DIOPHANTINE APPROXIMATION ON HECKE GROUPS

by J. LEHNER

Dedicated to Robert Rankin in admiration and respect

1. Introduction. If α is a real irrational number, there exist infinitely many reduced rational fractions p/q for which

$$\left|\alpha-\frac{p}{q}\right|<\frac{1}{\sqrt{5}\,q^2}, \tag{1.1}$$

and $\sqrt{5}$ is the best constant possible. This result is due to A. Hurwitz. The following generalization was proposed in [**2**]. Let Γ be a finitely generated fuchsian group acting on H^+, the upper half of the complex plane. Let $\mathscr{L}$ be the limit set of Γ and $\mathscr{P}$ the set of cusps (parabolic vertices). Assume $\infty\in\mathscr{P}$. Then if $\alpha\in\mathscr{L}-\mathscr{P}$, we have

$$\left|\alpha-\frac{p}{q}\right|<\frac{1}{kq^2}, \tag{1.2}$$

for infinitely many $p/q\in\Gamma(\infty)$, where k depends only on Γ. Attention centers on

$$h=h(\Gamma)=\sup k, \tag{1.3}$$

k running over the set for which (1.2) holds. We call h the Hurwitz constant for Γ. When $\Gamma=\Gamma(1)$, the modular group, (1.2) reduces to (1.1) and $h(\Gamma(1))=\sqrt{5}$. A proof of (1.2) when Γ is horocyclic (i.e., $\mathscr{L}=\mathbb{R}$, the real axis) was furnished by Rankin [**4**]; he also found upper and lower bounds for h. See also [**3**, pp. 334–5], where the theorem is proved for arbitrary finitely generated Γ.

In the present paper we consider the Hecke groups

$$G_q=\left\langle\begin{pmatrix}1 & \lambda_q\\ 0 & 1\end{pmatrix},\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right\rangle,$$

where

$$\lambda_q=2\cos\frac{\pi}{q},\qquad q=4,5,\ldots \tag{1.4}$$

These are horocyclic groups. It is our object to evaluate, or at least estimate

$$h(G_q)\equiv h_q. \tag{1.5}$$

The Hecke groups include $G_3=\Gamma(1)$, for which $h_3=\sqrt{5}$ as stated above, and

$$G_\infty=\left\langle\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix},\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right\rangle,$$

a subgroup of $\Gamma(1)$, for which $h_\infty=2$, as proved by W. T. Scott [**6**]—see also [**4**, p. 289]. We shall therefore confine our attention to $q\geq 4$.

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Our results are contained in the following theorems.

THEOREM 1. $h_{2l}=2,\ l=2,3,\ldots.$

THEOREM 2. $2\le h_{2l-1}\le\left[\left(\frac{2}{\lambda-1}-\lambda\right)^2+4\right]^{1/2},$

$$\lambda=\lambda_{2l-1},\qquad l=4,5,\ldots,$$

$$2\le h_5\le 2.036.$$

2. The lower bound. We wish to show first that

$$h_q\ge 2,\qquad q\ge 4 \tag{2.1}$$

as proved in [**2**]. Here we sketch the argument.

From now on we write

$$\lambda=\lambda_q,\qquad G=G_q.$$

A fundamental region for G_q is the triangle

$$R_0=\{z=x+iy: y>0, |x|\le\lambda/2, x^2+y^2\ge 1\}. \tag{2.2}$$

This may be regarded as a quadrilateral with i as one vertex, the sides being labelled $1, 2, 2', 1'$, starting with the left-hand vertical and proceeding counterclockwise. The sides $1, 1'$ are conjugate under G, and likewise $2, 2'$. The set of cusps $\mathcal{P}$ is $G(\infty)$, as ∞ is the only cusp in R_0. The elliptic vertices are i, $\rho=\exp \pi i/q$, and $-\bar{\rho}$.

Let $\alpha\in\mathbb{R}-G(\infty)$. The vertical line $L_\alpha=L:=\{\alpha+iy, y>0\}$ crosses infinitely many fundamental regions, say $R_1, R_2,\ldots$ in order. Let $\mu_i=L\cap R_i$ be the directed segment of L lying in R_i. The transformation $V_i^{-1}\in G$ maps R_i on R_0 and μ_i on the directed arc $\mu_i'=V_i^{-1}\mu_i$ lying in R_0.

