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SYMMETRY BREAKING FOR GROUND-STATE SOLUTIONS OF HÉNON SYSTEMS IN A BALL

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Abstract. We consider in this paper the problem

$$\begin{cases} -\Delta u = |x|^{\alpha} v^{p}, & x \in \Omega, \\ -\Delta v = |x|^{\beta} u^{q_\varepsilon}, & x \in \Omega, \\ u > 0, \quad v > 0, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is the unit ball in $\mathbb{R}^N$ centred at the origin, $0 \le \alpha < pN$, $\beta > 0$, $N \ge 3$. Suppose $q_\varepsilon \to q$ as $\varepsilon \to 0^+$ and q_ε, q satisfy, respectively,

$$\frac{N}{p+1} + \frac{N}{q_\varepsilon + 1} > N - 2, \qquad \frac{N}{p+1} + \frac{N}{q+1} = N - 2;$$

we investigate the asymptotic estimates of the ground-state solutions $(u_\varepsilon, v_\varepsilon)$ of (1) as $\beta \to +\infty$ with p, q_ε fixed. We also show the symmetry-breaking phenomenon with α, β fixed and $q_\varepsilon \to q$ as $\varepsilon \to 0^+$. In addition, the ground-state solution is non-radial provided that $\varepsilon > 0$ is small or β is large enough.

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1. Introduction. In this paper, we investigate the limiting behaviour of the ground-state solutions of the problem

$$\begin{cases} -\Delta u = |x|^{\alpha} v^{p}, & x \in \Omega, \\ -\Delta v = |x|^{\beta} u^{q_\varepsilon}, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{2}$$

where Ω is the unit ball in $\mathbb{R}^N$ centred at the origin, $0 \le \alpha < pN$, $\beta > 0$, $N \ge 3$. We assume in this paper that $q_\varepsilon \to q$ as $\varepsilon \to 0^+$ and q_ε, q satisfy, respectively,

$$\frac{N}{p+1} + \frac{N}{q_\varepsilon + 1} > N - 2, \qquad \frac{N}{p+1} + \frac{N}{q+1} = N - 2.$$

Problem (2) has two features. First, it is a Hénon-type system. The Hénon equation with Dirichlet boundary conditions

$$\begin{cases} -\Delta u = |x|^{\alpha} u^{p}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \tag{3}$$

was found in [10], which stems from rotating stellar structures. A standard compactness argument show that the infimum

$$\inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx}{(\int_\Omega |x|^\alpha |u|^{p+1}\,dx)^{\frac{2}{p+1}}} \tag{4}$$

is achieved for any $1 < p < 2^* - 1$, $\alpha > 0$. In 1982, Ni [14] proved that the infimum

$$\inf_{u\in H_{0,\mathrm{rad}}^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx}{(\int_\Omega |x|^\alpha |u|^{p+1}\,dx)^{\frac{2}{p+1}}} \tag{5}$$

is achieved for any $p \in (1, \frac{N+2+2\alpha}{N-2})$ by a function in $H_{0,\mathrm{rad}}^1(\Omega)$, the space of radial $H_0^1(\Omega)$ functions. Thus, radial solutions of (3) exist also for (Sobolev) supercritical exponents p. A natural question is whether any minimizer of (4) must be radially symmetric in the range $1 < p < \frac{N+2}{N-2}$ and $\alpha > 0$. Since the weight $|\cdot|^\alpha$ is an increasing function, neither rearrangement arguments nor the moving plane techniques of [7] can be applied.

For $\alpha > 0$, Smets et al. proved in [15] some symmetry-breaking results for (3). They proved that the minimizers of (4) (the so-called ground-state solutions, or least energy solutions) cannot be radial for α large enough. As a consequence, (3) has at least two solutions when α is sufficiently large (see also [16]).

Quite recently, Cao and Peng [3] proved that for $p+1$ sufficiently close to 2^*, the ground-state solutions of (3) possess a unique maximum point whose distance from $\partial\Omega$ tends to zero as $p \to \frac{N+2}{N-2}$.

For more results about symmetry breaking phenomena for solutions of problem (3) either α is large enough or $p \to \frac{N+2}{N-2}$, see for instance [2, 1] and references therein.

Second, the system in (2) is a Hamiltonian-type system, which is strongly indefinite. The existence of solutions of the Hamiltonian elliptic system

$$\begin{cases} -\Delta u = v^p, & x \in \Omega, \\ -\Delta v = u^q, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega \end{cases} \tag{6}$$

was first considered in [5] and [11] with $\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}$; the curve of $(p, q) \in \mathbb{R}^2$ satisfying $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$ is called critical hyperbola. Afterwards, various results were obtained in the literature. Extensions of problem (6) can be found in [6] and [13]. In [13], existence problems for Hardy-type systems and Hénon-type systems were established. Particularly, for Hénon-type systems

$$\begin{cases} -\Delta u = |x|^\alpha v^p, & x \in \Omega, \\ -\Delta v = |x|^\beta u^q, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{7}$$

the critical hyperbola is $\frac{1}{p+1}(1 + \frac{\alpha}{N}) + \frac{1}{q+1}(1 + \frac{\beta}{N}) = \frac{N-2}{N}$.

