SYMMETRY BREAKING FOR GROUND-STATE SOLUTIONS OF HÉNON SYSTEMS IN A BALL

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Abstract. We consider in this paper the problem
\[\begin{align*}
-\Delta u &= |x|^\alpha v^p, &x \in \Omega, \\
-\Delta v &= |x|^\beta u^q, &x \in \Omega, \\
u > 0, &v > 0, &x \in \Omega, \\
u = v = 0, &x \in \partial \Omega,
\end{align*}\]

where \(\Omega\) is the unit ball in \(\mathbb{R}^N\) centred at the origin, \(0 \leq \alpha < pN, \beta > 0, N \geq 3\). Suppose \(q_\varepsilon \to q\) as \(\varepsilon \to 0^+\) and \(q_\varepsilon, q\) satisfy, respectively,
\[\frac{N}{p + 1} + \frac{N}{q_\varepsilon + 1} > N - 2, \quad \frac{N}{p + 1} + \frac{N}{q + 1} = N - 2;\]
we investigate the asymptotic estimates of the ground-state solutions \((u_\varepsilon, v_\varepsilon)\) of (1) as \(\beta \to +\infty\) with \(p, q_\varepsilon\) fixed. We also show the symmetry-breaking phenomenon with \(\alpha, \beta\) fixed and \(q_\varepsilon \to q\) as \(\varepsilon \to 0^+\). In addition, the ground-state solution is non-radial provided that \(\varepsilon > 0\) is small or \(\beta\) is large enough.

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1. Introduction. In this paper, we investigate the limiting behaviour of the ground-state solutions of the problem
\[\begin{align*}
-\Delta u &= |x|^\alpha v^p, &x \in \Omega, \\
-\Delta v &= |x|^\beta u^q, &x \in \Omega, \\
u = v = 0, &x \in \partial \Omega,
\end{align*}\]

where \(\Omega\) is the unit ball in \(\mathbb{R}^N\) centred at the origin, \(0 \leq \alpha < pN, \beta > 0, N \geq 3\). We assume in this paper that \(q_\varepsilon \to q\) as \(\varepsilon \to 0^+\) and \(q_\varepsilon, q\) satisfy, respectively,
\[\frac{N}{p + 1} + \frac{N}{q_\varepsilon + 1} > N - 2, \quad \frac{N}{p + 1} + \frac{N}{q + 1} = N - 2.\]

Problem (2) has two features. First, it is a Hénon-type system. The Hénon equation with Dirichlet boundary conditions
\[\begin{align*}
-\Delta u &= |x|^\alpha u^p, &x \in \Omega, \\
u = 0, &x \in \partial \Omega
\end{align*}\]
was found in [10], which stems from rotating stellar structures. A standard compactness argument show that the infimum

$$\inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 \, dx}{\left(\int_{\Omega} |x|^\alpha |u|^{p+1} \, dx\right)^{\frac{2}{p+1}}}$$  (4)

is achieved for any $1 < p < 2^* - 1$, $\alpha > 0$. In 1982, Ni [14] proved that the infimum

$$\inf_{u \in H_0^{\text{rad}}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 \, dx}{\left(\int_{\Omega} |x|^\alpha |u|^{p+1} \, dx\right)^{\frac{2}{p+1}}}$$  (5)

is achieved for any $p \in \left(1, \frac{N+2+2\alpha}{N-2}\right)$ by a function in $H_0^1(\Omega)$, the space of radial $H_0^1(\Omega)$ functions. Thus, radial solutions of (3) exist also for (Sobolev) supercritical exponents $p$. A natural question is whether any minimizer of (4) must be radially symmetric in the range $1 < p < \frac{N+2}{N-2}$ and $\alpha > 0$. Since the weight $| \cdot |^\alpha$ is an increasing function, neither rearrangement arguments nor the moving plane techniques of [7] can be applied.

For $\alpha > 0$, Smets et al. proved in [15] some symmetry-breaking results for (3). They proved that the minimizers of (4) (the so-called ground-state solutions, or least energy solutions) cannot be radial for $\alpha$ large enough. As a consequence, (3) has at least two solutions when $\alpha$ is sufficiently large (see also [16]).

Quite recently, Cao and Peng [3] proved that for $p + 1$ sufficiently close to $2^*$, the ground-state solutions of (3) possess a unique maximum point whose distance from $\partial \Omega$ tends to zero as $p \rightarrow \frac{N+2}{N-2}$.

For more results about symmetry breaking phenomena for solutions of problem (3) either $\alpha$ is large enough or $p \rightarrow \frac{N+2}{N-2}$, see for instance [2, 1] and references therein.

