ON FUNCTIONS AND EQUATIONS IN DISTRIBUTIVE LATTICES

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Summary. In [1], R. L. Goodstein has extended some well-known theorems on functions and equations in a Boolean algebra to the case of a distributive lattice L with 0 and 1. The purpose of this paper is to prove that most of Goodstein's theorems, as well as some additional results, are still valid in the case when L is not required to have least and greatest elements.

Throughout this paper, we shall always assume that $\langle L, \cup, .\rangle$ is a distributive lattice.

The definition of a lattice function of n variables is as follows:

1. The elements $a, b, c, \ldots, A, B, C, \ldots$ of L are lattice functions.
2. The functions ε_i, defined by

$$\varepsilon_i(x_1, \ldots, x_n) = x_i \quad \forall x_1, \ldots, x_n \in L \quad (i = 1, 2, \ldots, n) \tag{1}$$

are lattice functions.

3. If $f, g: L^n \to L$ are lattice functions, then the functions $f \cup g$ and fg, defined by

$$(f \cup g)(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) \cup g(x_1, \ldots, x_n) \ \forall x_1, \ldots, x_n \in L, \tag{2}$$

$$(fg)(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) g(x_1, \ldots, x_n) \ \forall x_1, \ldots, x_n \in L, \tag{3}$$

are lattice functions.

Lemma 1. *The inequality*

$$a \cup bx \leqq c \cup dx \tag{4}$$

is equivalent to the following system of inequalities:

$$a \leqq c \cup d, \tag{5}$$

$$bx \leqq c \cup d, \tag{6}$$

$$a \leqq c \cup x. \tag{7}$$

Proof. Since $c \cup dx = (c \cup d)(c \cup x)$, the inequality (4) is equivalent to the system consisting of (5), (6), (7) and $bx \leqq c \cup x$; but the last inequality is identically satisfied.

Lemma 2. *The inequality* (4) *is identically satisfied if and only if*

$$a \leqq c \tag{8}$$

and

$$b \leqq c \cup d. \tag{9}$$

Proof. If (4) is identically satisfied so are the relations (6) and (7). Taking $x = b$ in (6) and $x = c$ in (7), we get (9) and (8), respectively. Conversely, the relations (8) and (9) imply that $a \cup bx \leqq c \cup (c \cup d)x = c \cup dx$ for all $x \in L$.

Lemma 3. *The equation*

$$a \cup bx = c \cup dx \tag{10}$$

is identically satisfied if and only if

$$a = c \tag{11}$$

and

$$a \cup b = c \cup d. \tag{12}$$

Proof. The identity (10) holds if and only if $a \leqq c$, $b \leqq c \cup d$, $c \leqq a$, $d \leqq a \cup b$, by Lemma 2. These inequalities imply, in turn, $a = c$ and

$$a \cup b = c \cup b \leqq c \cup d = a \cup d \leqq a \cup b.$$

Conversely, (11) and (12) imply

$$a \cup bx = a \cup (a \cup b)x = c \cup (c \cup d)x = c \cup dx.$$

We come now to the study of lattice functions. It was proved in [1] that every lattice function can be written in the form $f(x) = A \cup Bx$, where $A \leqq B$. Let $g(x) = C \cup Dx$, where $C \leqq D$, be another lattice function. Lemma 2 shows that the inequality $f(x) \leqq g(x)$ is identically satisfied if and only if $A \leqq C$ and $B \leqq D$. Hence $f = g$ if and only if $A = C$ and $B = D$.

For every $n+1$ elements $a, x_1, \ldots, x_n \in L$ and for every n indices $\alpha_1, \ldots, \alpha_n$ equal to 0 or 1, let us put

$$ax_1^{\alpha_1} \ldots x_n^{\alpha_n} = \begin{cases} ax_{i_1} \ldots x_{i_m} \text{ if } \alpha_{i_1} = \ldots = \alpha_{i_m} = 1, \text{ the other } \alpha_j = 0; \\ a, \text{ if all } \alpha_j = 0. \end{cases} \tag{13}$$

The above results can be generalized as follows.

