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POINTWISE CONTRACTION CRITERIA FOR THE EXISTENCE OF FIXED POINTS

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ABSTRACT We show that, in a complete metric space, every selfmap that is a "weak directional contraction" admits a fixed point.

1. **Introduction.** Let (X, ρ) be a complete metric space, and let a function $T: X \to X$ be given. The celebrated contraction principle of Banach asserts that if there exists a number σ in $(0, 1)$ such that

$$(*) \qquad \rho(Tx, Ty) \le \sigma\rho(x, y) \forall x, y \in X,$$

(T is then said to be a *contraction*) then T has a (unique) fixed point; i.e. a point x such that $Tx = x$.

Our purpose is to investigate what can be said if (*) holds only in some local sense. For example, suppose for each x in X there is some neighborhood $N(x)$ of x such that

$$(**) \qquad \rho(Tx, Ty) \le \sigma\rho(x, y) \forall y \in N(x).$$

Must T have a fixed point? That the answer is negative follows from the fact that *any* function T satisfies this condition when ρ is the discrete metric (i.e. when the range of ρ is $\{0, 1\}$). Thus any such "pointwise" criterion must be accompanied in some way by at least an indirect hypothesis concerning the metric structure.

In the next section we discuss the main result of this paper, a fixed point theorem for "weak directional contractions". The proof of this result is given in §3.

2. **Weak directional contractions.** Let x and y be points in X. The *open interval* between x and y, denoted (x, y), is given by

$$(x, y) = \{z \in X : z \ne x, z \ne y, \rho(x, z) + \rho(z, y) = \rho(x, y)\}.$$

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Let $T: X \to X$ be a given mapping. We define $\underline{D}T(x; y)$, the lower derivative of T at x in the direction of y, as follows:

$$\underline{D}T(x; y) = 0 \quad \text{if} \quad y = x, \quad \text{and otherwise}$$

$$\underline{D}T(x; y) = \liminf_{\substack{z \to x \\ z \in (x, y)}} \rho(Tz, Tx)/\rho(z, x).$$

This has the usual meaning: for each $\varepsilon > 0$, we take the infimum of $\rho(Tz, Tx)/\rho(z, x)$ over those z in (x, y) such that $\rho(x, z) < \varepsilon$ (this is $+\infty$ if no such z exist). The limit of these infima is $\underline{D}T(x; y)$.

Definition 1. T is said to be a weak directional contraction if T is continuous and if there exists a number σ in $[0, 1)$ such that $\underline{D}T(x; Tx) \leq \sigma$ for all x in X.

Remark 1. Note that in order for T to be a weak directional contraction, it is necessary that (x, Tx) contain points arbitrarily near x whenever $x \neq Tx$. Thus if ρ is the discrete metric, the only weak directional contraction on X is the identity mapping. This example shows that the fixed point whose existence is asserted in the following theorem need not be unique.

Theorem 1. *Every weak directional contraction on a complete metric space has a fixed point.*

Remark 2. M. Edelstein [2] [3] has investigated the question of fixed points for mappings which are contractions in a certain local and uniform sense, by adapting the Picard method of successive approximations (which is ineffective in the context of Theorem 1). Other extensions of the contraction principle are possible when a Banach space structure is present; we refer the reader to Chapter 5 of the monograph by D. R. Smart [6]. The following example lies outside the bounds of the results cited above.

Example. Let $X = R^2$, with the norm given by:

$$\|(x, y)\| = |x| + |y|.$$

If $\rho((x, y), (x', y')) = \|(x - x', y - y')\|$, then (X, ρ) is a complete metric space. It is easy to see that the open interval between any two distinct points (x_1, y_1) and (x_2, y_2) consists of the closed solid rectangle having the two given points as diagonally opposite corners, with those two points deleted (this reduces to a line segment in the usual sense if x_1 and x_2 or y_1 and y_2 coincide).

We define $T: X \to X$ as follows:

$$T(x, y) = (3x/2 - y/3, x + y/3).$$

It is easily seen that T is not a contraction (even in a local sense). However, T

is a weak directional contraction. For let $T(x, y) \neq (x, y)$. Then (setting $T(x, y) = (a, b)$) it follows that $b \neq y$, so that the open interval between (x, y) and $T(x, y)$ contains points of the form (x, z) with z arbitrarily close to y. But for such points we have:

$$\rho(T(x, z), T(x, y))/\rho((x, z), (x, y)) = 2/3.$$

Note that the fixed points of T are all the points of the form $(x, 3x/2)$, $x \in R$.

The following extension of Theorem 1 applies to certain cases in which $\underline{D}T(x; Tx)$ is not necessarily bounded away from 1.