LEMMA 1. *For each $\alpha\in\mathbb{R}-G(\infty)$ there exist infinitely many j, and corresponding points t_j' on μ_j', such that*

$$\operatorname{Im} t_j'>1. \tag{2.3}$$

Proof. Consider the sequence of μ_i'. Each μ_i' cuts two sides of R_0, say k_i, k_{i+1} in that order. Denote the conjugate of k by k' and set $(k')'=k$. Then μ_{i+1}' cuts k_{i+1}', k_{i+2}. Suppose k_i, k_{i+1} are not consecutive sides of R_0; since μ_i' (when extended) is orthogonal to $\mathbb{R}$, this means $(k_i, k_{i+1})=(1,2'), (2,1'), (1,1')$ or their reverses. Then it is clear there is a t_i' on μ_i' satisfying (2.3). Hence if the lemma is untrue we must assume the sequence of k's is eventually $k, l; l', k'; k, l;\ldots$, where $(k,l)=(1,2), (2',1')$, or their reverses. In the first case, for example, this would mean that infinitely many sides of fundamental regions issue from the elliptic vertex $-\bar{\rho}$. This is impossible.

Having proved the lemma, we now write $t_j=V_jt_j'\in L$ $(j=1,2,\ldots)$. Thus $t_j=\alpha+iy_j, j>0$. Let $V_j^{-1}=(q_j', -p_j': q_j, -p_j)$. Then

$$\operatorname{Im} t_j'=\frac{y_j}{(q_j\alpha-p_j)^2+q_j^2y_j^2}>1.$$

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Since the arithmetic mean is not less than the geometric mean, we get

$$y_j > (q_j\alpha - p_j)^2 + q_j^2 y_j^2 \geq 2\,|q_j\alpha - p_j|\,q_j y_j.$$

Hence

$$\left|\alpha - \frac{p_j}{q_j}\right| < \frac{1}{2q_j^2} \quad (j = 1, 2, \ldots),$$

which implies (2.1).

3. Approximation by λ-fractions. One of the standard ways of proving Hurwitz's theorem (1.1) is to use regular continued fractions. One first shows that if (1.1) is satisfied by any rational number P/Q in lowest terms, then P/Q must be a convergent in the expansion of α as a regular continued fraction. The problem is thus reduced to studying the approximation of α by its convergents.

We follow a similar route. For a given $\lambda = \lambda_q$, $q \geq 4$, a λ-fraction (λ CF) has been developed by D. Rosen [**5**]. A λ CF is a finite or infinite continued fraction

$$[r_0\lambda, \varepsilon_1/r_1\lambda, \ldots] = r_0\lambda + \frac{\varepsilon_1}{r_1\lambda + \ldots}, \tag{3.1}$$

where the integers r_i, $i \geq 1$, are positive and $\varepsilon = \pm 1$. If the fraction is finite, we can write

$$[r_0\lambda, \varepsilon_1/r_1\lambda, \ldots, \varepsilon_n/r_n\lambda] = S^{r_0}TS^{-\varepsilon_1 r_1} \ldots S^{-\varepsilon_n r_n}T(\infty),$$

where $S = (1, \lambda : 0, 1)$, $T = (0, -1 : 1, 0)$ are the generators of G. It follows that every finite λ CF has a value that is a cusp of G (belongs to $G(\infty)$) and conversely. An infinite convergent λ CF, then, converges to a point in $\mathbb{R} - G(\infty)$.

