2-GROUPS WITH EVERY AUTOMORPHISM CENTRAL

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Abstract

An infinite family of 2-groups is produced. These groups have no direct factors and have a non-abelian automorphism group in which all automorphisms are central.


The automorphisms of a group $G$ which induce the identity on $G/Z(G)$ are called central. This paper investigates groups which admit only central automorphisms. Various authors ([2], [5], [7], [9]) considered non-abelian $p$-groups with abelian automorphism groups in which necessarily every automorphism is central. Curran [1] and Malone [6] constructed non-abelian $p$-groups with non-abelian automorphism groups in which every automorphism is central. The groups constructed by Curran and Malone, however, all had direct factors. Malone [6] wondered if this condition were necessary. The groups $G_n$ ($n \geq 4$) described below show that direct factors are not necessary.

Define $G_n$ to be the group

$$\langle x_1, \ldots, x_n | x_i^{2^i} = 1, 1 \leq i \leq n, [x_i, x_{i+1}] = x_i^{2^i}, 1 \leq i < n, [x_i, x_j] = 1, 1 < i + 1 < j \leq n \rangle.$$ 

Then $G_n$ has order $2^{n(n+1)/2}$. The group $G_3$ was first considered by Miller [7] and is group 99 of the Hall and Senior tables [4].
THEOREM. For \( n \geq 3 \), \( G_n \) has no direct factors and only central automorphisms. The automorphism group of \( G_n \) has order \( 2^{p(n)} \), where \( p(n) = (n - 1)(2n^2 - n + 6)/6 \), and is non-abelian for \( n \geq 4 \).

PROOF. The centre and derived subgroups of \( G_n \) are \( \langle x_2^2, x_3^2, \ldots, x_n^2 \rangle \) and \( \langle x_2^2, x_3^4, \ldots, x_n^{2n-1} \rangle \), respectively. Let \( x(i_1, \ldots, i_n) \) denote \( x_1^{i_1} \cdots x_n^{i_n} \). Then every element of \( G_n \) may be uniquely represented as \( x(i_1, \ldots, i_n) \) with \( 0 \leq i_k < 2^k \). Commutator collection, together with the fact that \( G_n \) has class 2, yields the equation

\[
x(i_1, \ldots, i_n)^2 = x(0, 2i_2(1 + i_1), 2i_3(1 + 2i_2), \ldots, 2i_n(1 + 2^{n-2}i_{n-1})).
\]

Thus the square of every element is central. Since the centre has exponent \( 2^{n-1} \), \( G_n \) has exponent \( 2^n \). All elements of maximal order must have \( i_n \) odd.

Suppose \( G_n \) is the direct product \( A \times B \), where \( A \) has exponent \( 2^n \). If \( y = x_n^{2n-1} \), then the \( 2^{n-1} \)-th power of every element of order \( 2^n \) is \( y \). Thus \( y \) is an element of \( A \), and no element of \( B \) has order \( 2^n \). Hence every element of order \( 2^n \) is contained in \( A \), and so \( A \) contains \( x_1^{x_n}, \ldots, x_{n-1}^{x_n}, x_n \) and must therefore equal \( G_n \).

If \( i_{k,j} \) are integers, then the map taking \( x_1 \) to \( x(1, 2i_{1,2}, 4i_{1,3}, \ldots, 2^{n-1}i_{1,n}) \) and \( x_k \) to \( x(2i_{k,1}, \ldots, 2i_{k,k-1}, 2i_{k,k} + 1, 2^2i_{k,k+1}, \ldots, 2^{n-k+1}i_{k,n}) \), \( 1 < k < n \), defines a homomorphism of \( G_n \) which is the identity on \( G_n/Z(G_n) \). Indeed, because the centre of \( G_n \) equals the Frattini subgroup, these homomorphisms are automorphisms. There are \( 2^{p(n)} \) distinct automorphisms of the above form.

As \( G_n \) has no abelian direct factors, [8, Theorem 1] may be used to show that \( G_n \) has \( 2^{p(n)} \) central automorphisms. We shall prove by induction that all automorphisms of \( G_n \) are central for \( n \geq 3 \), and thus all have the above form. The case \( n = 3 \) is easy because \( p(3) = 7 \) and the Miller group, \( G_3 \), has \( 2^7 \) automorphisms (see [4, 7]).