In recent years, a study of the limiting behaviour of ground-state solutions of elliptic problems has attracted considerable attention. For the system (6), the limiting behaviour of solutions of (6) as $\frac{1}{p+1} + \frac{1}{q+1} \to \frac{N-2}{N}$ was discussed in [8]. For the system (7), Yang and He [9] proved that for $\frac{1}{p+1} + \frac{1}{q+1} \to \frac{N-2}{N}$, the ground-state solutions of (7) possess a unique maximum point whose distance from $\partial\Omega$ tends to zero as

$\frac{1}{p+1} + \frac{1}{q+1} \to \frac{N-2}{N}$. Such problems are closely related to solutions of the following problem:

$$\begin{cases} -\Delta U = V^p, & y \in \mathbb{R}^N, \\ -\Delta V = U^q, & y \in \mathbb{R}^N, \\ U(y) > 0, \quad V(y) > 0, & y \in \mathbb{R}^N, \\ U(0) = 1, \quad U \to 0, \quad V \to 0 \text{ as } |y| \to \infty, \end{cases} \tag{8}$$

where $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$. It was proved in **[12]** that $U \in D^{2,\frac{p+1}{p}}(\mathbb{R}^N)$, $V \in D^{2,\frac{q+1}{q}}(\mathbb{R}^N)$, where $D^{2,r}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|\Delta \cdot\|_r$. Actually, U and V are radially symmetric for $p \geq 1$ as showed in **[4]**. Moreover, U and V are unique and decreasing in r. In the discussion of one equation problem, one uses the instanton for the best Sobolev constant. However, no explicit form of (U, V) was found for $p > \frac{2}{N-2}$ up to now. Instead, the asymptotic behaviour of (U, V) as $r \to \infty$ is sufficient for this purpose. It was found in **[12]** that

$$\lim_{r\to\infty} r^{N-2}V(r) = a, \qquad \begin{cases} \lim\limits_{r\to\infty} r^{N-2}U(r) = b & \text{if } p > \frac{N}{N-2}, \\ \lim\limits_{r\to\infty} \dfrac{r^{N-2}}{\log r} U(r) = b & \text{if } p = \frac{N}{N-2}, \\ \lim\limits_{r\to\infty} r^{p(N-2)-2}U(r) = b & \text{if } \frac{2}{N-2} < p < \frac{N}{N-2}, \end{cases} \tag{9}$$

and

$$\lim_{r\to\infty} \frac{rV'(r)}{V(r)} = 2 - N, \qquad \begin{cases} \lim\limits_{r\to\infty} \frac{rU'(r)}{U(r)} = 2 - N & \text{if } p \geq \frac{N}{N-2}, \\ \lim\limits_{r\to\infty} \frac{rU'(r)}{U(r)} = 2 - p(N-2) & \text{if } p \leq \frac{N}{N-2}. \end{cases} \tag{10}$$

In this paper, we are interested in the symmetry of ground-states solutions of (2). Now, we denote

$$E_\alpha(\Omega) = \left\{ u \in W^{2,\frac{p+1}{p}} \cap W_0^{1,\frac{p+1}{p}}(\Omega) : \int_\Omega |x|^{-\frac{\alpha}{p}} |\Delta u|^{\frac{p+1}{p}}\, dx < \infty \right\}$$

and

$$E_\alpha^{\text{rad}}(\Omega) = \{u \in E_\alpha(\Omega) : u(x) = u(|x|)\}.$$

Our main results are as follows.

THEOREM 1.1. *Suppose $N \geq 3$, $0 \leq \alpha < pN$, $\beta > 0$, $p > \frac{2}{N-2}$, $q_\varepsilon > \frac{N+p}{Np-2p-1}$; then there exists $\beta^* > 0$ such that the ground-state solutions $u_{\alpha,\beta,\varepsilon}$ are non-radial provided $\beta > \beta^*$.*

THEOREM 1.2. *Suppose $N \geq 3$, $0 \leq \alpha < pN$, $\beta > 0$, $p > \frac{2}{N-2}$, $pq_\varepsilon > 1$; then there exists $\varepsilon^* > 0$ such that the ground-state solutions $u_{\alpha,\beta,\varepsilon}$ are non-radial provided $\varepsilon < \varepsilon^*$ or $q - q_\varepsilon < \varepsilon^*$.*

This paper is organized as follows. In section 2, we give some preliminaries which turn out to be essential. In section 3, we present some estimates for radial ground-state solutions of (2) with α, p, q_ε fixed and $\beta \to \infty$. This will lead us to get the first symmetry-breaking result, stating that for β sufficiently large, the ground-state solution of problem (2) is non-radial. In section 4, another symmetry-breaking result is proved, with α, β, p fixed and $q_\varepsilon \to q$ as $\varepsilon \to 0$.

2. Preliminaries. Before proving our main results, we want to introduce some simple calculus lemma which turns out to be essential:

LEMMA 2.1. *Let u be a radially symmetric function of Ω (unit ball in $\mathbb{R}^N$) with $u(1) = 0$, $u'(0)$exists. Then*

$$(i)\ |u(x)| \le \frac{1}{w_{N-1}^{\frac{p}{p+1}}(p(N-1)-1)^{\frac{1}{p+1}}} \frac{(\int_\Omega |\nabla u|^{\frac{p+1}{p}}\,dy)^{\frac{p}{p+1}}}{(|x|^{p(N-1)-1})^{\frac{1}{p+1}}}$$

$$(ii)\ \left|\frac{\partial u}{\partial r}\right| \le r^{1-N}\left(\frac{r^{N+\alpha}}{N+\alpha}\right)^{\frac{1}{p+1}}\left(\frac{1}{w_{N-1}}\int_{B_r(0)} |x|^{-\frac{\alpha}{p}}|\Delta u|^{\frac{p+1}{p}}\,dx\right)^{\frac{p}{p+1}},$$

where w_{N-1} is the surface area of the unit ball in $\mathbb{R}^N$.

