A REMARK ON PREINVEX FUNCTIONS

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In this paper, we show that the ratio of preinvex functions is invex. Hence, we give a positive answer to the open question which was proposed in a paper of Yang, Yang and Teo in (2003).

1. INTRODUCTION

Let $\mathbb{R}^n$ denotes $n$-dimension Euclidean space. In [2], Hanson considered the real differentiable function $f(x)$ on $\mathbb{R}^n$ whose gradient $\nabla f(x)$ satisfies the condition: for any $x, y \in \mathbb{R}^n$, there exists a vector $\eta(x, y) \in \mathbb{R}^n$ such that

$$f(x) \geq f(y) + \nabla f(y) \eta(x, y).$$


Let $K \subset \mathbb{R}^n$ and $f : K \rightarrow \mathbb{R}$. Then $f$ is preinvex if for any $x, y \in K$, there exists a vector $\eta(x, y) \in \mathbb{R}^n$, for all $\alpha \in [0,1)$, $y + \alpha \eta(x, y) \in K$

$$f(y + \alpha \eta(x, y)) \leq \alpha f(x) + (1 - \alpha) f(y).$$

It is easy to show that preinvexity is a generalisation of invexity for nondifferentiable function.

In [6], Yang and Chen presented a wider class of generalised convex functions, called semipreinvex functions as follows.

A set $K$ in $\mathbb{R}^n$ is said to satisfy the “semi-connected” property, if for any $x, y \in K$ and $\alpha \in [0,1]$, there exists a vector $\eta(x, y, \alpha) \in \mathbb{R}^n$, such that $y + \alpha \eta(x, y, \alpha) \in K$. Let $K$ be a set in $\mathbb{R}^n$ having the “semi-connected” property with $\eta(x, y, \alpha) : K \times K \times [0,1] \rightarrow \mathbb{R}^n$ and $f(x)$ be a real function on $K$. Then $f$ is called semi-preinvex with respect to $\eta(x, y, \alpha)$ if for $x, y \in K$ and $\alpha \in [0,1]$,

$$f(y + \alpha \eta(x, y, \alpha)) \leq \alpha f(x) + (1 - \alpha) f(y).$$
holds and \( \lim_{\alpha \to 0} \alpha \eta(x, y, \alpha) = 0 \).

The following result is due to Khan and Hanson [7] and Craven and Mond [8].

**Theorem 1.1.** Let \( X_0 \subseteq \mathbb{R}^n \) and let \( f \) and \( g \) be real-valued functions defined on \( X_0 \). If \( f(x) \geq 0, g(x) > 0, f(x) \) and \(-g(x)\) are invex with respect to a same \( \eta(x, y) \) on \( X_0 \), then \( f(x)/g(x) \) is an invex function with respect to \( \bar{\eta}(x, y) = (g(y)/g(x))\eta(x, y) \).

Yang, Yang and Teo [1] generalise Theorem 1.1 as follows.

**Theorem 1.2.** (See [1, Theorem 2.9] ) Let \( X_0 \subseteq \mathbb{R}^n \) and let \( f \) and \( g \) be real-valued differential functions defined on \( X_0 \). If \( f(x) \geq 0, g(x) > 0, f(x) \) and \(-g(x)\) are semipreinvex with respect to a same \( \eta(x, y, \alpha) \) on \( X_0 \), and \( \lim_{\alpha \to 0} \eta(x, y, \alpha) = \eta(x, y) \), then \( f(x)/g(x) \) is an invex function with respect to \( \bar{\eta}(x, y) = (g(y)/g(x))\eta(x, y) \).

Then, Yang, Yang and Teo [1] proposed an open question as follows:

Is there a similar result as that of Theorem 1.2 for preinvex functions?

In this paper, we show that the ratio of preinvex functions is invex. Hence, we give a positive answer to the open question in [1].

2. **Main Results**

First of all, we prove the following result which is a generalisation of Theorem 1.1 and a similar result with [1, Theorem 2.8].