In the λ CF (3.1) define

$$\begin{gathered} P_{-1} = 1, \qquad P_0 = r_0\lambda, \qquad Q_{-1} = 0, \qquad Q_0 = 1, \\ P_n = r_n\lambda P_{n-1} + \varepsilon_n P_{n-2}, \qquad Q_n = r_n\lambda Q_{n-1} + \varepsilon_n Q_{n-2}, \qquad n \geq 1. \end{gathered} \tag{3.2}$$

Then we derive the following equations.

$$\begin{gathered} [r_0\lambda, \varepsilon_1/r_1\lambda, \ldots, \varepsilon_n/r_n\lambda] = P_n/Q_n \\ P_nQ_{n-1} - Q_nP_{n-1} = (-1)^{n-1}\varepsilon_1\varepsilon_2 \ldots \varepsilon_n. \end{gathered} \tag{3.3}$$

If α is represented by the convergent λ CF (3.1), we have

$$\alpha = \lim_{n\to\infty} P_n/Q_n.$$

Defining the "tail" of (3.1) by

$$\alpha_n = [r_n\lambda, \varepsilon_{n+1}/r_{n+1}\lambda, \ldots] \quad (n \geq 0),$$

we get

$$\alpha = \left[r_0\lambda, \ldots, \frac{\varepsilon_{n-1}}{r_{n-1}\lambda}, \frac{\varepsilon_n}{\alpha_n}\right]. \tag{3.4}$$

Now this is a finite λ CF whose value is P_n/Q_n. Using (3.2) we obtain

$$\alpha = \frac{\alpha_n P_{n-1} + \varepsilon_n P_{n-2}}{\alpha_n Q_{n-1} + \varepsilon_n Q_{n-2}} \quad (n \geq 1) \tag{3.5}$$

$$\alpha - \frac{P_{n-1}}{Q_{n-1}} = \frac{(-1)^{n-1}\varepsilon_1\varepsilon_2 \ldots \varepsilon_n}{m_{n-1}Q_{n-1}^2} \tag{3.6}$$

with

$$m_{n-1}(\alpha) \equiv m_{n-1} = \alpha_n + \varepsilon_n Q_{n-2}/Q_{n-1}$$
$$= \left[r_n\lambda, \frac{\varepsilon_{n+1}}{r_{n+1}\lambda}, \ldots\right] + \varepsilon_n \Big/ \left[r_{n-1}\lambda, \frac{\varepsilon_{n-1}}{r_{n-2}\lambda}, \ldots, \frac{\varepsilon_2}{r_1\lambda}\right]. \tag{3.7}$$

Rosen calls (3.1) *reduced* if certain conditions on r_i and ε_i are fulfilled; these will be given later. He establishes the following properties of reduced λ CF.

(3.8) *An infinite reduced λ CF converges.*

(3.9) *Every real number α can be expanded uniquely by the "nearest integer algorithm" in a reduced λ CF. If the fraction is infinite, it converges to α.*

(3.10) *In a reduced λ CF we have $Q_n \geq 1$, Q_n is non-decreasing, and $Q_n \to \infty$ as $n \to \infty$.*

We denote the expansion of α in a reduced λ CF by λ CFα.

Remark. The λ CF is a type of semi-regular continued fraction, but as the above shows, it possesses properties beyond those of the general semi-regular continued fraction.

We are now ready to prove the following result.

Theorem 3. *For $P/Q \in G(\infty)$ and $\alpha \in \mathbb{R} - G(\infty)$ let*

$$|\alpha - P/Q| < 1/2Q^2. \tag{3.11}$$

Then P/Q is a convergent of the reduced λ CFα.

Proof. Write

$$\alpha - \frac{P}{Q} = \frac{\varepsilon\theta}{Q^2}, \qquad 0 < \theta < \tfrac{1}{2}, \qquad \varepsilon = \pm 1.$$

Expand P/Q in a reduced λ CF:

$$P/Q = [r_0\lambda, \varepsilon_1/r_1\lambda, \ldots, \varepsilon_{n-1}/r_{n-1}\lambda]; \tag{3.12}$$

the fraction is finite because $P/Q \in G(\infty)$. Call the convergents P_i/Q_i, so that

$$\frac{P}{Q} = \frac{P_{n-1}}{Q_{n-1}}.$$

Next define ω by

$$\alpha = \frac{P_{n-1}\omega + \varepsilon_n P_{n-2}}{Q_{n-1}\omega + \varepsilon_n Q_{n-2}}, \tag{3.13}$$