Proof. (i) For

$$\begin{aligned} u(1) - u(x) &= \int_{|x|}^1 |u'(t)|\,dt, \\ |u(x)| &\le \int_{|x|}^1 |u'(t)|\,dt \\ &\le \left(\int_{|x|}^1 |u'(t)|^{\frac{p+1}{p}} t^{N-1}\,dt\right)^{\frac{p}{p+1}} \left(\int_{|x|}^1 t^{-(N-1)p}\,dt\right)^{\frac{1}{p+1}}. \end{aligned}$$

Since

$$\int_{|x|}^1 t^{-(N-1)p}\,dt = \frac{1}{p(N-1)-1}\left(\frac{1}{|x|^{p(N-1)-1}} - 1\right) \le \frac{1}{p(N-1)-1}\frac{1}{|x|^{p(N-1)-1}}$$

and

$$\begin{aligned} \int_{|x|}^1 |u'(t)|^{\frac{p+1}{p}} t^{N-1}\,dt &= \frac{1}{w_{N-1}} \int \left(\int_{|x|}^1 |u'(t)|^{\frac{p+1}{p}} t^{N-1}\,dt\right) w(\theta)\,d\theta \\ &= \frac{1}{w_{N-1}} \int_{|x|\le|y|\le 1} |\nabla u|^{\frac{p+1}{p}}\,dy \\ &\le \frac{1}{w_{N-1}} \int_\Omega |\nabla u|^{\frac{p+1}{p}}\,dy, \end{aligned}$$

we find that

$$|u(x)| \le \frac{1}{w_{N-1}^{\frac{p}{p+1}}(p(N-1)-1)^{\frac{1}{p+1}}} \frac{(\int_\Omega |\nabla u|^{\frac{p+1}{p}}\,dy)^{\frac{p}{p+1}}}{(|x|^{p(N-1)-1})^{\frac{1}{p+1}}}.$$

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(*ii*) For

$$|r^{N-1}\frac{\partial u}{\partial r}| \le \int_0^r t^{N-1}|\Delta u|\,dt$$
$$\le \left(\int_0^r t^{\frac{(N-1+\alpha)(p+1)}{p+1}}\,dt\right)^{\frac{1}{p+1}}\left(\int_0^r (t^{\frac{p(N-1)-\alpha}{p+1}}|\Delta u|)^{\frac{p+1}{p}}\,dt\right)^{\frac{p}{p+1}}$$
$$= \left(\frac{r^{N+\alpha}}{N+\alpha}\right)^{\frac{1}{p+1}}\left(\frac{1}{w_{N-1}}\int_{B_r(0)}|x|^{-\frac{\alpha}{p}}|\Delta u|^{\frac{p+1}{p}}\,dx\right)^{\frac{p}{p+1}},$$

we have

$$\left|\frac{\partial u}{\partial r}\right| \le r^{1-N}\left(\frac{r^{N+\alpha}}{N+\alpha}\right)^{\frac{1}{p+1}}\left(\frac{1}{w_{N-1}}\int_{\Omega}|x|^{-\frac{\alpha}{p}}|\Delta u|^{\frac{p+1}{p}}\,dx\right)^{\frac{p}{p+1}}.$$

□

As a result of Lemma 2.1, we obtain the following corollary.

COROLLARY 2.1. *Under hypothesis of Lemma 2.1, and if* $\frac{N+\beta}{q+1}+\frac{N+\alpha}{p+1} > N-2$, *then*

$$\int_{\Omega}|x|^{\beta}|u(x)|^{q+1}\,dx \le C\left(\int_{\Omega}|x|^{-\frac{\alpha}{p}}|\Delta u|^{\frac{p+1}{p}}\,dx\right)^{\frac{p(q+1)}{p+1}},$$

where

$$C=\left(\frac{1}{w_{N-1}}\right)^{\frac{p(q+1)}{p+1}}\frac{1}{N+\beta-\left(N-2-\frac{N+\alpha}{p+1}\right)(q+1)}\left(\frac{1}{N-2-\frac{N+\alpha}{p+1}}\right)^{q+1}\left(\frac{1}{N+\alpha}\right)^{\frac{q+1}{p+1}}.$$

Proof. From the above Lemma 2.1, we have

$$|u(|x|)| \le \int_{|x|}^1 |u'(r)|\,dr \le \int_{|x|}^1\left(\frac{r^{N+\alpha}}{N+\alpha}\right)^{\frac{1}{p+1}} r^{1-N}\,dr\left(\frac{1}{w_{N-1}}\int_{\Omega}|x|^{-\frac{\alpha}{p}}|\Delta u|^{\frac{p+1}{p}}\,dx\right)^{\frac{p}{p+1}}.$$

