinite, hence finite by (1.1). Thus End (A) = Hom (A, A) would be equinumerous with End $(\Phi(A))$, hence also finite. But for each $n \in \mathbf{Z}$, the mapping $a \mapsto na$ is an endomorphism on A; moreover for all $m, n \in \mathbf{Z}$, $ma = na$ for all $a \in A$ if and only if $(m - n)r \in \mathbf{Z}$ for all rationals r if and only if $m = n$. Thus End (A) is infinite, a contradiction.

2.2. THEOREM. TAG *is not equivalent to a representable class of algebras.*

Proof. Let $\Phi : \text{TAG} \rightarrow \mathscr{L}$ be an equivalence, where $\mathscr{L}$ is representable. Let $F_1 \in \mathscr{L}$ be the free $\mathscr{L}$-algebra over a singleton set, and let $A_1 = \Phi^{-1}(F_1)$. Letting $P \subseteq \mathbf{Z}$ denote the set of positive primes, and letting

$$C_p(A) = \{a \in A : p^n a = 0 \text{ for some } n \in \omega\}$$

denote the "p-primary component" of an abelian group A ($p \in P$), we can write $A_1 = \sum_{p \in P} C_p(A_1)$ (A_1 is the direct sum (= coproduct) by the "p-primary decomposition theorem"). Thus $F_1 = \sum_{p \in P}^{\mathscr{L}} B_p$, where $B_p = \Phi(C_p(A_1))$.

For each $J \subseteq K \subseteq P$ let

$$\sigma_{JK} : \sum_{p \in J}^{\mathscr{L}} B_p \rightarrow \sum_{p \in K}^{\mathscr{L}} B_p$$

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denote the natural "injection". Because each $\Phi^{-1}(\sigma_{JK})$ has a left inverse in TAG (coproducts are subdirect products) we know that σ_{JK} is a coretraction, hence an embedding. Let ρ_{JK} be the left inverse for σ_{JK}. Since all algebras under consideration are finitary, it follows that F_1 is the direct limit of the algebras $\sum_{p\in J}^{\mathscr{L}} B_p$ for $J \subseteq P$ finite. Thus if $x \in F_1$ is the free generator we get that x is in the image of σ_{JP} for some finite $J \subseteq P$. Since F_1 is freely generated by x, it follows that ρ_{JP} is also a right inverse for σ_{JP}, hence A_1 can be written as a finite direct sum,

$$A_1 = C_{p_1}(A_1) \oplus \ldots \oplus C_{p_n}(A_1).$$

Let $p \in P$ exceed each p_i, $1 \leqq i \leqq n$. Then there is only the zero homomorphism from A_1 to $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$. Thus

$$|\mathrm{Hom}\ (F_1, \Phi(\mathbf{Z}_p))| = 1.$$

This means $|\Phi(\mathbf{Z}_p)| = 1$, so that $\Phi(\mathbf{Z}_p)$ is the terminal object of $\mathscr{L}$ (the trivial group $\{0\}$ is the terminal object of TAG). Thus we have $\mathbf{Z}_p = \{0\}$, a contradiction.

3. The category EDCH. Let $X \in$ EDCH, and let A be its Boolean algebra of clopen sets. Then a straightforward consequence of Stone duality is that X is extremally disconnected if and only if A is complete. Thus to establish (0.2 (ii)) it suffices to show that CBA is not equivalent to either an elementary P-class or an SP-class of order $<c$ (it is easy to see that CBA is itself a representable P-class, but neither an elementary class nor an S-class).

Let A be a Boolean algebra. We denote by $\mu : A \rightarrow (A)^m$ the "MacNeille completion" (= the injective hull, see [**2**]) of A. Dually, if $X \in$ ZDCH, we let $\gamma : (X)^g \rightarrow X$ denote the "Gleason space" over X (i.e., the projective cover of X, see [**14**]). We will establish (0.2 (ii)) in part by first proving that ultraproducts in CBA do not exist generally.

