



Convolution of Trace Class Operators over Locally Compact Quantum Groups

Zhiguo Hu, Matthias Neufang, and Zhong-Jin Ruan

Abstract. We study locally compact quantum groups \mathbb{G} through the convolution algebras $L_1(\mathbb{G})$ and $(T(L_2(\mathbb{G})), \triangleright)$. We prove that the reduced quantum group C^* -algebra $C_0(\mathbb{G})$ can be recovered from the convolution \triangleright by showing that the right $T(L_2(\mathbb{G}))$ -module $\langle K(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G})) \rangle$ is equal to $C_0(\mathbb{G})$. On the other hand, we show that the left $T(L_2(\mathbb{G}))$ -module $\langle T(L_2(\mathbb{G})) \triangleright K(L_2(\mathbb{G})) \rangle$ is isomorphic to the reduced crossed product $C_0(\widehat{\mathbb{G}}) \rtimes C_0(\mathbb{G})$, and hence is a much larger C^* -subalgebra of $B(L_2(\mathbb{G}))$.

We establish a natural isomorphism between the completely bounded right multiplier algebras of $L_1(\mathbb{G})$ and $(T(L_2(\mathbb{G})), \triangleright)$, and settle two invariance problems associated with the representation theorem of Junge–Neufang–Ruan (2009). We characterize regularity and discreteness of the quantum group \mathbb{G} in terms of continuity properties of the convolution \triangleright on $T(L_2(\mathbb{G}))$. We prove that if \mathbb{G} is semi-regular, then the space $\langle T(L_2(\mathbb{G})) \triangleright B(L_2(\mathbb{G})) \rangle$ of right \mathbb{G} -continuous operators on $L_2(\mathbb{G})$, which was introduced by Bekka (1990) for $L_\infty(G)$, is a unital C^* -subalgebra of $B(L_2(\mathbb{G}))$. In the representation framework formulated by Neufang–Ruan–Spronk (2008) and Junge–Neufang–Ruan, we show that the dual properties of compactness and discreteness can be characterized simultaneously via automatic normality of quantum group bimodule maps on $B(L_2(\mathbb{G}))$. We also characterize some commutation relations of completely bounded multipliers of $(T(L_2(\mathbb{G})), \triangleright)$ over $B(L_2(\mathbb{G}))$.

1 Introduction

Let $\mathbb{G} = (L_\infty(\mathbb{G}), \Gamma, \varphi, \psi)$ be a von Neumann algebraic locally compact quantum group and let $L_1(\mathbb{G})$ be the convolution quantum group algebra of \mathbb{G} . It is known that the right fundamental unitary of \mathbb{G} induces a co-multiplication on $B(L_2(\mathbb{G}))$. The pre-adjoint of this co-multiplication defines a completely contractive multiplication \triangleright on the space $T(L_2(\mathbb{G}))$ of trace class operators on $L_2(\mathbb{G})$ such that $L_1(\mathbb{G})$ is naturally a quotient algebra of $T(L_2(\mathbb{G}))$ (cf. [14]). We consider this right lifting convolution algebra of $L_1(\mathbb{G})$, with focus on the convolution $T(L_2(\mathbb{G}))$ -bimodule action on $B(L_2(\mathbb{G}))$ and its restriction to the C^* -algebra $K(L_2(\mathbb{G}))$ of compact operators on $L_2(\mathbb{G})$. This study in particular helps us to completely settle the following question (which was the main motivation of this paper):

When is $K(L_2(\mathbb{G}))$ invariant under the completely isometric representation

$$(1.1) \quad \Theta^f : RM_{cb}(L_1(\mathbb{G})) \xrightarrow{\cong} CB_{L_\infty(\widehat{\mathbb{G}})}^{\sigma, L_\infty(\mathbb{G})}(B(L_2(\mathbb{G})))$$

by Junge–Neufang–Ruan [16]?