where we have introduced ε_n by

$$\varepsilon = \varepsilon_1 \varepsilon_2 \ldots \varepsilon_n (-1)^{n-1}.$$

Then using (3.3) we get

$$\frac{\varepsilon\theta}{Q_{n-1}^2} = \alpha - \frac{P}{Q} = \frac{P_{n-1}\omega + \varepsilon_n P_{n-2}}{Q_{n-1}\omega + \varepsilon_n Q_{n-2}} - \frac{P_{n-1}}{Q_{n-1}}$$
$$= \frac{\varepsilon_n \varepsilon_1 \varepsilon_2 \ldots \varepsilon_{n-1}(-1)^{n-1}}{Q_{n-1}(Q_{n-1}\omega + \varepsilon_n Q_{n-2})},$$

and so

$$\theta = \frac{Q_{n-1}}{Q_{n-1}\omega + \varepsilon_n Q_{n-2}}.$$

Hence

$$\omega = \frac{Q_{n-1} - \theta\varepsilon_n Q_{n-2}}{\theta Q_{n-1}} > 0,$$

because $Q_{n-1} \geq Q_{n-2}$ by (3.10)—P_{n-1}/Q_{n-1} is a reduced λ CF—and $0 < \theta < 1/2$, $\varepsilon_n = \pm 1$.

We expand ω in a reduced λ CF

$$\omega = [r_n\lambda, \varepsilon_{n+1}/r_{n+1}\lambda, \ldots]. \tag{3.14}$$

Here $r_n = \{\omega/\lambda\}$ = nearest integer to ω/λ. Since

$$\omega = \frac{1}{\theta} - \frac{\varepsilon_n Q_{n-2}}{Q_{n-1}} > 2 - 1 = 1 > \frac{\lambda}{2},$$

it follows that $r_n \geq 1$.

We have two expressions for α:

$$\alpha = \frac{P_{n-1}\omega + \varepsilon_n P_{n-2}}{Q_{n-1}\omega + \varepsilon_n Q_{n-2}} = \frac{P_{n-1}\alpha_n + \varepsilon_n P_{n-2}}{Q_{n-1}\alpha_n + \varepsilon_n Q_{n-2}};$$

see (3.13), (3.5). Hence $\omega = \alpha_n$ and by (3.4), (3.14),

$$\alpha = \left[r_0\lambda, \frac{\varepsilon_1}{r_1\lambda}, \ldots, \frac{\varepsilon_{n-1}}{r_{n-1}\lambda}, \frac{\varepsilon_n}{\alpha_n}\right] = \left[r_0\lambda, \ldots, \frac{\varepsilon_{n-1}}{r_{n-1}\lambda}, \frac{\varepsilon_n}{r_n\lambda}, \frac{\varepsilon_{n+1}}{r_{n+1}\lambda}, \ldots\right].$$

It follows from (3.12) that

$$\frac{P}{Q} = \frac{P_{n-1}}{Q_{n-1}} = \left[r_0\lambda, \frac{\varepsilon_1}{r_1\lambda}, \ldots, \frac{\varepsilon_{n-1}}{r_{n-1}\lambda}\right],$$

and so P/Q is a convergent of α.

4. Theorem 3 permits us to confine our considerations to approximations of α by the convergents of its reduced λ CF. By (3.6) we have

$$\left|\alpha - \frac{P_{n-1}}{Q_{n-1}}\right| = \frac{1}{m_{n-1}Q_{n-1}^2},$$

$m_{n-1} \equiv m_{n-1}(\alpha)$ being given by (3.7). Let

$$M(\alpha) = \overline{\lim_{n\to\infty}}\, m_{n-1}(\alpha); \tag{4.1}$$

then we have

$$h_q = \inf_\alpha M(\alpha), \qquad \alpha \in \mathbb{R} - G_q(\infty).$$

In this section we treat the case $q=2l$, where $l \geq 2$. To prove Theorem 1, it is obviously sufficient to exhibit one α, say α_0, for which

$$M(\alpha_0) \leq 2, \tag{4.2}$$

for this would imply $h_q \leq 2$ and we already know $h_q \geq 2$ from (2.1).

At this point we introduce the definition of a *reduced* λ CF when $q = 2l$. Let

$$s = [(q-3)/2] = l - 2 \geq 0.$$

(4.3) The fraction $[r_0\lambda, \varepsilon_1/r_1\lambda, \ldots]$ is said to be reduced ([**5**, p. 555]) if the inequality $r_i\lambda + \varepsilon_{i+1} < 1$ is satisfied for no more than s consecutive values of i, say $i = j, j+1, \ldots, j+s-1; j \geq 1$.