Since

$$\int_{|x|}^1\left(\frac{r^{N+\alpha}}{N+\alpha}\right)^{\frac{1}{p+1}} r^{1-N}\,dr = \left(\frac{1}{N+\alpha}\right)^{\frac{1}{p+1}}\int_{|x|}^1 r^{\frac{N+\alpha}{p+1}}r^{1-N}\,dr$$
$$\le \frac{1}{N-2-\frac{N+\alpha}{p+1}}\left(\frac{1}{N+\alpha}\right)^{\frac{1}{p+1}}\frac{1}{|x|^{N-2-\frac{N+\alpha}{p+1}}},$$

we have

$$r^{\beta}r^{N-1}|u(r)|^{q+1}$$
$$\le r^{\beta}r^{N-1}\left(\frac{1}{N-2-\frac{N+\alpha}{p+1}}(\frac{1}{N+\alpha})^{\frac{1}{p+1}}\frac{1}{r^{N-2-\frac{N+\alpha}{p+1}}}\right)^{q+1}\left(\frac{1}{w_{N-1}}\int_{\Omega}|x|^{-\frac{\alpha}{p}}|\Delta u|^{\frac{p+1}{p}}\,dx\right)^{\frac{p(q+1)}{p+1}},$$

which implies

$$\int_0^1 r^\beta r^{N-1}|u(r)|^{q+1}\,dr \le \left(\frac{1}{N-2-\frac{N+\alpha}{p+1}}\left(\frac{1}{N+\alpha}\right)^{\frac{1}{p+1}}\right)^{q+1}$$
$$\times \int_0^1 \left(\frac{1}{r^{N-2-\frac{N+\alpha}{p+1}}}\right)^{q+1} r^\beta r^{N-1}\,dr \left(\frac{1}{w_{N-1}}\int_\Omega |x|^{-\frac{\alpha}{p}}|\Delta u|^{\frac{p+1}{p}}\,dx\right)^{\frac{p(q+1)}{p+1}}.$$

Thus, the conclusion holds. □

3. Asymptotic estimates. Consider the minimization problem

$$S^{\text{rad}}_{\alpha,\,\beta,\,\varepsilon} = \inf_{u\in E^{\text{rad}}_\alpha(\Omega)\setminus\{0\}} R_{\alpha,\beta,\varepsilon}(u), \tag{11}$$

where

$$R_{\alpha,\beta,\varepsilon}(u) = \frac{\int_\Omega |x|^{-\frac{\alpha}{p}}|\Delta u|^{\frac{p+1}{p}}\,dx}{(\int_\Omega |x|^\beta |u|^{q_\varepsilon+1}\,dx)^{\frac{p+1}{p(q_\varepsilon+1)}}}, \qquad u\in E_\alpha(\Omega)\setminus\{0\}, \tag{12}$$

is the Rayleigh quotient associated with (2). Similar to **[14]**, we can also prove that

$$S^{\text{rad}}_{\alpha,\beta,\varepsilon}(\Omega) = \inf_{u\in E^{\text{rad}}_\alpha(\Omega)\setminus\{0\}} \frac{\int_\Omega |x|^{-\frac{\alpha}{p}}|\Delta u|^{\frac{p+1}{p}}\,dx}{(\int_\Omega |x|^\beta |u|^{q_\varepsilon+1}\,dx)^{\frac{p+1}{p(q_\varepsilon+1)}}}$$

is attained by some positive function $u^{\text{rad}}_{\alpha,\beta,\varepsilon}$. After scaling, $u^{\text{rad}}_{\alpha,\beta,\varepsilon}$ is also a solution of (2).

Now, we provide an estimate of the energy $S^{\text{rad}}_{\alpha,\beta,\varepsilon}$ as $\beta\to\infty$.

LEMMA 3.1. *If $N\ge 3$, there exists $C>0$ depending on N, p such that*

$$S^{\text{rad}}_{\alpha,\,\beta,\,\varepsilon} \ge C\beta^{\frac{p+2+q_\varepsilon}{p(q_\varepsilon+1)}} \text{ as } \beta\to\infty.$$

Proof. Let $u\in E^{\text{rad}}_\alpha(\Omega)$ and define the rescaled function $v(|x|)=u(|x|^s)$, where $s=\frac{N}{\beta+N}$. Then

$$\int_\Omega |x|^\beta|u|^{q_\varepsilon+1}\,dx = w_{N-1}\int_0^1 r^{\beta+N-1}|u(r)|^{q_\varepsilon+1}\,dr = s\int_\Omega |x|^{s(\beta+N)-N}|v(x)|^{q_\varepsilon+1}\,dx,$$

and from Lemma 2.1, we have

$$\int_\Omega |x|^{-\frac{\alpha}{p}}|\Delta u|^{\frac{p+1}{p}}\,dx \ge w_{N-1}^{\frac{p+1}{p}}(N+\alpha)^{\frac{1}{p}} r^{\frac{(p+1)(N-1)-(N+\alpha)}{p}}\left|\frac{\partial u}{\partial r}\right|^{\frac{p+1}{p}},$$

which implies that

$$\int_\Omega |x|^{-\frac{\alpha}{p}} |\Delta u|^{\frac{p+1}{p}}\, dx \geq w_{N-1}^{\frac{p+1}{p}} (N+\alpha)^{\frac{1}{p}} \int_0^1 r^{\frac{(p+1)(N-1)-(N+\alpha)}{p}} \left|\frac{\partial u}{\partial r}\right|^{\frac{p+1}{p}} dr$$
$$= w_{N-1}^{\frac{p+1}{p}} (N+\alpha)^{\frac{1}{p}} \int_0^1 r^{\frac{(p+1)(N-1)-(N+\alpha)}{p}} s^{-\frac{p+1}{p}} r^{\frac{1-s}{s}\frac{p+1}{p}} \left|\frac{\partial v}{\partial t}\right|^{\frac{p+1}{p}} dr$$
$$= w_{N-1}^{\frac{1}{p}} (N+\alpha)^{\frac{1}{p}} s^{-\frac{1}{p}} \int_\Omega |x|^{\frac{p(N-1)(s-1)-s(1+\alpha)+(1-s)}{p}} |\nabla v|^{\frac{p+1}{p}}\, dx.$$