Second, the system in (2) is a Hamiltonian-type system, which is strongly indefinite. The existence of solutions of the Hamiltonian elliptic system

$$\begin{cases}
-\Delta u = v^p, & x \in \Omega, \\
-\Delta v = u^q, & x \in \Omega, \\
u = v = 0, & x \in \partial \Omega
\end{cases}$$  (6)

was first considered in [5] and [11] with $\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}$; the curve of $(p, q) \in \mathbb{R}^2$ satisfying $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$ is called critical hyperbola. Afterwards, various results were obtained in the literature. Extensions of problem (6) can be found in [6] and [13]. In [13], existence problems for Hardy-type systems and Hénon-type systems were established. Particularly, for Hénon-type systems

$$\begin{cases}
-\Delta u = |x|^{\alpha} v^p, & x \in \Omega, \\
-\Delta v = |x|^{\beta} u^q, & x \in \Omega, \\
u = v = 0, & x \in \partial \Omega
\end{cases}$$  (7)

the critical hyperbola is $\frac{1}{p+1}(1 + \frac{\alpha}{N}) + \frac{1}{q+1}(1 + \frac{\beta}{N}) = \frac{N-2}{N}$.

In recent years, a study of the limiting behaviour of ground-state solutions of elliptic problems has attracted considerable attention. For the system (6), the limiting behaviour of solutions of (6) as $\frac{1}{p+1} + \frac{1}{q+1} \rightarrow \frac{N-2}{N}$ was discussed in [8]. For the system (7), Yang and He [9] proved that for $\frac{1}{p+1} + \frac{1}{q+1} \rightarrow \frac{N-2}{N}$, the ground-state solutions of (7) possess a unique maximum point whose distance from $\partial \Omega$ tends to zero as
\( \frac{1}{p+1} + \frac{1}{q+1} \to \frac{N-2}{N} \). Such problems are closely related to solutions of the following problem:

\[
\begin{align*}
\{ & -\Delta U = V^p, & & y \in \mathbb{R}^N, \\
& -\Delta V = U^q, & & y \in \mathbb{R}^N, \\
& U(y) > 0, & & V(y) > 0, \\
& U(0) = 1, & & U \to 0, & V \to 0 \text{ as } |y| \to \infty,
\end{align*}
\]

where \( \frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N} \). It was proved in [12] that \( U \in D^{2, \frac{p+1}{r}}(\mathbb{R}^N), \ V \in D^{2, \frac{q+1}{r}}(\mathbb{R}^N) \), where \( D^2(\mathbb{R}^N) \) is the completion of \( C_0^\infty(\mathbb{R}^N) \) with respect to the norm \( \| \Delta \cdot \|_r \). Actually, \( U \) and \( V \) are radially symmetric for \( p \geq 1 \) as showed in [4]. Moreover, \( U \) and \( V \) are unique and decreasing in \( r \). In the discussion of one equation problem, one uses the instanton for the best Sobolev constant. However, no explicit form of \( (U, V) \) was found for \( p > \frac{N}{N-2} \) up to now. Instead, the asymptotic behaviour of \( (U, V) \) as \( r \to \infty \) is sufficient for this purpose. It was found in [12] that

\[
\lim_{r \to \infty} r^{N-2} V(r) = a,
\]

and

\[
\lim_{r \to \infty} \frac{rV'(r)}{V(r)} = 2 - N,
\]

where

\[
\begin{align*}
\lim_{r \to \infty} r^{N-2} U(r) &= b \quad \text{if } p > \frac{N}{N-2}, \\
\lim_{r \to \infty} \frac{r^{N-2}}{\log r} U(r) &= b \quad \text{if } p = \frac{N}{N-2}, \\
\lim_{r \to \infty} r^{p(N-2)-2} U(r) &= b \quad \text{if } \frac{2}{N-2} < p < \frac{N}{N-2},
\end{align*}
\]

\[
\begin{align*}
\lim_{r \to \infty} \frac{r^N U(r)}{U(r)} &= 2 - N \quad \text{if } p \geq \frac{N}{N-2}, \\
\lim_{r \to \infty} \frac{r^N U(r)}{U(r)} &= 2 - p(N-2) \quad \text{if } p \leq \frac{N}{N-2}.
\end{align*}
\]

In this paper, we are interested in the symmetry of ground-states solutions of (2). Now, we denote

\[
E_\alpha(\Omega) = \left\{ u \in W^{2, \frac{p+1}{r}} \cap W^{1, \frac{q+1}{r}}_0(\Omega) : \int_\Omega |x|^{-\beta} |\Delta u|^\frac{p+1}{r} \, dx < \infty \right\}
\]

and

\[
E^\text{rad}_\alpha(\Omega) = \{ u \in E_\alpha(\Omega) : u(x) = u(|x|) \}.
\]

Our main results are as follows.

**Theorem 1.1.** Suppose \( N \geq 3, \ 0 \leq \alpha < pN, \ \beta > 0, \ p > \frac{2}{N-2}, \ q \epsilon > \frac{N+p}{Np-2p-1} \); then there exists \( \beta^* > 0 \) such that the ground-state solutions \( u_{\alpha, \beta, \epsilon} \) are non-radial provided \( \beta > \beta^* \).

**Theorem 1.2.** Suppose \( N \geq 3, \ 0 \leq \alpha < pN, \ \beta > 0, \ p > \frac{2}{N-2}, \ p q \epsilon > 1 \); then there exists \( \epsilon^* > 0 \) such that the ground-state solutions \( u_{\alpha, \beta, \epsilon} \) are non-radial provided \( \epsilon < \epsilon^* \) or \( q-q \epsilon < q \).