This inequality is equivalent to $r_i = 1, \varepsilon_{i+1} = -1$.

We shall define α_0 as a (pure) *periodic* λ CF. The infinite λ CF $[r_0\lambda, \varepsilon_1/r_1\lambda, \ldots]$ is periodic if for some integer $p \geq 1$ (the period) and $\nu \geq 0$ we have

$$r_{\nu+p} = r_\nu, \qquad \varepsilon_{\nu+p+1} = \varepsilon_{\nu+1}. \tag{4.4}$$

We shall write the periodic fraction as

$$\left[r_0\lambda, \frac{\varepsilon_1}{r_1\lambda}, \ldots, \frac{\varepsilon_{p-1}}{r_{p-1}\lambda}, \frac{\varepsilon_p}{r_0\lambda}, \frac{\varepsilon_1}{r_1\lambda}, \ldots\right] = \left[\overline{r_0\lambda, \ldots, \frac{\varepsilon_{p-1}}{r_{p-1}\lambda}}; \frac{\varepsilon_p}{r_0\lambda}\right]. \tag{4.5}$$

Note that it is necessary to exhibit the term $\varepsilon_p/r_0\lambda$, since (4.4) does not distinguish between $\varepsilon_p = 1$ and $\varepsilon_p = -1$. The tails of a periodic λ CF are also periodic:

$$\alpha_0 = \alpha_p = \alpha_{2p} = \ldots \tag{4.6}$$

Now consider the periodic λ CF

$$\alpha_0 = \left[\overline{2\lambda, -\frac{1}{\lambda}, \ldots, -\frac{1}{\lambda}}; \frac{-1}{2\lambda}\right] = \alpha_{s+1} = \ldots \tag{4.7}$$

of period $s+1$. This fraction satisfies (4.3) and so is reduced. By (3.8) it converges. Along

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with α_0 we introduce the reduced λ CF

$$\alpha_1 \equiv \beta_0 = \left[\overline{\lambda, -\frac{1}{\lambda}, \ldots, -\frac{1}{2\lambda}; -\frac{1}{\lambda}}\right] = \beta_{s+1} = \ldots, \tag{4.8}$$

$$\alpha_0 = 2\lambda - 1/\beta_0. \tag{4.9}$$

Let

$$\alpha'_n = \left[r_n\lambda, \frac{\varepsilon_n}{r_{n-1}\lambda}, \ldots, \frac{\varepsilon_2}{r_1\lambda}\right], \tag{4.10}$$

so that by (3.7)

$$m_{n-1} = \alpha_n + \varepsilon_n/\alpha'_{n-1}. \tag{4.11}$$

In the present case, with $p = s+1$, we can write

$$m_{np-1} = \alpha_{np} + \varepsilon_{np}/\alpha'_{np-1} = \alpha_0 - 1/\alpha'_{np-1},$$

$$\lim_{n\to\infty} m_{np-1} = \alpha_0 - 1/\beta_0 = \alpha_0 + \alpha_0 - 2\lambda = 2(\alpha_0 - \lambda),$$

using (4.9).

We shall prove that

$$\alpha_0 = \lambda + 1, \tag{4.12}$$

so that

$$\lim_{n\to\infty} m_{np-1} = 2.$$

On the other hand, if $j \not\equiv 0 \pmod p$,

$$m_{j-1} = \alpha_j - 1/\alpha'_{n-1} < \alpha_j < \lambda < 2 = \lim_{n\to\infty} m_{np-1}.$$

Hence (4.2) follows, and with it, Theorem 1.