Thus, we obtain

$$\frac{\int_\Omega |x|^{-\frac{\alpha}{p}} |\Delta u|^{\frac{p+1}{p}}\, dx}{(\int_\Omega |x|^\beta |u|^{q_\varepsilon+1}\, dx)^{\frac{p+1}{p(q_\varepsilon+1)}}} \geq \frac{w_{N-1}^{\frac{1}{p}} (N+\alpha)^{\frac{1}{p}} s^{-\frac{1}{p}} \int_\Omega |x|^{\frac{p(N-1)(s-1)-s(1+\alpha)+(1-s)}{p}} |\nabla v|^{\frac{p+1}{p}}\, dx}{(s\int_\Omega |v(x)|^{q_\varepsilon+1}\, dx)^{\frac{p+1}{p(q_\varepsilon+1)}}}.$$

It follows that

$$S^{\mathrm{rad}}_{\alpha,\beta,\varepsilon} \geq w_{N-1}^{\frac{1}{p}} (N+\alpha)^{\frac{1}{p}} s^{-\frac{1}{p}} s^{-\frac{p+1}{p(q_\varepsilon+1)}} \inf_{v\in W^{1,\frac{p+1}{p}}_{0,\mathrm{rad}}(\Omega)} \frac{\int_\Omega |x|^{\frac{p(N-1)(s-1)-s(1+\alpha)+(1-s)}{p}} |\nabla v|^{\frac{p+1}{p}}\, dx}{(\int_\Omega |v(x)|^{q_\varepsilon+1}\, dx)^{\frac{p+1}{p(q_\varepsilon+1)}}}.$$

Now, we claim that for every $0 \leq s \leq 1$, we have $p(N-1)(s-1) - s(1+\alpha) + (1-s) < 0$. Indeed, for

$$p(N-1)(s-1) - s(1+\alpha) + (1-s)$$
$$= (p(N-1) - 2 - \alpha)s - (p(N-1) - 1),$$

if $p(N-1) - 2 \leq \alpha < pN$, we have $p(N-1)(s-1) - s(1+\alpha) + (1-s) < 0$; if $\alpha < p(N-1) - 2 < p(N-1) - 1$, then for every $0 \leq s \leq 1$, we also have $p(N-1)(s-1) - s(1+\alpha) + (1-s) < 0$. Therefore,

$$\int_\Omega |\nabla v|^{\frac{p+1}{p}}\, dx \leq \int_\Omega |x|^{\frac{p(N-1)(s-1)-s(1+\alpha)+(1-s)}{p}} |\nabla v|^{\frac{p+1}{p}}\, dx,$$

which implies that

$$c_s = \inf_{v\in W^{1,\frac{p+1}{p}}_{0,\mathrm{rad}}(\Omega)} \frac{\int_\Omega |x|^{\frac{p(N-1)(s-1)-s(1+\alpha)+(1-s)}{p}} |\nabla v|^{\frac{p+1}{p}}\, dx}{(\int_\Omega |v(x)|^{q_\varepsilon+1}\, dx)^{\frac{p+1}{p(q_\varepsilon+1)}}}$$

is achieved by standard arguments. Since $|x| \leq 1$, if $p(N-1) - 2 \leq \alpha < pN$, $p(N-1)(s-1) - s(1+\alpha) + (1-s) \leq -(p(N-1) - 1)$, which implies that $c_s \geq c_0$; if $\alpha < p(N-1) - 2$, $p(N-1)(s-1) - s(1+\alpha) + (1-s) = (p(N-1) - 2 - \alpha)s - (p(N-1) - 1)$, then for every $0 \leq s \leq 1$, $|x|^{\frac{p(N-1)(s-1)-s(1+\alpha)+(1-s)}{p}}$ is non-increasing, which implies that c_s is non-increasing on [0, 1]; then $c_s \geq c_1$. Thus,

$$S^{\mathrm{rad}}_{\alpha,\beta,\varepsilon} \geq C(N+\alpha)^{\frac{1}{p}} s^{-\frac{p+2+q_\varepsilon}{p(q_\varepsilon+1)}}, \qquad \beta \to \infty.$$

□

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By the assumptions on p, q_ε, the inclusion $W^{2,\frac{p+1}{p}}(\Omega) \hookrightarrow L^{q_\varepsilon+1}(\Omega)$ is compact. It implies that

$$S_{\alpha,\beta,\varepsilon}(\Omega) = \inf_{u\in E_\alpha(\Omega)\setminus\{0\}} \frac{\int_\Omega |x|^{-\frac{\alpha}{p}}|\Delta u|^{\frac{p+1}{p}}\,dx}{(\int_\Omega |x|^\beta |u|^{q_\varepsilon+1}\,dx)^{\frac{p+1}{p(q_\varepsilon+1)}}}$$

is attained by some positive function $u_{\alpha,\beta,\varepsilon}$. After scaling, $u_{\alpha,\beta,\varepsilon}$ is a solution of (2).