THEOREM 2. *Let T be a continuous selfmap on a complete metric space X such that* $\underline{D}T(x; Tx) < 1$ *for all x. Suppose that every sequence* $\{x_n\}$ *in X such that* $\underline{D}T(x_n; Tx_n)$ *is not bounded away from* 1 *has a cluster point. Then T has a fixed point.*

REMARK 3. The example $X = [1, \infty)$, $\rho =$ Euclidean metric, $Tx = x + 1/x$ shows that the cluster point condition cannot be dispensed with. To see that Theorem 2 is indeed more general than Theorem 1, consider a differentiable function $f : [0, 1] \to [0, 1]$ such that $|f'| < 1$ but $|f'|$ is not bounded away from 1.

A metric space X is said to be (metrically) *convex* if $(x, y) \neq \phi$ for every pair (x, y) of distinct points. A convex subset of a Banach space has this property.

T is called a *pointwise contraction* if for some σ in $[0, 1)$ we have, for all x,

$$\limsup_{\substack{y \to x \\ y \neq x}} \rho(Ty, Tx)/\rho(y, x) \leq \sigma.$$

COROLLARY 1. *Every pointwise contraction on a complete convex metric space has a fixed point.*

That this follows from Theorem 1 is a consequence of the following: (a) every pointwise contraction is continuous and (b) in a complete convex space, (x, y) contains points arbitrarily near x whenever $x \neq y$. These imply that a pointwise contraction of a complete convex space is a weak directional contraction. In fact, in the case of convex metric spaces, a stronger result is true. In the original version of this report [1], we asked the following question: is every pointwise contraction on a complete convex metric space a global contraction? A positive response to this question has now been given by W. A. Kirk and W. O. Ray [5].

3. **Proof of the theorems.** It suffices to prove Theorem 2. We now state for

convenience the following theorem of Ivar Ekeland [4]:

THEOREM. *Let $F: X \to [0, \infty)$ be a continuous function bounded below, and let $\varepsilon > 0$ be given. Then there is a point u such that*

(i) $F(u) < \inf_X F + \varepsilon$,

(ii) $F(x) - F(u) \geq -\varepsilon\rho(x, u) \forall x \in X$.

Let us define $F: X \to [0, \infty)$ as follows:

$$F(x) = \rho(Tx, x).$$

Since T is continuous, it follows that F is continuous. Applying Ekeland's theorem, we deduce the existence, for each positive integer K, of a point u_K such that

$$F(u_K) < \inf_X F + 1/K, \tag{1}$$

$$F(x) + \rho(x, u_K)/K \geq F(u_K) \forall x \in X. \tag{2}$$

If for any K we have $F(u_K) = 0$, then u_K is a fixed point and we are done. So let us suppose that $F(u_K)$ is positive for each K.

CLAIM. $\underline{D}T(u_K; Tu_K) \geq 1 - 1/K$.

Since $u_K \neq T(u_K)$ there exists a sequence $\{x_n\}$ in (u_K, Tu_K) such that $\rho(u_K, x_n)$ converges to 0 as $n \to \infty$, and

$$\lim_{n \to \infty} \rho(Tx_n, Tu_K)/\rho(x_n, u_K) = \underline{D}T(u_K; Tu_K). \tag{3}$$

By definition,

$$\rho(u_K, Tu_K) = \rho(u_K, x_n) + \rho(x_n, Tu_K). \tag{4}$$

We find (in light of (2)):

$$\begin{aligned}\rho(u_K, Tu_K) &\leq \rho(x_n, Tx_n) + \rho(x_n, u_K)/K \\ &\leq \rho(x_n, Tu_K) + \rho(Tu_K, Tx_n) + \rho(x_n, u_K)/K \\ &\leq \rho(x_n, Tu_K) + \underline{D}T(u_K; Tu_K)\rho(x_n, u_K) + o(\rho(x_n, u_K)) + \rho(x_n, u_K)/K,\end{aligned}$$

where $o(\rho(x_n, u_K))/\rho(x_n, u_K) \to 0$ as $n \to \infty$.

Combining this with (4), we arrive at:

$$(1 - 1/K)\rho(x_n, u_K) \leq \underline{D}T(u_K, Tu_K)\rho(x_n, u_K) + o(\rho(x_n, u_K)). \tag{5}$$

Dividing across by $\rho(x_n, u_K)$ and letting n tend to ∞, we obtain the required inequality.

The hypotheses now imply that the sequence $\{u_K\}$ has a cluster point u. In

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view of (1), we have

$$\rho(x, Tx) \geq \rho(u, Tu) \forall x \in X. \tag{6}$$

If $u = Tu$ we are done, so let us suppose the contrary and show that (6) leads to a contradiction. Arguing as we did to obtain (5), we obtain a sequence $\{x_n\}$ in (u, Tu) such that $\rho(x_n, u)$ tends to 0 as $n \to \infty$, and

$$\rho(x_n, u) \leq \underline{D}T(u; Tu)\rho(x_n, u) + o(\rho(x_n, u)),$$

where $o(\rho(x_n, u))/\rho(x_n, u) \to 0$ as $n \to \infty$. This implies

$$\underline{D}T(u; Tu) \geq 1,$$

which contradicts the hypotheses. Q.E.D.

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