This paper is organized as follows. In section 2, we give some preliminaries which turn out to be essential. In section 3, we present some estimates for radial ground-state solutions of (2) with \( \alpha, \ p, \ q \epsilon \) fixed and \( \beta \to \infty \). This will lead us to get the first
symmetry-breaking result, stating that for $\beta$ sufficiently large, the ground-state solution of problem (2) is non-radial. In section 4, another symmetry-breaking result is proved, with $\alpha, \beta, p$ fixed and $q_\varepsilon \to q$ as $\varepsilon \to 0$.

2. Preliminaries. Before proving our main results, we want to introduce some simple calculus lemma which turns out to be essential:

**Lemma 2.1.** Let $u$ be a radially symmetric function of $\Omega$ (unit ball in $\mathbb{R}^N$) with $u(1) = 0, u'(0)$ exists. Then

\[
(i) \quad |u(x)| \leq \frac{1}{w_{N-1}^\frac{1}{p}(p(N-1) - 1)^{\frac{1}{p}} \left(|x|^p|N-1| - 1\right)^{\frac{1}{p}}} \left(\int_\Omega |\nabla u|^{\frac{p+1}{p}} \frac{1}{\Omega} \right)^{\frac{1}{p}} + \frac{1}{w_{N-1}} \left(\int_{B(0)} |x|^{-\frac{p}{2}} |\Delta u|^{\frac{p+1}{p}} \frac{1}{\Delta} \right)^{\frac{1}{p}},
\]

(ii) \quad \left| \frac{\partial u}{\partial r} \right| \leq \frac{1}{r^{1-N}} \left(\int_\Omega |\nabla u|^{\frac{p+1}{p}} \frac{1}{\Omega} \right)^{\frac{1}{p}} \left(\int_{B(0)} |x|^{-\frac{p}{2}} |\Delta u|^{\frac{p+1}{p}} \frac{1}{\Delta} \right)^{\frac{1}{p}} ,
\]

where $w_{N-1}$ is the surface area of the unit ball in $\mathbb{R}^N$.

**Proof.** (i) For

\[
|u(1)| = \left| \int_0^1 |u'(t)| \, dt, \quad |u(x)| \leq \int_0^1 |u'(t)| \, dt \leq \left(\int_0^1 |u'(t)|^\frac{p+1}{p} \, t^{N-1} \, dt\right)^{\frac{p}{p+1}} \left(\int_0^1 t^{-(N-1)p} \, dt\right)^{\frac{1}{p+1}}.
\]

Since

\[
\int_0^1 t^{-(N-1)p} \, dt = \frac{1}{p(N-1) - 1} \left(\frac{1}{|x|^p|N-1|-1} - 1\right) \leq \frac{1}{p(N-1) - 1} \frac{1}{|x|^p|N-1|-1}
\]

and

\[
\int_0^1 |u'(t)|^\frac{p+1}{p} \, t^{N-1} \, dt = \frac{1}{w_{N-1}} \left(\int_0^1 |u'(t)|^\frac{p+1}{p} \, t^{N-1} \, dt\right) w(\theta) \, d\theta \leq \frac{1}{w_{N-1}} \int_{|x| \leq |y| \leq 1} |\nabla u|^{\frac{p+1}{p}} \, dy \leq \frac{1}{w_{N-1}} \int_\Omega |\nabla u|^{\frac{p+1}{p}} \, dy,
\]

we find that

\[
|u(x)| \leq \frac{1}{w_{N-1}^\frac{1}{p}(p(N-1) - 1)^{\frac{1}{p}} \left(|x|^p|N-1| - 1\right)^{\frac{1}{p}}} \left(\int_\Omega |\nabla u|^{\frac{p+1}{p}} \frac{1}{\Omega} \right)^{\frac{1}{p}} + \frac{1}{w_{N-1}} \left(\int_{B(0)} |x|^{-\frac{p}{2}} |\Delta u|^{\frac{p+1}{p}} \frac{1}{\Delta} \right)^{\frac{1}{p}}.
\]
(ii) For
\[ |r^{N-1} \frac{\partial u}{\partial r}| \leq \int_0^r r^{N-1} |\Delta u| \, dt \]
\[ \leq \left( \int_0^r t^{\frac{(N-1+\alpha)\gamma+1}{p+1}} \, dt \right)^{\frac{1}{p+1}} \left( \int_0^r \left( \frac{r^{N-1} |\Delta u|}{N+\alpha} \right)^{\frac{p+1}{p}} \, dt \right)^{\frac{p}{p+1}} \]
\[ = \left( \frac{r^{N+\alpha}}{N+\alpha} \right)^{\frac{1}{p+1}} \left( \frac{1}{w_{N-1}} \int_{B_r(0)} |x|^{-\frac{\beta}{\sigma}} |\Delta u|^{\frac{p+1}{p}} \, dx \right)^{\frac{p}{p+1}}, \]
we have
\[ \left| \frac{\partial u}{\partial r} \right| \leq r^{1-N} \left( \frac{r^{N+\alpha}}{N+\alpha} \right)^{\frac{1}{p+1}} \left( \frac{1}{w_{N-1}} \int_{\Omega} |x|^{-\frac{\beta}{\sigma}} |\Delta u|^{\frac{p+1}{p}} \, dx \right)^{\frac{p}{p+1}}. \]

□

As a result of Lemma 2.1, we obtain the following corollary.