LEMMA 3.2. *Assume $N \geq 3$, for any p, q_ε satisfy $\frac{N}{p+1} + \frac{N}{q_\varepsilon+1} > N-2$ with $p > \frac{2}{N-2}$, $q_\varepsilon > \frac{N+p}{Np-2p-1}$; there exists $\beta^* \geq 0$ such that $S_{\alpha,\beta,\varepsilon} < S^{\mathrm{rad}}_{\alpha,\beta,\varepsilon}$ provided $\beta > \beta^*$.*

Proof. For any fixed $u \in C_0^\infty(\Omega)$, define $u_\beta(x) = u(\beta(x-x_\beta))$, where $x_\beta = (1-\frac{1}{\beta}, 0, \dots, 0)$. For $|\beta(x-x_\beta)| \leq 1$, that is, $|x-x_\beta| \leq \frac{1}{\beta}$, then $|x| \geq |x_\beta| - \frac{1}{\beta} = 1-\frac{2}{\beta}$. One has

$$\int_\Omega |x|^{-\frac{\alpha}{p}}|\Delta u_\beta|^{\frac{p+1}{p}}\,dx \leq \left(1-\frac{2}{\beta}\right)^{-\frac{\alpha}{p}} \beta^{\frac{2(p+1)}{p}-N} \int_\Omega |\Delta u|^{\frac{p+1}{p}}\,dx,$$

and

$$\int_\Omega |x|^\beta |u_\beta|^{q_\varepsilon+1}\,dx \geq \left(1-\frac{2}{\beta}\right)^\beta \beta^{-N} \int_\Omega |u|^{q_\varepsilon+1}\,dx.$$

Hence by definition, one obtains

$$\begin{aligned} S_{\alpha,\beta,\varepsilon} &\leq \frac{\int_\Omega |x|^{-\frac{\alpha}{p}}|\Delta u_\beta|^{\frac{p+1}{p}}\,dx}{(\int_\Omega |x|^\beta |u_\beta|^{q_\varepsilon+1}\,dx)^{\frac{p+1}{p(q_\varepsilon+1)}}} \\ &\leq \frac{\left(1-\frac{2}{\beta}\right)^{-\frac{\alpha}{p}} \beta^{\frac{2(p+1)}{p}-N} \int_\Omega |\Delta u|^{\frac{p+1}{p}}\,dx}{\left(\left(1-\frac{2}{\beta}\right)^\beta \beta^{-N} \int_\Omega |u|^{q_\varepsilon+1}\,dx\right)^{\frac{p+1}{p(q_\varepsilon+1)}}} \\ &\leq C_1 \beta^{\frac{2(p+1)}{p}-N+\frac{N(p+1)}{p(q_\varepsilon+1)}} \frac{\int_\Omega |\Delta u|^{\frac{p+1}{p}}\,dx}{(\int_\Omega |u|^{q_\varepsilon+1}\,dx)^{\frac{p+1}{p(q_\varepsilon+1)}}}. \end{aligned}$$

Since u is fixed and $\frac{\int_\Omega |\Delta u|^{\frac{p+1}{p}}\,dx}{(\int_\Omega |u|^{q_\varepsilon+1}\,dx)^{\frac{p+1}{p(q_\varepsilon+1)}}}$ is independent of β, we have

$$S_{\alpha,\beta,\varepsilon} \leq C\beta^{\frac{2(p+1)}{p}-N+\frac{N(p+1)}{p(q_\varepsilon+1)}}.$$

From Lemma 3.1 $S^{\mathrm{rad}}_{\alpha,\beta,\varepsilon} \geq C\beta^{\frac{p+2+q_\varepsilon}{p(q_\varepsilon+1)}}$ as $\beta \to \infty$, and $\frac{p+2+q_\varepsilon}{p(q_\varepsilon+1)} > \frac{2(p+1)}{p} - N + \frac{N(p+1)}{p(q_\varepsilon+1)}$, that is, $q_\varepsilon > \frac{N+p}{Np-2p-1}$. Hence $S_{\alpha,\beta,\varepsilon} < S^{\mathrm{rad}}_{\alpha,\beta,\varepsilon}$ as $\beta \to \infty$. □

4. Analysis for ε close to 0. In this section, we analyse the case where ε is close to 0, that is, q_ε is close to q. We will show that for any fixed $0 \leq \alpha < pN$, $\beta > 0$, the minimizer of $R_{\alpha,\beta,\varepsilon}$ is non-radial provided that ε is sufficiently small.

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LEMMA 4.1. *If $N \geq 3$, there exists $c_0 > 0$, such that for every q_ε and for every $0 \leq \alpha < pN$, $\beta > 0$,*

$$c_0 \beta^{\frac{p+1}{p(q_\varepsilon+1)}} \leq S^{rad}_{\alpha,\beta,\varepsilon}.$$

Proof. From Lemma 2.1, we obtain

$$\int_\Omega |x|^\beta |u(x)|^{q_\varepsilon+1}\, dx \leq C\bigg(\int_\Omega |x|^{-\frac{\alpha}{p}} |\Delta u|^{\frac{p+1}{p}}\, dx\bigg)^{\frac{p(q_\varepsilon+1)}{p+1}},$$

where

$$C = \left(\frac{1}{w_{N-1}}\right)^{\frac{p(q_\varepsilon+1)}{p+1}} \frac{1}{N+\beta-(N-2-\frac{N+\alpha}{p+1})(q_\varepsilon+1)} \left(\frac{1}{N-2-\frac{N+\alpha}{p+1}}\right)^{q_\varepsilon+1} \left(\frac{1}{N+\alpha}\right)^{\frac{q_\varepsilon+1}{p+1}}.$$

