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LOWER BOUNDS FOR THE RAMSEY NUMBERS

BY

PIERRE ROBILLARD

ABSTRACT. A lower bound for a family of Ramsey numbers is derived using a geometrical argument.

The Ramsey number $N(q_1, q_2, \ldots, q_t, r)$ is defined as the least n such that for every t-ary partition $A_1 \cup A_2 \cup \cdots \cup A_t$ of the $\binom{n}{r}$ unordered r-subsets in an n-element set, there must exist one i for which A_i contains all the $\binom{q_i}{r}$ r-subsets of a q_i-subset. We want to find a lower bound for a family of Ramsey numbers.

In order to prove that n^* is a lower bound for the number $N(q_1, q_2, \ldots, q_t, r)$ it is sufficient to produce a partition $A_1^* \cup A_2^* \cup \cdots \cup A_t^*$ of the $\binom{n^*}{r}$ unordered r-subsets in an n^*-element set where it is impossible to find a set A_i containing all the $\binom{q_i}{r}$ r-subsets of a q_i-subset.

Let us consider the finite projective geometry PG$(r-1, q)$ of dimension $(r-1)$ over the field $GF(q)$ (q is a prime number or the power of a prime number). A set of points in PG$(r-1, q)$ is said to possess the property P_d if no d-subset of them are linearly dependent. We denote by $m_d(r, q)$ the maximum number of points we can choose in PG$(r-1, q)$ so that no d are dependent. The number $m_d(r, q)$ arises in connection with some problems of the theory of confounded factorial designs [2] and error correcting codes [3]. The evaluation of $m_d(r, q)$ is known as 'the packing problem'.

The geometry PG$(r-1, q)$ contains $(q^r-1)/(q-1)=N$ points. Let S denote the set of all $\binom{N}{r}$ unordered r-subsets of PG$(r-1, q)$. Let A_1 consist of all the r-subsets of points of S with the property P_r. Let A_2 consist of all the r-subsets of points of $S-A_1$ with the property P_{r-1} and in general let A_v consist of all the r-subsets of points in $S-\bigcup_{i=1}^{v-1} A_i$ with property P_{r-v+1} for $v=2, 3, \ldots, r-1$.

From the definition of $m_r(r, q)$ it follows easily that no $(m_r(r, q)+1)$-subset of points exists with all its r-subsets contained in A_1. For each v-subset T of points in PG$(r-1, q)$ whose $\binom{v}{r}$ unordered r-subsets are contained in A_2 we can associate an $r \times v$ matrix $M(T)$ the columns of which represent (in some given order) the v points of T. From the definition of A_2 it follows that $M(T)$ has rank $r-1$. Thus one can premultiply $M(T)$ by a nonsingular matrix A and obtain the matrix $M^*(T)=A \cdot M(T)$ the last row of which is null. If we delete this last row, each column of $M^*(T)$ represents a point of PG$(r-2, q)$; this new set of v points has property P_{r-1}. The maximum number of points of PG$(r-1, q)$ one can choose such that all its r-subsets are contained in A_2 is then $m_{r-1}(r-1, q)$. A similar argument applies for the set A_v and the maximum number is $m_{r-v+1}(r-v+1, q)$.

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As $m_s(s,q)=s+1$ for $q\leq s$ then $m_w(w,q)\to\infty$ as $w\to\infty$, and there exist an integer $w\geq 2$ for which the inequality $m_w(w,q)\geq r$ holds. We now have the following

THEOREM 1.

$$N(m_r(r,q)+1, m_{r-1}(r-1,q)+1,\ldots,m_w(w,q)+1,r) > (q^r-1)/(q-1) \tag{1}$$

COROLLARY. *The inequality* (1) *remains true if some* $m_s(s,q)$ *are replaced by a value* $m_s^*(s,q)>m_s(s,q)$.

We now apply the result of the theorem when $r=3$ and consider the geometry PG$(2,q)$; A_1 consists of all the independent triplets of points of PG$(2,q)$ and A_2 of all the dependent triplets of points. In this case we have

$$m_2(2,q) = q+1 \tag{2}$$

$$m_3(3,q) = q+1 \quad \text{if } q \text{ is odd}$$
$$= q+2 \quad \text{if } q \text{ is even.} \tag{3}$$

Then

$$N(q+2,q+2,3) > q^2+q+1 \quad \text{if } q \text{ is odd}$$
$$N(q+3,q+2,2) > q^2+q+1 \quad \text{if } q \text{ is even.}$$

For example this gives

$$N(5,4,3) > 7$$
$$N(5,5,3) > 13$$
$$N(7,6,3) > 21$$
$$N(7,7,3) > 31$$

for $q=2, 3, 4, 5$ respectively.

Gulati [6] has proved that $m_t(t,2)=t+1$ for $t\geq 4$. This, with (2) and (3), leads to the inequality

$$N(r+2,r+1,\ldots,6,5,4,3) > 2^r-1 \quad \text{for } r\geq 4.$$

The upper bounds for $m_t(t,q)$ derived in [1, 4, 5, 6] can also be used to obtain some lower bounds for the Ramsey numbers.

A generalization. We consider again the geometry PG$(r-1,q)$ and S denotes the set of all the unordered r-subsets of points. A_1 now consists of all the r-subsets of points S with property P_{r_1} where $r\geq r_1\geq 2$. A_2 consists of all the r-subsets of points of $S-A_1$ with property P_{r_2} where $r_1>r_2\geq 2$ and in general A_v consists of all the r-subsets of points $S-\bigcup_{i=1}^{v-1}A_i$ with property P_{r_v} where $r_{v-1}>r_v\geq 2$. We now have

THEOREM 2.

$N(m_{r_1}(r,q)+1, m_{r_2}(r_1-1,q)+1,\ldots,m_{r_w}(r_{w-1}-1,q)+1,r_1) > (q^r-1)/(q-1)$ *where all the quantities* $m_s(t,q)$ *in the left-hand side are greater than or equal to* r.

The proof of this theorem follows the lines of the proof of Theorem 1.

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As an application of Theorem 2 we consider the values $r=4$, $r_1=3$, and $r_2=2$. It is known that

$$m_3(4, q) = q^2+1$$

and

$$m_2(2, q) = q+1.$$

Thus

$$N(q^2+2, q+2, 3) > q^3+q^2+q+1 \quad \text{for } q > 1.$$

Then

$$N(6, 4, 3) > 15$$

$$N(11, 5, 3) > 40$$

$$N(18, 6, 3) > 85$$

for $q=2, 3, 4$ respectively.

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DÉPARTEMENT D' INFORMATIQUE
UNIVERSITÉ DE MONTRÉAL,
MONTRÉAL, QUÉBEC