**Corollary 2.1.** Under hypothesis of Lemma 2.1, and if \( \frac{N+\beta}{q+1} + \frac{N+\alpha}{p+1} > N - 2 \), then
\[ \int_{\Omega} |x|^{\beta} |u(x)|^{q+1} \, dx \leq C \left( \int_{\Omega} |x|^{-\frac{\beta}{\sigma}} |\Delta u|^{\frac{p+1}{p}} \, dx \right)^{\frac{q+1}{p+1}}, \]
where
\[ C = \left( \frac{1}{w_{N-1}} \right)^{\frac{q+1}{p+1}} \frac{1}{N+\beta - \left( N - 2 - \frac{N+\alpha}{p+1} \right) (q+1)} \left( \frac{1}{N - 2 - \frac{N+\alpha}{p+1}} \right)^{\frac{q+1}{p+1}} \left( \frac{1}{N+\alpha} \right)^{\frac{q+1}{p+1}}. \]

**Proof.** From the above Lemma 2.1, we have
\[ |u(x)| \leq \int_{|x|}^1 |u(r)| \, dr \leq \int_{|x|}^1 \left( \frac{r^{N+\alpha}}{N+\alpha} \right)^{\frac{1}{p+1}} r^{1-N} \, dr \left( \frac{1}{w_{N-1}} \int_{\Omega} |x|^{-\frac{\beta}{\sigma}} |\Delta u|^{\frac{p+1}{p}} \, dx \right)^{\frac{p}{p+1}}. \]
Since
\[ \int_{|x|}^1 \left( \frac{r^{N+\alpha}}{N+\alpha} \right)^{\frac{1}{p+1}} r^{1-N} \, dr = \left( \frac{1}{N+\alpha} \right)^{\frac{1}{p+1}} \int_{|x|}^1 r^{\frac{N+\alpha}{p+1}} r^{1-N} \, dr \]
\[ \leq \frac{1}{N - 2 - \frac{N+\alpha}{p+1}} \left( \frac{1}{N+\alpha} \right)^{\frac{1}{p+1}} \frac{1}{|x|^{N-2-\frac{N+\alpha}{p+1}}}, \]
we have
\[ r^\beta r^{N-1} |u(r)|^{q+1} \]
\[ \leq r^\beta r^{N-1} \left( \frac{1}{N-2-\frac{N+\alpha}{p+1}} \right)^{\frac{q+1}{p+1}} \left( \frac{1}{w_{N-1}} \int_{\Omega} |x|^{-\frac{\beta}{\sigma}} |\Delta u|^{\frac{p+1}{p}} \, dx \right)^{\frac{q+1}{p+1}}. \]
which implies
\[
\int_0^1 r^\beta r^{N-1} |u(r)|^{q+1} dr \leq \left( \frac{1}{N - 2 - \frac{N+\alpha}{p+1}} \left( \frac{1}{N + \alpha} \right)^{\frac{1}{p+1}} \right)^{q+1} \times \int_0^1 \left( \frac{1}{r^{N-2 - \frac{N+\alpha}{p+1}}} \right)^{q+1} r^\beta r^{N-1} \left( \frac{1}{w_{N-1}} \int_\Omega |x|^{-\frac{p+1}{p}} \Delta u \frac{p+1}{r} dx \right)^{\frac{p+1}{p+1}}.
\]

Thus, the conclusion holds.

\[\square\]

3. Asymptotic estimates. Consider the minimization problem
\[
S^{rad}_{\alpha, \beta, \varepsilon} = \inf_{u \in E^{rad}_{\alpha}(\Omega) \setminus \{0\}} R_{\alpha, \beta, \varepsilon}(u), \tag{11}
\]
where
\[
R_{\alpha, \beta, \varepsilon}(u) = \frac{\int_\Omega |x|^{-\frac{p+1}{p}} |\Delta u |^{\frac{p+1}{p}} dx}{(\int_\Omega |x|^{\beta}|u|^{q+1} dx)^{\frac{p+1}{p+1}}}, \quad u \in E_{\alpha}(\Omega) \setminus \{0\}, \tag{12}
\]
is the Rayleigh quotient associated with (2). Similar to [14], we can also prove that
\[
S^{rad}_{\alpha, \beta, \varepsilon}(\Omega) = \inf_{u \in E^{rad}_{\alpha}(\Omega) \setminus \{0\}} \frac{\int_\Omega |x|^{-\frac{p+1}{p}} |\Delta u |^{\frac{p+1}{p}} dx}{(\int_\Omega |x|^{\beta}|u|^{q+1} dx)^{\frac{p+1}{p+1}}}
\]
is attained by some positive function \(u^{rad}_{\alpha, \beta, \varepsilon}\). After scaling, \(u^{rad}_{\alpha, \beta, \varepsilon}\) is also a solution of (2).