Since $u \in E^{\mathrm{rad}}_\alpha(\Omega)$ is arbitrary,

$$c_0\bigg(N+\beta-(N-2-\frac{N+\alpha}{p+1})(q_\varepsilon+1)\bigg)^{\frac{(p+1)}{p(q_\varepsilon+1)}} \leq S^{\mathrm{rad}}_{\alpha,\beta,\varepsilon},$$

which ends the proof. □

Let us denote by S the classical Sobolev constant

$$S = \inf_{u\in W^{2,\frac{p+1}{p}} \cap W_0^{1,\frac{p+1}{p}}(\Omega)} \frac{\int_\Omega |\Delta u|^{\frac{p+1}{p}}\, dx}{(\int_\Omega |u|^{q+1}\, dx)^{\frac{p+1}{p(q+1)}}}.$$

It is standard that this Rayleigh quotient is invariant under translations and dilations.

LEMMA 4.2. *If $N \geq 3$ and $0 \leq \alpha < pN$, $\beta > 0$, then*

$$S = S_{\alpha,\beta,0} < S^{rad}_{\alpha,\beta,0}.$$

Proof. Using Corollary 2.1, it is easy to verify that $S^{\mathrm{rad}}_{\alpha,\beta,0}$ is achieved, so that $S < S^{\mathrm{rad}}_{\alpha,\beta,0}$. Now, we claim that $S = S_{\alpha,\beta,0}$. From the definition of $S_{\alpha,\beta,0}$, we know that $S \leq S_{\alpha,\beta,0}$. Thus, we will prove that $S_{\alpha,\beta,0} \leq S$. Indeed, for $\delta > 0$, we can choose $x_\delta = (1 - \frac{1}{|\ln\delta|}, 0, \ldots, 0)$, $U_\delta(x) = U(\frac{x-x_\delta}{\delta})$ and $V_\delta(x) = V(\frac{x-x_\delta}{\delta})$, where (U, V) is the solution of (8). Let $\varphi_\delta \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function satisfying

$$\varphi_\delta(x) = \begin{cases} 1, & x \in B\big(x_\delta, \frac{1}{2|\ln\delta|}\big), \\ 0, & x \in \mathbb{R}^N \setminus B\big(x_\delta, \frac{1}{|\ln\delta|}\big), \end{cases}$$

$0 \leq \varphi_\delta(x) \leq 1$, $|\nabla\varphi_\delta(x)| \leq C|\ln\delta|$, $|\Delta\varphi_\delta| \leq C|\ln\delta|^2$ in $\mathbb{R}^N$, where $C > 0$, is independent of δ. Set $w_\delta = \varphi_\delta U_\delta$, similar to **[9]**, we have

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \frac{\int_\Omega |x|^{-\frac{\alpha}{p}} |\Delta w_\delta|^{\frac{p+1}{p}}\, dx}{(\int_\Omega |x|^\beta |w_\delta|^{q_\varepsilon+1}\, dx)^{\frac{1}{q_\varepsilon+1}\frac{p+1}{p}}} \leq S,$$

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and

$$\lim_{\varepsilon\to 0}\frac{\int_\Omega |x|^{-\frac{\alpha}{p}}|\Delta w_\delta|^{\frac{p+1}{p}}\,dx}{(\int_\Omega |x|^\beta |w_\delta|^{q_\varepsilon+1}\,dx)^{\frac{1}{q_\varepsilon+1}\frac{p+1}{p}}}$$
$$\geq \lim_{\varepsilon\to 0}\frac{\int_\Omega |x|^{-\frac{\alpha}{p}}|\Delta u_{\alpha,\beta,\varepsilon}|^{\frac{p+1}{p}}\,dx}{(\int_\Omega |x|^\beta |u_{\alpha,\beta,\varepsilon}|^{q_\varepsilon+1}\,dx)^{\frac{1}{q_\varepsilon+1}\frac{p+1}{p}}}$$
$$\geq \lim_{\varepsilon\to 0}\frac{\int_\Omega |\Delta u_{\alpha,\beta,\varepsilon}|^{\frac{p+1}{p}}\,dx}{(\int_\Omega |u_{\alpha,\beta,\varepsilon}|^{q+1}\,dx)^{\frac{1}{q+1}\frac{p+1}{p}}},$$

where $u_{\alpha,\beta,\varepsilon}$ is a minimizer of $S_{\alpha,\beta,\epsilon}$; thus $S_{\alpha,\beta,0} \leq S$. □

LEMMA 4.3. *Assume that $N \geq 3$. For any $n \in \mathbb{N}$ there exists $\delta_n > 0$ such that $S_{\alpha,\beta,\varepsilon} < S^{\text{rad}}_{\alpha,\beta,\varepsilon}$ provided $\beta \geq \frac{1}{n}$ and $\varepsilon < \delta_n$.*

Proof. By contradiction, assume that there exists $n \in \mathbb{N}$ and sequences $\beta_k \geq \frac{1}{n}$ and $\delta_k \to 0$ such that

$$S_{\alpha,\beta_k,\delta_k} = S^{\text{rad}}_{\alpha,\beta_k,\delta_k}. \tag{13}$$

From Lemma 3.2, there exists c_1 independent of q_{δ_k}, such that

$$S_{\alpha,\beta,\delta_k} \leq c_1 \beta_k^{\frac{2(p+1)}{p}-N+\frac{N(p+1)}{p(q_{\delta_k}+1)}}.$$