3.1. THEOREM. *Let $\langle A_i : i \in I\rangle$ be a family of complete Boolean algebras, with D an ultrafilter on I. Then $\prod_D^{\mathrm{CBA}} A_i$ exists if and only if $\prod_D A_i$ is complete.*

Proof. Certainly if $\prod_D A_i$ is complete then it is the ultraproduct in CBA (since CBA is a P-class). For the converse we show first that if $\prod_D^{\mathrm{CBA}} A_i$ exists then $\prod_D^{\mathrm{CBA}} A_i = (\prod_D A_i)^m$. So for each $J \in D$ let

$$\alpha_J : \prod_{i\in J} A_i \rightarrow \prod_D^{\mathrm{CBA}} A_i$$

be the direct limit morphism, and let

$$\beta_J : \prod_{i\in J} A_i \rightarrow \prod_D A_i$$

be the usual ultraproduct homomorphism

$$f \upharpoonright J \mapsto [f]_D, \quad (f \in \prod_{i\in I} A_i).$$

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Then there is a unique

$$\lambda : \prod_D A_i \to \prod_D^{\mathrm{CBA}} A_i$$

such that $\lambda \circ \beta_J = \alpha_J$ for all $J \in D$, and there is a unique

$$\delta : \prod_D^{\mathrm{CBA}} A_i \to (\prod_D A_i)^m$$

such that $\delta \circ \alpha_J = \mu \circ \beta_J$ for all $J \in D$. We show first that $\delta \circ \lambda = \mu$. Indeed, pick $[f]_D \in \prod_D A_i$. Then for some $J \in D$,

$$\delta(\lambda([f]_D)) = \delta(\lambda(\beta_J(f \upharpoonright J))) = \delta(\alpha_J(f \upharpoonright J)) = \mu(\beta_J(f \upharpoonright J)) = \mu([f]_D).$$

Since μ is one-one, so also is λ. We next show that δ is one-one as well. By definition of the MacNeille completion there is an

$$\eta : (\prod_D A_i)^m \to \prod_D^{\mathrm{CBA}} A_i$$

with $\eta \circ \mu = \lambda$. Now for all $J \in D$,

$$\eta \circ \delta \circ \alpha_J = \eta \circ \mu \circ \beta_J = \lambda \circ \beta_J = \alpha_J.$$

By definition of direct limit, this forces $\eta \circ \delta =$ identity, whence δ is indeed an embedding. We then conclude that δ is an isomorphism since $(\prod_D A_i)^m$ is the minimal complete extension of $\prod_D A_i$.

To finish the proof, suppose B is any Boolean algebra which is not complete, and let X be the Stone space of B. Then $\gamma : (X)^g \to X$ is a surjective map which is not a homeomorphism, so there is an $x \in X$ with more than one γ-preimage. This means that there is always a homomorphism $\nu : B \to 2$ with more than one extension to $(B)^m$. This failure of uniqueness prevents $(\prod_D A_i)^m$ from being a direct limit in the proper sense whenever $\prod_D A_i$ is incomplete. Therefore $\prod_D^{\mathrm{CBA}} A_i$ fails to exist for $\prod_D A_i$ incomplete.

3.2. COROLLARY. *Let $A \in$ CBA be infinite, and let D be a countably incomplete ultrafilter. Then $A_{\mathrm{CBA}}^{(D)}$ does not exist.*

Proof. By (3.1), we need only show that $A^{(D)}$ is not complete. Indeed, since D is countably incomplete, $A^{(D)}$ is ω_1-saturated (in the sense of [**5**]). Since $A^{(D)}$ is also infinite, there are countable subsets with no maximal elements. By ω_1-saturatedness, these subsets can have no suprema either.

3.3. COROLLARY. CBA *is not a category with ultraproducts. Thus* CBA *cannot be equivalent to an elementary* P-*class.*

3.4. *Remark.* That $(A^{(D)})^m$ fails in general to be the D-ultrapower of A in CBA is ironic, since the natural candidate for the D-diagonal morphism, $\mu \circ \Delta_D$, is an elementary embedding in the parlance of model theory. To see this, note that every complete algebra is "separable", meaning that its set of atoms has a supremum. Now the class of separable Boolean algebras (see [**11**]) is elementary, and its theory admits quantifier elimination by the addition of one predicate which says of an element that it is atomic, and other predicates which say that an element contains n atoms, $n = 1, 2, \ldots$. It thus follows that embeddings of models of this expanded theory are elementary, whence $\mu : A \to (A)^m$ is elementary if and only if A is separable.

To finish the proof of (0.2 (ii)) we have the following more general result.

3.5. THEOREM. *Let $\mathscr{A}$ be a full subcategory of* CBA *which contains both the two-element and the four-element Boolean algebras. Then $\mathscr{A}$ cannot be equivalent to an* SP-*class of order* $< c$.