It remains to prove (4.12). From (3.5) and (4.8) we get

$$\beta_0 = \frac{P_s\beta_0 - P_{s-1}}{Q_s\beta_0 - Q_{s-1}},$$

$$Q_s\beta_0^2 - (Q_{s-1} + P_s)\beta_0 + P_{s-1} = 0, \tag{4.13}$$

P_i/Q_i being the convergents of β_0. The value of Q_i can be found in [**1**, p. 7] or is easily checked by induction:

$$Q_i = \frac{\sin(i+1)\pi/q}{\sin \pi/q} \quad (i \le s-1),$$

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so $(s = l-2)$

$$Q_{s-1} = \frac{\sin((l-2)\pi/2l}{\sin \pi/2l} = \frac{\cos \pi/l}{\sin \pi/2l} \quad (l \geq 2),$$

$$Q_{s-2} = \frac{\sin((l-3)\pi/2l}{\sin \pi/2l} = \frac{\cos 3\pi/2l}{\sin \pi/2l}.$$

By induction from (3.2),

$$P_i = Q_{i+1} \quad (i \leq s-2).$$

Hence

$$\begin{aligned} P_{s-1} &= \lambda P_{s-2} - P_{s-3} = \lambda Q_{s-1} - Q_{s-2} = Q_s - \lambda Q_{s-1}, \\ P_s &= 2\lambda P_{s-1} - P_{s-2} = 2\lambda(Q_s - \lambda Q_{s-1}) - Q_{s-1} \\ &= 2\lambda Q_s - (2\lambda^2+1)Q_{s-1}, \\ Q_{s-1} + P_s &= 2\lambda(Q_s - \lambda Q_{s-1}). \end{aligned}$$

We shall shortly see that $Q_s \neq 0$. Substituting these values in (4.3) and dividing by Q_s, we get

$$\beta_0^2 - 2\lambda\left(1 - \lambda \frac{Q_{s-1}}{Q_s}\right)\beta_0 + 1 - \lambda \frac{Q_{s-1}}{Q_s} = 0. \tag{4.14}$$

Now with $\zeta = \exp(\pi i/2l)$, $\lambda = \zeta + \zeta^{-1}$,

$$\begin{aligned} Q_s = 2\lambda Q_{s-1} - Q_{s-2} &= \frac{2(\zeta+\zeta^{-1})(\zeta^2+\zeta^{-2}) - (\zeta^3+\zeta^{-3})}{2 \sin \pi/2l} \\ &= \frac{\lambda(\lambda^2-1)}{2 \sin \pi/2l} \neq 0, \\ \frac{Q_{s-1}}{Q_s} &= \frac{\lambda^2-2}{\lambda(\lambda^2-1)}. \end{aligned}$$

Thus (4.14) becomes

$$\beta_0^2 - \frac{2\lambda}{\lambda^2-1}\beta_0 + \frac{1}{\lambda^2-1} = 0,$$

the roots of which are $\beta_0 = 1/(\lambda-1)$, $1/(\lambda+1)$. This gives $\alpha_0 = \lambda+1$, $\lambda-1$. However, from the definition (4.7), $\alpha_0 > 2\lambda - 1 > \lambda - 1$, and so we have proved (4.12). Theorem 1 is now established.

5. We turn now to the case q odd, $q > 5$:

$$q = 2l-1, \qquad l \geq 4; \qquad s = l-2.$$

There are additional conditions besides (4.3) in the definition of an infinite reduced λ CF when q is odd.

(5.1) If $r_i\lambda+\varepsilon_{i+1}<1$ for $i=j, j+1, \ldots, j+s-1$, then $r_{j+s}\geq 2$.

(5.2) If $(B(s), -1/2\lambda, -1/B(s))$ occurs in the expansion, the succeeding sign is plus. Here

$$B(n)=\left(\lambda, -\frac{1}{\lambda}, \ldots, -\frac{1}{\lambda}\right)$$

with n partial quotients.

One way to avoid these rather complicated conditions is to use fractions with less than the maximum allowable number of consecutive sequences $r_i\lambda+\varepsilon_{i+1}<1$. For example, let

$$\alpha_0=\left[\overline{\lambda, -\frac{1}{\lambda}, \ldots, -\frac{1}{\lambda}; \frac{1}{\lambda}}\right]=\lim_{n\to\infty}\frac{P_n}{Q_n} \tag{5.3}$$

of period s, in which there are $s-1$ consecutive sequences. Then

$$\alpha_0=\alpha_s=\frac{P_{s-1}\alpha_0+P_{s-2}}{Q_{s-1}\alpha_0+Q_{s-2}}.$$

A calculation quite similar to that of Section 4 yields

$$\alpha_0^2+\left(\lambda-\frac{2}{\lambda-1}\right)\alpha_0-1=0, \tag{5.4}$$

where we have used

$$Q_{s-1}=\frac{\sin(s\pi/(2l-1))}{\sin(\pi/(2l-1))}=\frac{\cos(3\pi/2(2l-1))}{\sin(\pi/(2l-1))}\neq 0.$$