Now, we provide an estimate of the energy \(S^{rad}_{\alpha, \beta, \varepsilon}\) as \(\beta \to \infty\).

**Lemma 3.1.** If \(N \geq 3\), there exists \(C > 0\) depending on \(N, p\) such that
\[
S^{rad}_{\alpha, \beta, \varepsilon} \geq C \beta^{\frac{p+2+q}{p+1}} \quad \text{as} \quad \beta \to \infty.
\]

**Proof.** Let \(u \in E^{rad}_{\alpha}(\Omega)\) and define the rescaled function \(v(|x|) = u(|x|^s)\), where \(s = \frac{N}{\beta + N}\). Then
\[
\int_\Omega |x|^{\beta}|u|^{q+1} dx = w_{N-1} \int_0^1 r^{\beta+N-1}|u(r)|^{q+1} dr = s \int_\Omega |x|^{(\beta+N)-N} |v(x)|^{q+1} dx,
\]
and from Lemma 2.1, we have
\[
\int_\Omega |x|^{-\frac{\beta}{p}} |\Delta u |^{\frac{p+1}{p}} dx \geq w_{N-1}^{\frac{p+1}{p}} (N + \alpha)^{\frac{p+1}{p}} \left( \frac{p+1}{r} \right)^{\frac{p+1}{p}} \left( \frac{1 \delta \left( \frac{p+1}{r} \right)^{\frac{p+1}{p}} |\partial u|^{\frac{p+1}{p}} \right). 
\]
which implies that
\[
\int_{\Omega} \left| x \right|^{-\frac{\alpha}{\beta}} |\Delta u|^{\frac{p+1}{p}} \, dx \geq w_{N-1}^{p+1}(N+\alpha)^{\frac{1}{p}} \int_{0}^{1} r \left[ \frac{(p+1)(N-1)-s(N+\alpha)}{p} \right] \frac{\partial u}{\partial r}^{\frac{p+1}{p}} \, dr
\]
\[
= w_{N-1}^{p+1}(N+\alpha)^{\frac{1}{p}} \int_{0}^{1} r \left[ \frac{(p+1)(N-1)-s(N+\alpha)}{p} \right] \frac{1}{r-\frac{p+1}{p}} \frac{\partial v}{\partial r}^{\frac{p+1}{p}} \, dr
\]
\[
= w_{N-1}^{\frac{1}{p}}(N+\alpha)^{\frac{1}{p}} s^{-\frac{1}{p}} \int_{\Omega} \left| x \right|^{\frac{(p+1)(N-1)-s(N+\alpha)}{p}} |\nabla v|^{\frac{p+1}{p}} \, dx.
\]

Thus, we obtain
\[
\frac{\int_{\Omega} \left| x \right|^{-\frac{\alpha}{\beta}} |\Delta u|^{\frac{p+1}{p}} \, dx}{(\int_{\Omega} \left| x \right|^{\beta} |u|^{q+1} \, dx)^{\frac{p+1}{p}q^{p+1}}} \geq \frac{w_{N-1}^{\frac{1}{p}}(N+\alpha)^{\frac{1}{p}} s^{-\frac{1}{p}} \int_{\Omega} \left| x \right|^{\frac{(p+1)(N-1)-s(N+\alpha)}{p}} |\nabla v|^{\frac{p+1}{p}} \, dx}{(s \int_{\Omega} |v(x)|^{q+1} \, dx)^{\frac{p+1}{p}q^{p+1}}}.
\]

It follows that
\[
S_{u, \beta, \varepsilon}^{rad} \geq w_{N-1}^{\frac{1}{p}}(N+\alpha)^{\frac{1}{p}} s^{-\frac{1}{p}} \inf_{v \in W_{0, rad}^{p+1}} \frac{\int_{\Omega} \left| x \right|^{\frac{(p+1)(N-1)-s(N+\alpha)}{p}} |\nabla v|^{\frac{p+1}{p}} \, dx}{(s \int_{\Omega} |v(x)|^{q+1} \, dx)^{\frac{p+1}{p}q^{p+1}}}.
\]