Proof. Let $\Phi : \text{CBA} \to \mathscr{L}$ be an equivalence witnessing the negation of the theorem. Then $\mathscr{L}$ has free algebras, so let α be a cardinal and denote by F_α the free $\mathscr{L}$-algebra over an α-element set, with $A_\alpha = \Phi^{-1}(F_\alpha)$. Since F_0 is the initial object of $\mathscr{L}$, A_0 is the initial object of $\mathscr{A}$. Therefore, by hypothesis, $A_0 = 2$, the two-element algebra; and $A_0{}^2 = 2^2$, the four-element algebra, is the free CBA-algebra over a singleton (so $|A| = |\text{Hom}\,(A_0{}^2, A)|$ for all $A \in \mathscr{A}$).

Since $\mathscr{L}$ has order $< c$, we know that $|F_\alpha| < c$ for all $\alpha \leqq \omega$. We claim that F_1 is finite. For suppose otherwise. Then

$$\omega \leqq |F_1| = |\text{End}\,(F_1)| = |\text{End}\,(A_1)|,$$

whence A_1 is an infinite complete Boolean algebra. By a theorem of R. S. Pierce [**13**], $|A_1| \geqq c$. Let X_1 be the Stone space of A_1. Then $|X_1| \geqq |A_1|$, so

$$c \leqq |A_1| \leqq |X_1| \leqq |\text{End}\,(X_1)| = |\text{End}\,(A_1)| = |\text{End}\,(F_1)| = |F_1|,$$

a contradiction. Therefore, F_1 is finite; and since F_0 is a retract of F_1, F_0 is finite as well.

Finally, End (F_ω) is infinite. Therefore, End (A_ω) is infinite; whence $|A_\omega| \geqq c$, again by Pierce's theorem. Thus

$$c \leqq |A_\omega| = |\text{Hom}\,(A_0{}^2, A_\omega)| = |\text{Hom}\,(F_0{}^2, F_\omega)|.$$

But $|F_\omega| < c$ and $|F_0{}^2| < \omega$. This gives a contradiction.

4. The categories CH and CCH. Let $X \in \text{CH}$ and let $A = C(X)$ be its ring of continuous real-valued functions. By the duality theorem of A. O. Gel'fand and A. N. Kolmogorov (see [**7**]), this correspondence establishes a duality between CH and the class of rings $\text{RCF} = \{C(X) : X \in \text{CH}\}$. Moreover, X is connected if and only if A has no idempotents other than

0 and 1 (i.e., A is "idempotent free", or an object of IFRCF). Now neither RCF nor IFRCF is an S-class or a P-class and neither class is elementary. (One way to see this last statement is to note that only the reals $\mathbf{R} = C(\{\text{point}\})$ is a field. Since every proper ultrapower extension of $\mathbf{R}$ is also a field, it cannot be in RCF.)

What is more important for our purposes is that CCH does not have coproducts (there is no coproduct of two singleton spaces, for example). Thus (0.2 (iii)) is immediate. To establish (0.2 (iv, v)), we will need to study "ultracoproducts" in CH, i.e., ultraproducts in CH*. (In [**4**] a deeper study is made of this construction, but we will need very little of the finer topological details here.)

Let $\langle X_i : i \in I \rangle$ be an indexed family of compact Hausdorff spaces, with D an ultrafilter on I. Then the *D-ultracoproduct* $\sum_D X_i = \sum_D^{\mathrm{CH}} X_i$ is the inverse limit of the functor $\Phi : D \to \mathrm{CH}$; where for each $J \in D$,

$$\Phi(J) = \sum_{i \in J}^{\mathrm{CH}} X_i$$

(the coproduct in CH) and for $J \supseteq K \in D$, $\Phi(J, K)$ is the natural "injection" from $\sum_{i \in K}^{\mathrm{CH}} X_i$ to $\sum_{i \in J}^{\mathrm{CH}} X_i$. Noting that

$$\sum_{i \in I}^{\mathrm{CH}} X_i = \beta\left(\dot{\bigcup_{i \in I}} X_i\right)$$

(= the Stone-Čech compactification of the disjoint union) and using basic facts about inverse limits in CH, it is easy to verify that $\sum_D X_i$ is naturally homeomorphic to the subspace of $\beta(\dot{\bigcup}_{i \in I} X_i)$ consisting of all ultrafilters p of zero sets from $\dot{\bigcup}_{i \in I} X_i$ which "extend" D in the sense that $\dot{\bigcup}_{i \in J} X_i \in p$ for each $J \in D$. In the case of ultracopowers we use the notation $X(D)$ and let $\nabla_D : X(D) \to X$ denote the "D-codiagonal" map. We define $X \in \mathrm{CH}$ to be CH-*ultracofinite* if $\nabla_D : X(D) \to X$ is a homeomorphism for all ultrafilters D.