Continuing,

$$\alpha'_{ns-1}=\left[\lambda, -\frac{1}{\lambda}, \ldots, -\frac{1}{\lambda}, \frac{1}{\lambda}, -\frac{1}{\lambda}, \ldots, \frac{1}{\lambda}\right]$$
$$\to\left[\overline{\lambda, -\frac{1}{\lambda}, \ldots, -\frac{1}{\lambda}; \frac{1}{\lambda}}\right]=\alpha_0.$$

Hence

$$m_{ns-1}\to\alpha_0+\frac{1}{\alpha_0}, \qquad \text{as} \quad n\to\infty.$$

Now

$$(\alpha_0+1/\alpha_0)^2=(\alpha_0-\bar{\alpha}_0)^2=(\alpha_0+\bar{\alpha}_0)^2-4\alpha_0\bar{\alpha}_0=\left(\frac{2}{\lambda-1}-\lambda\right)^2+4,$$

$\bar{\alpha}_0$ being the other root of (5.4). So we get

$$\lim_{n\to\infty} m_{ns-1}=M_q,$$

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where

$$M_q = \alpha_0 + \frac{1}{\alpha_0} = \left[\left(\frac{2}{\lambda-1} - \lambda\right)^2 + 4\right]^{1/2}. \tag{5.5}$$

Again, for $j \not\equiv 0, 1 \pmod s$, $m_{j-1} = \alpha_j - 1/\alpha'_{j-1} < \alpha_j < 2 < M_q$. For $j = ns-1$,

$$m_{j-1} = m_{ns-2} = \alpha_{ns-1} - 1/\alpha'_{ns-2}$$
$$= \lambda + \frac{1}{\alpha_s} + \alpha'_{ns-1} - \lambda \to \frac{1}{\alpha_0} + \alpha_0 = M_q.$$

It follows that

$$h_{2l-1} \leq M(\alpha_0) \leq M_q, \tag{5.6}$$

as asserted in Theorem 2. For example, $M_7 \simeq 2{\cdot}12$. We observe that $M_q \to 2$ as $q \to \infty$.

The final case, $q=5 (l=3)$, can be treated in the same way, the result being $M_5 \simeq 2{\cdot}57$. We can get a better result, however, by taking

$$\alpha_0 = \left[\overline{\lambda, -\frac{1}{2\lambda}, -\frac{1}{2\lambda}}; -\frac{1}{\lambda}\right]$$

of period 3, which is reduced. The equation for α_0 is

$$\alpha_0^2 - \lambda\alpha_0 + \frac{2\lambda-1}{5} = 0, \tag{5.7}$$

where we have replaced the rational functions of λ that occur by polynomials, using $\lambda^2 - \lambda - 1 = 0$, $\lambda = \lambda_5 = (1+\sqrt{5})/2$. The roots of (5.7) are $(1{\cdot}264, 0{\cdot}354)$, α_0 being equal to the larger one. Now

$$m_{3n} = \alpha_{3n+1} - \frac{1}{\alpha'_{3n}} = \alpha_1 - \frac{1}{\alpha_0} = \frac{1}{\lambda - \alpha_0} - \frac{1}{\alpha_0},$$

$$m_{3n+1} = \alpha_{3n+2} - \frac{1}{\alpha'_{3n+1}} \to \alpha_2 - \frac{1}{\alpha_2}, \alpha_2 = 2\lambda - \frac{1}{\alpha_0},$$

$$m_{3n+2} = \alpha_{3n+3} - \frac{1}{\alpha'_{3n+2}} \to \alpha_0 - \frac{1}{\alpha_1} = 2\alpha_0 - \lambda.$$

Calculation shows that the largest of the right members is

$$\lim m_{3n+1} \simeq 2{\cdot}036,$$

which gives the upper bound for h_5 in Theorem 2.

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