Now, we claim that for every $0 \leq s \leq 1$, we have $p(N-1)(s-1) - s(1+\alpha) + (1-s) < 0$. Indeed, for
\[
p(N-1)(s-1) - s(1+\alpha) + (1-s)
\]
\[
= (p(N-1) - 2 - \alpha)s - (p(N-1) - 1),
\]
if $p(N-1) - 2 \leq \alpha < pN$, we have $p(N-1)(s-1) - s(1+\alpha) + (1-s) < 0$ if $\alpha < p(N-1) - 2 < p(N-1) - 1$, then for every $0 \leq s \leq 1$, we also have $p(N-1)(s-1) - s(1+\alpha) + (1-s) < 0$. Therefore,
\[
\int_{\Omega} |\nabla v|^{\frac{p+1}{p}} \, dx \leq \int_{\Omega} \left| x \right|^{\frac{(p+1)(N-1)-s(N+\alpha)}{p}} |\nabla v|^{\frac{p+1}{p}} \, dx,
\]
which implies that
\[
c_{s} = \inf_{v \in W_{0, rad}^{p+1}} \frac{\int_{\Omega} \left| x \right|^{\frac{(p+1)(N-1)-s(N+\alpha)}{p}} |\nabla v|^{\frac{p+1}{p}} \, dx}{(s \int_{\Omega} |v(x)|^{q+1} \, dx)^{\frac{p+1}{p}q^{p+1}}}
\]
is achieved by standard arguments. Since $|x| \leq 1$, if $p(N-1) - 2 \leq \alpha < pN$, $p(N-1)(s-1) - s(1+\alpha) + (1-s) \leq -(p(N-1) - 1)$, which implies that $c_{s} \geq c_{0}$; if $\alpha < p(N-1) - 2$, $p(N-1)(s-1) - s(1+\alpha) + (1-s) = (p(N-1) - 2 - \alpha)s - (p(N-1) - 1)$, then for every $0 \leq s \leq 1$, $|x|^{\frac{(p+1)(N-1)-s(N+\alpha)}{p}}$ is non-increasing, which implies that $c_{s}$ is non-increasing on $[0, 1]$; then $c_{s} \geq c_{1}$. Thus,
\[
S_{u, \beta, \varepsilon}^{rad} \geq C(N+\alpha)^{\frac{1}{2}} s^{-\frac{\alpha+\theta}{p+\theta}} \quad \beta \to \infty.
\]
By the assumptions on \( p, q_e \), the inclusion \( W^{2, \frac{p+1}{p}}(\Omega) \hookrightarrow L^{q_e+1}(\Omega) \) is compact. It implies that

\[
S_{\alpha, \beta, \epsilon}(\Omega) = \inf_{u \in E_\alpha(\Omega) \setminus \{0\}} \frac{\int_\Omega |x|^{-\frac{q_e}{p}} |\Delta u|^{\frac{p+1}{p}} dx}{(\int_\Omega |x|^\beta |u|^{q_e+1} dx)^{\frac{p+1}{p}}}
\]

is attained by some positive function \( u_{\alpha, \beta, \epsilon} \). After scaling, \( u_{\alpha, \beta, \epsilon} \) is a solution of (2).

**Lemma 3.2.** Assume \( N \geq 3 \), for any \( p, q_e \) satisfy \( N = \frac{N+1}{p+1} + \frac{N}{q_e+1} > N - 2 \) with \( p > \frac{2}{N-2} \), \( q_e > \frac{N+p}{Np-2p-1} \); there exists \( \beta^* \geq 0 \) such that \( S_{\alpha, \beta, \epsilon} < S_{\alpha, \beta, \epsilon}^{\text{rad}} \) provided \( \beta > \beta^* \).

**Proof.** For any fixed \( u \in C^0_0(\Omega) \), define \( u_\beta(x) = u(\beta(x - x_\beta)) \), where \( x_\beta = (1 - \frac{1}{\beta}, 0, \ldots, 0) \). For \( |\beta(x - x_\beta)| \leq 1 \), that is, \( |x - x_\beta| \leq \frac{1}{\beta} \), then \( |x| \geq |x_\beta| - \frac{1}{\beta} = 1 - \frac{2}{\beta} \). One has

\[
\int_\Omega |x|^{-\frac{q_e}{p}} |\Delta u_\beta|^{\frac{p+1}{p}} dx \leq \left( 1 - \frac{2}{\beta} \right)^{\frac{q_e}{p}} \beta^{\frac{2(p+1)}{p}} \int_\Omega |\Delta u|^{\frac{p+1}{p}} dx,
\]

and

\[
\int_\Omega |x|^\beta |u_\beta|^{q_e+1} dx \geq \left( 1 - \frac{2}{\beta} \right)^{\beta} \beta^{-N} \int_\Omega |u|^{q_e+1} dx.
\]

Hence by definition, one obtains

\[
S_{\alpha, \beta, \epsilon} \leq \frac{\int_\Omega |x|^{-\frac{q_e}{p}} |\Delta u_\beta|^{\frac{p+1}{p}} dx}{(\int_\Omega |x|^\beta |u_\beta|^{q_e+1} dx)^{\frac{p+1}{p}}}
\]

\[
\leq \frac{(1 - \frac{2}{\beta})^{-\frac{q_e}{p}} \beta^{\frac{2(p+1)}{p}} - N \int_\Omega |\Delta u|^{\frac{p+1}{p}} dx}{(1 - \frac{2}{\beta})^\beta \beta^{-N} \int_\Omega |u|^{q_e+1} dx}^{\frac{p+1}{p}}
\]

\[
\leq C_1 \beta^{-\frac{2(p+1)}{p} - N + N(p+1)} \frac{\int_\Omega |\Delta u|^{\frac{p+1}{p}} dx}{(\int_\Omega |u|^{q_e+1} dx)^{\frac{p+1}{p}}}
\]