**Theorem 2.1.** Let \( X_0 \subseteq \mathbb{R}^n \) and let \( f \) and \( g \) be real-valued functions defined on \( X_0 \). If \( f(x) \geq 0, g(x) > 0, f(x) \) and \(-g(x)\) are preinvex with respect to a same \( \eta(x, y) \) on \( X_0 \), then \( f(x)/g(x) \) is a semipreinvex function with respect to \( \eta^*(x, y, \alpha) = [(\alpha g(y))/(\alpha g(y) + (1 - \alpha)g(x))]\eta(x, y) \).

**Proof:** Since \( f(x) \) and \(-g(x)\) are preinvex with respect to a same \( \eta(x, y) \) and \( f(x) \geq 0, g(x) > 0 \), we have, for all \( x, y \in X_0 \) and \( \alpha \in [0, 1] \), \( y + \alpha \eta^*(x, y, \alpha) \in X_0 \), and

\[
\left( \frac{f}{g} \right)(y + \alpha \eta^*(x, y, \alpha)) \leq \frac{f(y + \alpha \eta^*(x, y, \alpha))}{g(y + \alpha \eta^*(x, y, \alpha))} \leq \frac{f(y + [(\alpha g(y))/(\alpha g(y) + (1 - \alpha)g(x))]\eta(x, y))}{g(y + [(\alpha g(y))/(\alpha g(y) + (1 - \alpha)g(x))]\eta(x, y))} \leq \frac{(\alpha g(y))/(\alpha g(y) + (1 - \alpha)g(x))f(x) + ((1 - \alpha)g(x))/(\alpha g(y) + (1 - \alpha)g(x))f(y)}{(\alpha g(y))/(\alpha g(y) + (1 - \alpha)g(x))g(x) + ((1 - \alpha)g(x))/(\alpha g(y) + (1 - \alpha)g(x))g(y)} \leq \frac{\alpha g(y)f(x) + (1 - \alpha)g(x)f(y)}{\alpha g(y)g(x) + (1 - \alpha)g(x)g(y)} \leq \frac{\alpha f(x) + (1 - \alpha)f(y)}{g(x)} \leq \frac{\alpha f(x) + (1 - \alpha)f(y)}{g(x)} \leq \frac{\alpha f(x) + (1 - \alpha)f(y)}{g(y)} \leq \frac{\alpha f(x) + (1 - \alpha)f(y)}{g(y)}
\]
A remark on preinvex functions

That is, \( f(x)/g(x) \) is a semipreinvex function with respect to \( \eta^*(x, y, \alpha) \).

The following result gives a positive answer to the open question in [1].

**Theorem 2.2.** Let \( X_0 \subset \mathbb{R}^n \) and let \( f \) and \( g \) be real-valued differential functions defined on \( X_0 \). If \( f(x) \geq 0, g(x) > 0 \), \( f(x) \) and \( -g(x) \) are preinvex with respect to a same \( \eta(x, y) \) on \( X_0 \), then \( f(x)/g(x) \) is an invex function with respect to \( \eta(x, y) = [g(y)/g(x)]\eta(x, y) \).

**Proof:** By Theorem 2.1, we know that \( f(x)/g(x) \) is a semipreinvex function with respect to \( \eta^*(x, y, \alpha) = [(g(y))/(\alpha g(y) + (1 - \alpha)g(x))]\eta(x, y) \). That is, for all \( x, y \in X_0 \) and \( \alpha \in [0, 1] \),

\[
\left( \frac{f}{g} \right)(y + \alpha \eta^*(x, y, \alpha)) \leq \alpha \left( \frac{f}{g} \right)(x) + (1 - \alpha) \left( \frac{f}{g} \right)(y).
\]

Then,

\[
\frac{(f/g)(y + \alpha \eta^*(x, y, \alpha)) - (f/g)(y)}{\alpha} \leq \left( \frac{f}{g} \right)(x) - \left( \frac{f}{g} \right)(y).
\]

Let \( \alpha \to 0 \), and note that \( \lim_{\alpha \to 0} \eta^*(x, y, \alpha) = \eta(x, y) \), we have

\[
\bigtriangledown \left( \frac{f}{g} \right)(y) \eta(x, y) \leq \left( \frac{f}{g} \right)(x) - \left( \frac{f}{g} \right)(y).
\]

Hence, \( f(x)/g(x) \) is an invex function with respect to \( \eta(x, y) \).

**References**