4.1. *Remark.* It is easy to show that CH has ultraproducts, but $\prod_D^{\mathrm{CH}} X_i$ is a singleton whenever D is a free filter (i.e., $\bigcap D = \emptyset$) and $\{i : X_i \neq \emptyset\} \in D$. Thus X is CH-ultrafinite if and only if $|X| \leqq 1$. In [**4**] it is proved that X is CH-ultracofinite if and only if X is a finite space. We will need only an easy special case of this fact here, however.

4.2. LEMMA. *Every singleton space is* CH-*ultracofinite.*

Proof. Let $X = \{x\}$. Then $p = \{\{x\} \times J : J \in D\}$ is the only member of $X(D)$, since D is an ultrafilter.

The next theorem settles the first clause of (0.2 (iv)) and then some.

4.3. THEOREM. *Let $\mathscr{A}$ be a full subcategory of* CH *which is closed-hereditary (i.e., closed under closed subspaces) and closed under* CH-*coproducts.*

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If $\mathscr{A}$ *is dual to a representable elementary* P-*class* (*such as an elementary* SP-*class*) *then* $\mathscr{A} \subseteq$ ZDCH.

Proof. By the above discussion, $\mathscr{A}$ is closed under CH-ultra-coproducts. Hence by (4.2), singletons are $\mathscr{A}$-ultracofinite objects. Suppose $\Phi : \mathscr{A} \rightarrow \mathscr{L}$ is a duality where $\mathscr{L}$ is a representable elementary P-class, and assume $\mathscr{A} \nsubseteq$ ZDCH. Then there is a connected $C \in \mathscr{A}$ with more than one point, hence C is infinite. Since constant maps are continuous, we know

$$\omega \leqq |C| \leqq |\text{End }(C)| = |\text{End }(\Phi(C))|;$$

whence $\Phi(C)$ must be infinite. Let $S \in \mathscr{A}$ be a singleton. S is $\mathscr{A}$-ultracofinite, so $\Phi(S)$, being $\mathscr{L}$-ultrafinite, is finite. Let $F_1 \in \mathscr{L}$ be the free $\mathscr{L}$-algebra over a singleton and let $X_1 = \Phi^{-1}(F_1)$. Then

$$|X_1| = |\text{Hom }(S, X_1)| = |\text{Hom }(F_1, \Phi(S))| = |\Phi(S)|.$$

Thus X_1 is finite, so only the constant maps from C to X_1 are continuous. Therefore

$$|X_1| = |\text{Hom }(C, X_1)| = |\text{Hom }(F_1, \Phi(C))| = |\Phi(C)|,$$

a contradiction since $\Phi(C)$ was proved to be infinite.

To finish with (0.2 (iv)) we prove the following.

4.4. THEOREM. *Let* $\mathscr{A}$ *be a full subcategory of* CH *which is closed-hereditary and closed under* CH-*products* (*i.e.*, *Tichonov products*). *If* $\mathscr{A}$ *is dual to a representable class* $\mathscr{L}$ *of order* $< c$ *then* $\mathscr{A} \subseteq$ ZDCH.

Proof. Let $\Phi : \mathscr{A} \rightarrow \mathscr{L}$ be a duality where $\mathscr{A}$ and $\mathscr{L}$ are as above, and assume $\mathscr{A} \nsubseteq$ ZDCH. Then there is an infinite connected $C \in \mathscr{A}$. Since $\mathscr{A}$ is closed under Tichonov products, there are connected objects in $\mathscr{A}$ of arbitrarily large cardinality.

Let $F_1 \in \mathscr{L}$ be the free $\mathscr{L}$-algebra over a singleton with $X_1 = \Phi^{-1}(F_1)$. We claim that $|X_1| < c$. Indeed otherwise we would have

$$c \leqq |X_1| \leqq |\text{End }(X_1)| = |\text{End }(F_1)| = |F_1|,$$

contradicting the fact that $\mathscr{L}$ is of order $< c$. Now compact Hausdorff spaces of cardinality $< c$ are totally disconnected. Therefore for any connected $C \in \mathscr{A}$,

$$|X_1| = |\text{Hom }(C, X_1)| = |\text{Hom }(F_1, \Phi(C))| = |\Phi(C)|.$$

This contradicts the fact that there is a proper class of homeomorphism types of connected objects of $\mathscr{A}$.