Since \( u \) is fixed and \( \frac{\int_\Omega |\Delta u|^{\frac{p+1}{p}} dx}{(\int_\Omega |u|^{q_e+1} dx)^{\frac{p+1}{p}}} \) is independent of \( \beta \), we have

\[
S_{\alpha, \beta, \epsilon} \leq C \beta^{-\frac{2(p+1)}{p} - N + N(p+1)}
\]

From Lemma 3.1 \( S_{\alpha, \beta, \epsilon}^{\text{rad}} \geq C \beta^{\frac{p+2+q_e}{p(q_e+1)}} \) as \( \beta \to \infty \), and \( \frac{p+2+q_e}{p(q_e+1)} > \frac{2(p+1)}{p} - N + \frac{N(p+1)}{p(q_e+1)} \), that is, \( q_e > \frac{N+p}{Np-2p-1} \). Hence \( S_{\alpha, \beta, \epsilon} < S_{\alpha, \beta, \epsilon}^{\text{rad}} \) as \( \beta \to \infty \).

**4. Analysis for \( \epsilon \) close to 0.** In this section, we analyse the case where \( \epsilon \) is close to 0, that is, \( q_e \) is close to \( q \). We will show that for any fixed \( 0 < \alpha < pN, \beta > 0 \), the minimizer of \( R_{\alpha, \beta, \epsilon} \) is non-radial provided that \( \epsilon \) is sufficiently small.
Lemma 4.1. If $N \geq 3$, there exists $c_0 > 0$, such that for every $q_\varepsilon$ and for every $0 \leq \alpha < pN$, $\beta > 0$,  
\[ c_0 \beta^{\frac{p+1}{p\alpha + 1}} \leq S_{\alpha, \beta, \varepsilon}^{\text{rad}}. \]

Proof. From Lemma 2.1, we obtain  
\[ \int_{\Omega} |x|^{\beta} |u(x)|^{q_\varepsilon + 1} \, dx \leq C \left( \int_{\Omega} |x|^{-\frac{p}{\beta}} |\Delta u|^{\frac{p+1}{p}} \, dx \right)^{\frac{p(q_\varepsilon + 1)}{p+1}}, \]
where
\[ C = \left( \frac{1}{w_{N-1}} \right)^{\frac{p(q_\varepsilon + 1)}{p+1}} \frac{1}{N + \beta - (N - 2 - \frac{N + \alpha}{p+1})(q_\varepsilon + 1)} \left( \frac{1}{N - 2 - \frac{N + \alpha}{p+1}} \right)^{q_\varepsilon + 1} \left( \frac{1}{N + \alpha} \right)^{\frac{q_\varepsilon + 1}{p+1}}. \]

Since $u \in E_{\alpha}^{\text{rad}}(\Omega)$ is arbitrary,
\[ c_0 \left( N + \beta - (N - 2 - \frac{N + \alpha}{p+1})(q_\varepsilon + 1) \right)^{\frac{p(q_\varepsilon + 1)}{p+1}} \leq S_{\alpha, \beta, \varepsilon}^{\text{rad}}, \]
which ends the proof.

Let us denote by $S$ the classical Sobolev constant
\[ S = \inf_{u \in W^{1, p+1}_\delta \cap W^{1, \frac{p+1}{\beta}}_\delta(\Omega)} \frac{\int_{\Omega} |\Delta u|^{\frac{p+1}{\beta}} \, dx}{\left( \int_{\Omega} |u|^{q_\varepsilon + 1} \, dx \right)^{\frac{p(q_\varepsilon + 1)}{p+1}}}. \]

It is standard that this Rayleigh quotient is invariant under translations and dilations.

Lemma 4.2. If $N \geq 3$ and $0 \leq \alpha < pN$, $\beta > 0$, then
\[ S = S_{\alpha, \beta, 0} < S_{\alpha, \beta, 0}^{\text{rad}}. \]

Proof. Using Corollary 2.1, it is easy to verify that $S_{\alpha, \beta, 0}^{\text{rad}}$ is achieved, so that $S < S_{\alpha, \beta, 0}^{\text{rad}}$. Now, we claim that $S = S_{\alpha, \beta, 0}$. From the definition of $S_{\alpha, \beta, 0}$, we know that $S \leq S_{\alpha, \beta, 0}$. Thus, we will prove that $S_{\alpha, \beta, 0} \leq S$. Indeed, for $\delta > 0$, we can choose $x_\delta = (1 - \frac{1}{|\ln \delta|}, 0, \ldots, 0)$, $U_\delta(x) = U(\frac{x-x_\delta}{\delta})$ and $V_\delta(x) = V(\frac{x-x_\delta}{\delta})$, where $(U, V)$ is the solution of (8). Let $\varphi_\delta \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function satisfying
\[ \varphi_\delta(x) = \begin{cases} 1, & x \in B(x_\delta, \frac{1}{2|\ln \delta|}), \\ 0, & x \in \mathbb{R}^N \setminus B(x_\delta, \frac{1}{|\ln \delta|}), \end{cases} \]