Theorem 1. *Every lattice function $f: L^n \to L$ can uniquely be written in the canonical form*

$$f(x_1, \ldots, x_n) = \bigcup_{i_1, \ldots, i_n} F(i_1, \ldots, i_n) x_1^{i_1} \ldots x_n^{i_n}, \tag{14}$$

where $F(i_1, \ldots, i_n)$ are elements of L such that

$$i_1 \leqq j_1, \ldots, i_n \leqq j_n \text{ imply } F(i_1, \ldots, i_n) \leqq F(j_1, \ldots, j_n). \tag{15}$$

Theorem 2. *Let* (14) *and*

$$g(x_1, \ldots, x_n) = \bigcup_{i_1, \ldots, i_n} G(i_1, \ldots, i_n) x_1^{i_1} \ldots x_n^{i_n} \tag{16}$$

be the canonical forms of the functions f and g. The inequality $f \leqq g$, that is

$$f(x_1, \ldots, x_n) \leqq g(x_1, \ldots, x_n) \quad \forall x_1, \ldots, x_n \in L \tag{17}$$

holds if and only if

$$F(i_1, \ldots, i_n) \leqq G(i_1, \ldots, i_n) \quad \forall i_1, \ldots, i_n \in \{0, 1\}. \tag{18}$$

Proof of Theorems 1 and 2. For $n = 1$, the theorems were proved before. The next step of the inductive proof is carried out as follows.

The function f can be written in the form

$$f(x_1, \ldots, x_n) = f'(x_1, \ldots, x_{n-1}) \cup f''(x_1, \ldots, x_{n-1})x_n, \tag{19}$$

where f' and f'' are lattice functions satisfying the identity

$$f'(x_1, \ldots, x_{n-1}) \leqq f''(x_1, \ldots, x_{n-1}). \tag{20}$$

In view of the inductive hypothesis, we have

$$F'(i_1, \ldots, i_{n-1}) \leqq F''(i_1, \ldots, i_{n-1}) \quad \forall i_1, \ldots, i_{n-1} \in \{0, 1\}, \tag{21}$$

where F' and F'' are the coefficients of the canonical forms of the functions f' and f'', respectively.

It follows that the function f can be written in the form (14), with

$$F(i_1, \ldots, i_{n-1}, 0) = F'(i_1, \ldots, i_{n-1})$$

and

$$F(i_1, \ldots, i_{n-1}, 1) = F''(i_1, \ldots, i_{n-1}).$$

By the inductive hypothesis, both F' and F'' have the property (15); taking into account (21), we see that the constants F have the property (15) too.

Furthermore, let $g(x_1, \ldots, x_n) = g'(x_1, \ldots, x_{n-1}) \cup g''(x_1, \ldots, x_{n-1})x_n$, where $g' \leqq g''$, be another lattice function. The inequality $f \leqq g$ holds identically if and only if the inequalities $f' \leqq g'$ and $f'' \leqq g''$ hold identically, i.e. if and only if

$$F'(i_1, \ldots, i_{n-1}) \leqq G'(i_1, \ldots, i_{n-1}) \text{ and } F''(i_1, \ldots, i_{n-1}) \leqq G''(i_1, \ldots, i_{n-1})$$

$\forall i_1, \ldots, i_{n-1} \in \{0, 1\}$. This means that the relation (17) is equivalent to (18).

Hence we deduce the uniqueness of the representation (14), which we state separately, thus completing the proof:

Corollary 1. *The identity $f = g$ holds if and only if*

$$F(i_1, \ldots, i_n) = G(i_1, \ldots, i_n) \quad \forall i_1, \ldots, i_n \in \{0, 1\}, \tag{22}$$

where F and G are the coefficients occurring in the canonical forms of the functions f and g, respectively.

Theorem 2 and Corollary 1 generalize the so-called " verification theorem " due to Löwenheim [4]. Theorem 1 and Theorem 3 below are also generalizations of a well-known result on Boolean functions.

Let us now determine the canonical forms of the functions $f \cup g$ and fg, defined by (2) and (3), respectively.