In order to prove (0.2 (v)) we will need some more facts about ultra-coproducts. If $\langle X_i : i \in I \rangle$ is a family of compact Hausdorff spaces and

D is an ultrafilter on I then the topology on $\sum_D X_i$, inherited from that of $\beta(\dot{\bigcup}_{i \in I} X_i)$, is basically generated by sets of the form $\sigma_D M_i$; where $M_i \subseteq X_i$ is cozero, and $\sigma_D S_i$ denotes $\{p \in \sum_D X_i : \dot{\bigcup}_{i \in I} S_i$ extends a member of $p\}$.

4.6. LEMMA. *Let $\langle X_i : i \in I \rangle$ and D be as above, and let A_i be the clopen set algebra of X_i, $i \in I$. If A is the clopen set algebra of $\sum_D X_i$, then $A \cong \prod_D A_i$. In particular, $\sum_D X_i$ is connected if and only if*

$$\{i : X_i \text{ is connected}\} \in D.$$

Proof. Let $f \in \prod_{i \in I} A_i$ ($f(i)$ is a clopen subset of X_i). We define $\eta(f) = \sigma_D f(i)$. It is then straightforward to verify that $\eta(f) = \eta(g)$ if and only if $\{i : f(i) = g(i)\} \in D$, that $\eta(f \cup g) = \eta(f) \cup \eta(g)$, and that $\eta(f^c) = (\eta(f))^c$ (where $(\cdot)^c$ is complementation). Thus η is an embedding of $\prod_D A_i$ into A. To check that η is onto, we show that every clopen subset of $\sum_D X_i$ is of the form $\sigma_D C_i$ where $C_i \subseteq X_i$ is clopen, $i \in I$. So let $C \subseteq \sum_D X_i$ be clopen. Then C is a finite union of basic clopen sets,

$$C = (\sigma_D M_{i1}) \cup \ldots \cup (\sigma_D M_{in}) = \sigma_D(M_{i1} \cup \ldots \cup M_{in}),$$

$M_{ij} \subseteq X_i$ cozero for $1 \leqq j \leqq n$, $i \in I$. Then $M_i = M_{i1} \cup \ldots \cup M_{in}$ is cozero. Another easy verification gives the fact that $\sigma_D M_i$ is clopen if and only if

$$\{i : M_i \text{ is clopen}\} \in D;$$

and η is thus an isomorphism.

Finally $\sum_D X_i$ is connected if and only if $A = 2 \Leftrightarrow \{i : A_i = 2\} \in D$ $\Leftrightarrow \{i : X_i \text{ is connected}\} \in D$.

4.7. THEOREM. *If* CH *is dual to an elementary* P-*class* $\mathscr{L}$ *then* CCH *is dual to an elementary class* $\mathscr{L}' \subseteq \mathscr{L}$.

Proof. Let $\Phi : \mathrm{CH} \rightarrow \mathscr{L}$ be a duality, where $\mathscr{L}$ is an elementary P-class; and let $\mathscr{L}' \subseteq \mathscr{L}$ be the Φ-image of CCH. To show that $\mathscr{L}'$ is an elementary class it suffices, by the well known characterization of elementary classes in terms of ultraproducts [**5**], to show that ultraproducts of algebras in $\mathscr{L}'$ are also in $\mathscr{L}'$; and that $\mathscr{L}'$ is closed under elementary equivalence. Now (4.4) plus our duality assumption assures that $\mathscr{L}'$ is closed under $\mathscr{L}$-ultraproducts. As for closure under elementary equivalence, we use in addition the Keisler-Shelah Ultrapower Theorem; that two relational structures are elementarily equivalent if and only if an ultrapower of one is isomorphic to an ultrapower of the other.

5. Some questions. The following are some of the more interesting questions arising during the course of this investigation.

5.1. Is CH dual to an elementary P-class?

5.2. Can the cardinality restrictions be removed from the hypotheses of (3.5) and (4.4)?

5.3. Are there any dual pairs $(\mathscr{S} \cdot \mathscr{K}, \mathscr{T} \cdot \mathscr{L})$ where $\mathscr{S} = \mathscr{T} =$ CH ($\mathscr{S} =$ EDCH, $\mathscr{T} =$ S) and $\mathscr{K}, \mathscr{L}$ are nontrivial elementary SP-classes?

5.4. Is TFAG equivalent to an equational class?

Added in Proof. B. Banaschewski has recently shown that:

(i) Any SP-class which is equivalent to an equational class is itself an equational class, hence the answer to (5.4) is no; and

(ii) CH is not dual to an SP-class, thus providing a companion for our results (4.3, 4.4).

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