$0 \leq \varphi_\delta(x) \leq 1$, $|\nabla \varphi_\delta(x)| \leq C |\ln \delta|$, $|\Delta \varphi_\delta| \leq C |\ln \delta|^2$ in $\mathbb{R}^N$, where $C > 0$, is independent of $\delta$. Set $w_\delta = \varphi_\delta U_\delta$, similar to [9], we have
\[ \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \frac{\int_{\Omega} |x|^{-\frac{p}{\beta}} |\Delta w_\delta|^{\frac{p+1}{\beta}} \, dx}{\left( \int_{\Omega} |x|^{\beta} |w_\delta|^{q_\varepsilon + 1} \, dx \right)^{\frac{p(q_\varepsilon + 1)}{p+1}}} \leq S, \]
and
\[
\lim_{\varepsilon \to 0} \int_\Omega |x|^{-\frac{p}{2}} |\Delta w_\varepsilon|^\frac{p+1}{p} \, dx \\
\geq \lim_{\varepsilon \to 0} \int_\Omega |x|^{-\frac{p}{2}} |\Delta u_{\alpha, \beta, \varepsilon}|^\frac{p+1}{p} \, dx \\
\geq \lim_{\varepsilon \to 0} \int_\Omega |\Delta u_{\alpha, \beta, \varepsilon}|^\frac{p+1}{p} \, dx \\
\geq \lim_{\varepsilon \to 0} \int_\Omega |u_{\alpha, \beta, \varepsilon}|^\frac{p+1}{p} \, dx \\
\int_\Omega |x|^\alpha |\Delta u_{\alpha, \beta, \varepsilon}|^\frac{p+1}{p} \, dx
\]

where \(u_{\alpha, \beta, \varepsilon}\) is a minimizer of \(S_{\alpha, \beta, \varepsilon}\); thus \(S_{\alpha, \beta, 0} \leq S\). \(\square\)

**Lemma 4.3.** Assume that \(N \geq 3\). For any \(n \in \mathbb{N}\) there exists \(\delta_n > 0\) such that \(S_{\alpha, \beta, \varepsilon} < S_{\alpha, \beta, 0}\) provided \(\beta \geq \frac{1}{n}\) and \(\varepsilon < \delta_n\).

**Proof.** By contradiction, assume that there exists \(n \in \mathbb{N}\) and sequences \(\beta_k \geq \frac{1}{n}\) and \(\delta_k \to 0\) such that
\[
S_{\alpha, \beta_k, \delta_k} = S_{\alpha, \beta_k, \delta_k}.
\]
From Lemma 3.2, there exists \(c_1\) independent of \(q_{\delta_k}\), such that
\[
S_{\alpha, \beta_k} \leq c_1 \beta_k^{\frac{p+1}{q_{\delta_k}+1}}.
\]
Lemma 4.1 implies that
\[
c_0 \beta_k^{\frac{p+1}{q_{\delta_k}+1}} \leq S_{\alpha, \beta_k}.
\]
Since
\[
\frac{p+1}{p(q_{\delta_k}+1)} - \frac{2(p+1)}{p} + N - \frac{N(p+1)}{p(q_{\delta_k}+1)} = \frac{(pN - 2p - 2)(q_{\delta_k}+1) - (p+1)(N-1)}{p(q_{\delta_k}+1)},
\]
we have
\[
\beta_k^{\frac{p+1}{(pN - 2p - 2)(q_{\delta_k}+1) - (p+1)(N-1)}} \leq c_1.
\]
From \(q + 1 = \frac{N(p+1)}{Np-2p-2} = \frac{(N-1)(p+1)}{Np-2p-2}\), and \(q_{\delta_k} \to q\) as \(k \to +\infty\), it is implies that \((pN - 2p - 2)(q_{\delta_k}+1) - (p+1)(N-1) > 0\) as \(k \to +\infty\). Thus, \(\beta_k\) is bounded. We can assume that \(\beta_k \to \beta \geq \frac{1}{n}\) as \(k \to +\infty\).

Claim that
\[
S_{\alpha, \beta, 0} = \lim_{k \to +\infty} S_{\alpha, \beta_k, \delta_k}.
\]
Indeed, by upper semi-continuity, it follows that
\[
S_{\alpha, \beta, 0} \geq \limsup_{k \to +\infty} S_{\alpha, \beta_k, \delta_k}.
\]
On the other hand, from
\[
\int_{\Omega} |x|^\beta |u_k|^{q_k+1} \, dx \leq \left( \int_{\Omega} |x|^\beta |u_k|^{q+1} \, dx \right)^{\frac{q_k+1}{q+1}} \left( \int_{\Omega} |x|^\beta \, dx \right)^{\frac{q-q_k}{q+1}},
\]
we have
\[
S_{rad}^{\alpha, \beta, 0} \leq \liminf_{k \to +\infty} S_{rad}^{\alpha, \beta, \delta_k}.
\]
Similarly, by upper continuity,
\[
S_{\alpha, \beta, 0} \geq \limsup_{k \to +\infty} S_{\alpha, \beta, \delta_k}. \tag{15}
\]
We obtain from (13)–(15), \(S_{\alpha, \beta, 0} \geq S_{rad}^{\alpha, \beta, 0}\), which contradicts Lemma 4.2. \(\square\)

REFERENCES