Theorem 3. *Let* (14) *and* (16) *be the canonical forms of the functions f and g, respectively. Then*

$$(f \cup g)(x_1, \ldots, x_n) = \bigcup_{i_1, \ldots, i_n} [F(i_1, \ldots, i_n) \cup G(i_1, \ldots, i_n)] x_1^{i_1} \ldots x_n^{i_n} \tag{23}$$

and

$$(fg)(x_1, \ldots, x_n) = \bigcup_{i_1, \ldots, i_n} F(i_1, \ldots, i_n)G(i_1, \ldots, i_n)x_1^{i_1}\ldots x_n^{i_n} \tag{24}$$

are the canonical forms of the functions $f \cup g$ *and* fg, *respectively.*

Proof. Relation (23) results immediately from (14) and (16). Since $ax^i x^j = ax^{i \cup j}$, it follows also that

$$(fg)(x_1, \ldots, x_n) = \bigcup_{i_1, \ldots, i_n} [\bigcup_{j \cup k = i} F(j_1, \ldots, j_n)G(k_1, \ldots, k_n)]x_1^{i_1}\ldots x_n^{i_n}, \tag{25}$$

where $\bigcup_{j \cup k = i}$ means that the join is extended over those indices

$j_1, \ldots, j_n, i_1, \ldots, i_n \in \{0, 1\}$ which satisfy $j_1 \cup k_1 = i_1, \ldots, j_n \cup k_n = i_n$.

Since both F and G have the property (15), it follows that

$$\bigcup_{j \cup k = i} F(j_1, \ldots, j_n)G(k_1, \ldots, k_n) = F(i_1, \ldots, i_n)G(i_1, \ldots, i_n) \tag{26}$$

and hence (25) reduces to (24).

Since the constants $F \cup G$, as well as the constants FG, have obviously the property (15), it follows that (23) and (24) are actually the canonical forms of $f \cup g$ and fg, respectively.

The above theorems can be applied to the study of lattice equations.

As was remarked in [**1**], any equation $A = B$ is equivalent to the inequality $A \cup B \leqq AB$. Hence we shall focus our attention on inequalities of the form

$$f(x_1, \ldots, x_n) \leqq g(x_1, \ldots, x_n). \tag{27}$$

We begin with the following

Lemma 4. *The inequality*

$$f(x) \leqq g(x) \tag{28}$$

is solvable if and only if the relation

$$F(0) \leqq G(1) \tag{29}$$

holds. If this condition is fulfilled, then an element $x \in L$ *is a solution of* (28) *if and only if*

$$F(1)x \leqq G(1) \tag{30}$$

and

$$F(0) \leqq G(0) \cup x. \tag{31}$$

Proof. The result follows immediately from Lemma 1 and Theorem 1.

Corollary 2. *If* x' *and* x'' *satisfy the inequality* (28) *and* $x \in [x', x'']$, *i.e.*

$$x'x'' \leqq x \leqq x' \cup x'', \tag{32}$$

then x *is also a solution of* (28).

Proof. It follows from Lemma 4 that $F(1)x' \leqq G(1)$ and $F(1)x'' \leqq G(1)$; hence $F(1)x \leqq F(1)(x' \cup x'') \leqq G(1)$. The inequality (31) is proved similarly. Therefore x satisfies (28), again by Lemma 4.

Let us now associate with each lattice function $f(x_1, \ldots, x_n)$, the lattice functions

$$F(x_1, \ldots, x_m; i_{m+1}, \ldots, i_n) = \bigcup_{i_1, \ldots, i_m} F(i_1, \ldots, i_m, i_{m+1}, \ldots, i_n)x_1^{i_1} \ldots x_m^{i_m}. \quad (33)$$

Theorem 4. *The inequality* (27) *is solvable if and only if relation*

$$F(0, \ldots, 0) \leqq G(1, \ldots, 1) \quad (34)$$

holds. If this condition is fulfilled, then a vector $(x_1, \ldots, x_n) \in L^n$ *is a solution of* (27) *if and only if it satisfies the relations*

$$F(x_1, \ldots, x_{k-1}, 0, 0, \ldots, 0) \leqq G(x_1, \ldots, x_{k-1}, 0, 1, \ldots, 1) \cup x_k \quad (35)$$

and

$$F(x_1, \ldots, x_{k-1}, 1, 0, \ldots, 0)x_k \leqq G(x_1, \ldots, x_{k-1}, 1, 1, \ldots, 1) \quad (36)$$

for $k = 1, 2, \ldots, n$.