Lemma 4.1 implies that

$$c_0 \beta_k^{\frac{p+1}{p(q_{\delta_k}+1)}} \leq S^{\text{rad}}_{\alpha,\beta,\delta_k}.$$

Since

$$\frac{p+1}{p(q_{\delta_k}+1)} - \frac{2(p+1)}{p} + N - \frac{N(p+1)}{p(q_{\delta_k}+1)} = \frac{(pN-2p-2)(q_{\delta_k}+1)-(p+1)(N-1)}{p(q_{\delta_k}+1)},$$

we have

$$\beta_k^{\frac{(pN-2p-2)(q_{\delta_k}+1)-(p+1)(N-1)}{p(q_{\delta_k}+1)}} \leq \frac{c_1}{c_0}.$$

From $q+1 = \frac{N(p+1)}{Np-2p-2} > \frac{(N-1)(p+1)}{Np-2p-2}$ and $q_{\delta_k} \to q$ as $k \to +\infty$, it is implies that $(pN-2p-2)(q_{\delta_k}+1)-(p+1)(N-1) > 0$ as $k \to +\infty$. Thus, β_k is bounded. We can assume that $\beta_k \to \beta \geq \frac{1}{n}$ as $k \to +\infty$.

Claim that

$$S^{\text{rad}}_{\alpha,\beta,0} = \lim_{k\to+\infty} S^{\text{rad}}_{\alpha,\beta_k,\delta_k}. \tag{14}$$

Indeed, by upper semi-continuity, it follows that

$$S^{\text{rad}}_{\alpha,\beta,0} \geq \limsup_{k\to+\infty} S^{\text{rad}}_{\alpha,\beta_k,\delta_k},$$

On the other hand, from

$$\int_\Omega |x|^{\beta_k}|u_k|^{q_{\delta_k}+1}\,dx \le \left(\int_\Omega |x|^{\beta_k}|u_k|^{q+1}\,dx\right)^{\frac{q_{\delta_k}+1}{q+1}}\left(\int_\Omega |x|^{\beta_k}\,dx\right)^{\frac{q-q_{\delta_k}}{q+1}},$$

we have

$$S^{\mathrm{rad}}_{\alpha,\beta,0} \le \liminf_{k\to+\infty} S^{\mathrm{rad}}_{\alpha,\beta_k,\delta_k}.$$

Similarly, by upper continuity,

$$S_{\alpha,\beta,0} \ge \limsup_{k\to+\infty} S_{\alpha,\beta_k,\delta_k}. \qquad (15)$$

We obtain from (13)–(15), $S_{\alpha,\beta,0} \ge S^{\mathrm{rad}}_{\alpha,\beta,0}$, which contradicts Lemma 4.2. □

REFERENCES

1. J. Byeon and Z-Q. Wang, On the Hénon equation: Asymptotic profile of ground state II, *J. Differ. Equ.* **216** (2005), 78–108.

2. J. Byeon and Z-Q. Wang, On the Hénon equation: Asymptotic profile of ground state I, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **23** (2006), 803–828.

3. D. Cao and S. Peng, The asymptotic behavior of the ground state solutions for Hénon equation, *J. Math. Anal. Appl.* **278** (2003), 1–17.

4. X. Chen, C. Li and B. Ou, Classification of solution for a system of integral equations, *Commun. Partial Differ. Equ.* **30** (2005), 59–65.

5. D. G. De Figueiredo and P. L. Felmer, On superquadratic elliptic systems, *Trans. Am. Math. Soc.* **343** (1994), 99–106.

6. D. G. de Figueiredo, I. Peral and J. D. Rossi, The critical hyperbola for a hamiltonian elliptic system with weights, *Ann. Mat. Pura Appl.* **187** (2008), 531–545.

7. B. Gidas, W. M. Ni and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^N$, in *Mathematical analysis and applications*: Part A, in Adv. in Math. Suppl. Stud., Vol. 7 (Academic Press, New York, 1981) pp. 369–402.

8. I. A. Guerra, Solutions of an elliptic system with a nearly critical exponent, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **25** (2008), 181–200.

9. H. He and J. Yang, Asymptotic behavior of Solutions for Hénon systems with nearly critical exponent, *J. Math. Anal. Appl.* **347** (2008), 459–471.

10. M. Hénon, Numerical experiments on the stability of spherical stellar systems, *Astronom. Astrophys. Lib.* **24** (1973), 229–238.

11. J. Hulshof and R. C. A. M. van der Vorst, Differential systems with strongly indefinite variational struture, *J. Funct. Anal.* **114** (1993), 32–58.

12. J. Hulshof and R. C. A. M. van der Vorst, Asymptotic behavior of ground states, *Proc. Am. Math. Soc*. **124** (1996), 2423–2431.

13. F. Liu and J. Yang, Nontrivial solutions of Hardy-Hénon type elliptic systems, *Acta Math. Sci.* **27** (2007), 673–688.

14. W. M. Ni, A nonlinear Dirichlet problem on the unit ball and its applications, *Indiana Univ. Math. J.* **31** (1982), 801–807.

15. D. Smets, J. B. Su and M. Willem, Non-radial ground states for the Hénon equation, *Comm. Contemp. Math.* **4** (2002), 467–480.

16. D. Smets and M. Willem, Partial symmetry and asymptotic behavior for some elliptic variational problem, *Calc. Var. Partial Differ. Equ.* **18** (2005), 57–75.