Suppose \( n > 3 \). The above remarks show that \( y \) is fixed by all automorphisms of \( G_n \). There is an isomorphism \( \iota \) from \( G_n/\langle y \rangle \) to \( G_{n-1} \times Z \) such that the elements \( x_1', \ldots, x_{n-1}' \) generate \( G_{n-1} \), and such that the element \( x_n' \) of order \( 2^{n-1} \) generates \( Z \). The homomorphism \( \iota \) from \( \text{Aut}(G_n) \) to \( \text{Aut}(G_{n-1} \times Z) \) and the inductive hypothesis give information about \( \text{Aut}(G_n) \).

Let \( \epsilon \) be the inclusion map from \( G_{n-1} \) to \( G_{n-1} \times Z \), \( \pi \) the projection map from \( G_{n-1} \times Z \) to \( G_{n-1} \), and \( \phi \) an automorphism of \( G_n \). Then, arguing as in Malone [6], we see that \( \iota \phi \epsilon \pi \) is an automorphism of \( G_{n-1} \). So any automorphism of \( G_n \) takes \( x_1 \) to \( x(1, 2i_{1,2}, \ldots, 2^{n-2}i_{1,n-1}, j_{1,n}) \) and \( x_k \) to \( x(2i_{k,1}, \ldots, 2i_{k,k-1}, 2i_{k,k} + 1, 2^2i_{k,k+1}, \ldots, 2^{n-k}i_{k,n-1}, j_{k,n}) \), \( 1 < k < n \). But \( x_k \) is mapped to an element of order \( 2^k \), so \( 2^{n-k} \) divides \( j_{k,n} \) for \( 1 \leq k < n \). Since \( x_n' \) is central and \( x_n \) is not, \( x_n \) is mapped to \( x(2i_{n,1}, \ldots, 2i_{n,n-1}, 2i_{n,n} + 1) \). Thus all automorphisms of \( G_n \) are central, which completes the inductive proof.
Finally, Aut\( (G_n) \) is non-abelian for \( n \geq 4 \). Let \( \phi \) be the automorphism which maps \( x_4 \) to \( x_3^2 x_4 \) and which fixes the remaining generators. Let \( \psi \) map \( x_3 \) to \( x_3 x_4^4 \) and fix the other generators. Then \( (x_4)\phi \psi = x_3^2 x_4^5 \) and \( (x_4)\psi \phi = x_3^2 x_4 \). Thus \( \phi \) and \( \psi \) do not commute, and so the proof of the theorem is complete.

**Further Examples.** We exhibit some further examples of 2-groups with all automorphisms central. Let \( H_n \) and \( K_n \) denote the groups 

\[
H_n = \left\langle x_1, \ldots, x_n | x_i^2 = 1, 1 \leq i \leq n, [x_i, x_n] = x_{i+1}^2, 1 \leq i < n \right\rangle,
\]

and

\[
K_n = \left\langle x_0, \ldots, x_n | x_0^2 = x_n^2 = 1, 1 \leq i < n, [x_0, x_i] = x_{i+1}^2, 1 \leq i < n - 1, [x_0, x_{n-1}] = x_n \right\rangle,
\]

where the commutators \([x_i, x_j] (1 \leq i < j \leq n)\) not shown above are trivial. A proof of the statements made below appears in [3], a copy of which may be obtained from the author. The groups \( H_n \) and \( K_n \) have order \( 2^{2n} \), and their automorphism groups are elementary abelian of order \( 2^{2n} \). Furthermore, \( H_n \) and \( K_n \) are not isomorphic either to themselves or to any of Jonah and Konvisser’s groups [5].

The groups \( G_n \), like the groups \( H_n \) and \( K_n \), may be generalized to give different groups whose automorphisms are all central. Let \( G(l_1, \ldots, l_n) \) denote the group 

\[
\left\langle x_1, \ldots, x_n | x_i^{m_i} = 1, 1 \leq i \leq n, [x_{i-1}, x_i] = x_{i-1}^{m_i/2}, 1 < i \leq n, [x_i, x_j] = 1, 1 < i + 1 < j \leq n \right\rangle,
\]

where \( n \geq 3 \), \( m_i = 2^{l_i} \) and \( 0 < l_1 < \cdots < l_n \). Then every automorphism of \( G(l_1, \ldots, l_n) \) is central if and only if \( l_1 = 1 \) and \( l_2 = 2 \). The groups \( G(1, 2, l) \) are isomorphic to those considered by Struik [9], and the group \( G(1, \ldots, n) \) is isomorphic to \( G_n \). Finally, \( G(1, 2, \ldots, l_n) \) has no direct factors and, when \( n > 3 \), has a non-abelian automorphism group of order \( 2^p \), where \( p = 2(n - 1) + \Sigma_{k=3}^n [(l_1 + l_2 + \cdots + l_k) - k + (n - k)(k - 1)] \).

**References**


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