Proof. For $n = 1$, Theorem 4 reduces to Lemma 4. The proof is easily completed by induction.

Corollary 3. *If the condition* (34) *is fulfilled, then every vector* $(x_1, \ldots, x_n) \in L^n$ *satisfying*

$$F(x_1, \ldots, x_{k-1}, 0, 0, \ldots, 0) \leqq x_k \leqq G(x_1, \ldots, x_{k-1}, 1, 1, \ldots, 1) \quad (37)$$

for $k = 1, 2, \ldots, n$, *is a solution of* (27).

Theorem 4 can be specialized in the case when the lattice L is *biresiduated*, i.e. when it is residuated with respect to the meet and join operations. In other words, this means that for every two elements $a, b \in L$, there exists an element $a{:}b \in L$ and an element $a{::}b \in L$ such that $bx \leqq a$ if and only if $x \leqq a{:}b$, and $a \leqq b \cup x$ if and only if $a{::}b \leqq x$. Boolean algebras and totally ordered sets with 0 and 1 are examples of biresiduated lattices; the Cartesian product $L_1 \times L_2$ of two biresiduated lattices L_1 and L_2 is also biresiduated.

Theorem 5. *Assume the lattice L is biresiduated. If the condition* (34) *is fulfilled, then the solutions of the inequality* (27) *are given by*

$$\begin{aligned} F(x_1, \ldots, x_{k-1}, 0, 0, \ldots, 0){::}G(x_1, \ldots, x_{k-1}, 0, 1, \ldots, 1) &\leqq x_k \\ &\leqq G(x_1, \ldots, x_{k-1}, 1, 1, \ldots, 1){:}F(x_1, \ldots, x_{k-1}, 1, 0, \ldots, 0), \end{aligned} \quad (38)$$

for $k = 1, 2, \ldots, n$.

Proof. The result follows immediately from Theorem 4.

Theorem 5 generalizes a result proved by M. Gotō [2] for the two-element Boolean algebra and by the present author [5], [6] for arbitrary Boolean algebras; see also V. N. Grebenščikov [3].

The next theorem refers again to the general case of an arbitrary distributive lattice; it generalizes a theorem on Boolean functions which goes back to A. N. Whitehead [7].

Theorem 6. *Every lattice function* $f{:}L^n \to L$ *maps* L^n *onto the interval*

$$[F(0, \ldots, 0), F(1, \ldots, 1)].$$

Proof (essentially given in [**1**]). Let c be an element satisfying

$$F(0, \ldots, 0) \leqq c \leqq G(1, \ldots, 1); \tag{39}$$

we have to prove that the equation $f(x_1, \ldots, x_n) = c$, which is equivalent to the inequality

$$f(x_1, \ldots, x_n) \cup c \leqq f(x_1, \ldots, x_n)c, \tag{40}$$

is solvable. Taking into account Theorem 3, we see that the condition (34) for the inequality (40) becomes $F(0, \ldots, 0) \cup c \leqq F(1, \ldots, 1)c$ and it is satisfied, because (39) implies that $F(0, \ldots, 0) \cup c = c = F(1, \ldots, 1)c$.

Conversely, it follows from Theorem 1 that

$$F(0, \ldots, 0) \leqq f(x_1, \ldots, x_n) \leqq \bigcup_{i_1, \ldots, i_n} F(i_1, \ldots, i_n) = F(1, \ldots, 1).$$

Assume now that the lattice L has least and greatest elements and denote them by 0 and 1, respectively. Reasoning as in the proof of Theorem D in [**1**], we see that the coefficients $F(i_1, \ldots, i_n)$ occurring in the canonical form of a lattice function $f(x_1, \ldots, x_n)$ are simply $F(i_1, \ldots, i_n) = f(i_1, \ldots, i_n)$. Hence the theorems proved in Sections 1-2 of [**1**] are particular cases of our results.

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