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Here, $RM_{cb}(L_1(\mathbb{G}))$ is the completely bounded right multiplier algebra of $L_1(\mathbb{G})$, and $CB_{L_\infty(\widehat{\mathbb{G}})}^{\sigma, L_\infty(\mathbb{G})}(B(L_2(\mathbb{G})))$ is the algebra of completely bounded normal $L_\infty(\widehat{\mathbb{G}})$ -bimodule maps on $B(L_2(\mathbb{G}))$ that map $L_\infty(\mathbb{G})$ into $L_\infty(\mathbb{G})$. See [22] for the representation (1.1) over commutative and co-commutative locally compact quantum groups.

We recall in Section 2 some definitions and results on locally compact quantum groups. Section 3 is devoted to the study of the restriction of the convolution $T(L_2(\mathbb{G}))$ -bimodule action on $B(L_2(\mathbb{G}))$ to $K(L_2(\mathbb{G}))$. First, comparing with the equality $\langle B(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G})) \rangle = LUC(\mathbb{G})$ (cf. [14]), we obtain that $\langle K(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G})) \rangle = C_0(\mathbb{G})$. Then we prove that ${}_{\triangleright}K(L_2(\mathbb{G})) = \langle T(L_2(\mathbb{G})) \triangleright K(L_2(\mathbb{G})) \rangle$ is a C^* -subalgebra of $B(L_2(\mathbb{G}))$. In terms of the convolution \triangleright on $T(L_2(\mathbb{G}))$, we present a quantum group version of the Stone–von Neumann theorem by showing that $C_0(\widehat{\mathbb{G}})_r \rtimes C_0(\mathbb{G}) \cong {}_{\triangleright}K(L_2(\mathbb{G}))$, noticing that ${}_{\triangleright}K(L_2(\mathbb{G})) = K(L_2(\mathbb{G}))$ if \mathbb{G} is regular in the sense of Baaj–Skandalis [2] (in particular, if \mathbb{G} is the commutative quantum group $L_\infty(G)$ over a locally compact group G). Here, $C_0(\widehat{\mathbb{G}})_r \rtimes C_0(\mathbb{G})$ is the reduced crossed product induced by the co-multiplication on $C_0(\mathbb{G})$. Consequently, we obtain that \mathbb{G} is discrete if and only if the convolution \triangleright on $T(L_2(\mathbb{G}))$ is separately w^* -continuous. Results in Section 3 indicate that the C^* -algebra ${}_{\triangleright}K(L_2(\mathbb{G}))$ also encodes some “cross relations” between the right and left fundamental unitaries of \mathbb{G} .

In Section 4, with the help of the representation (1.1), we obtain a canonical completely isometric algebra isomorphism $RM_{cb}(L_1(\mathbb{G})) \cong RM_{cb}(T(L_2(\mathbb{G})))$ between the completely bounded right multiplier algebras of $L_1(\mathbb{G})$ and $T(L_2(\mathbb{G}))$. As an application of this identification and the results established in Sections 3, we settle the above invariance problem associated with (1.1). Namely, we show that $K(L_2(\mathbb{G}))$ is invariant under the representation Θ^r if and only if \mathbb{G} is regular, which is true if and only if the convolution \triangleright is w^* -continuous on the left. This investigation provides in turn an interesting formulation for the representation (1.1), which reflects both the convolution $T(L_2(\mathbb{G}))$ -module structure and the duality between $C_0(\mathbb{G})$ and $C_0(\widehat{\mathbb{G}})$.

In Section 5, we consider the large and complicated space

$${}_{\triangleright}X(L_2(\mathbb{G})) = \langle T(L_2(\mathbb{G})) \triangleright B(L_2(\mathbb{G})) \rangle$$

of right \mathbb{G} -continuous operators on $L_2(\mathbb{G})$. We prove in particular that if \mathbb{G} is semi-regular, then ${}_{\triangleright}X(L_2(\mathbb{G}))$ is indeed a unital C^* -subalgebra of $B(L_2(\mathbb{G}))$. For commutative locally compact quantum groups, the space ${}_{\triangleright}X(L_2(\mathbb{G}))$ was studied by Bekka [5] and Neufang [21].

In Section 6, we prove that compactness of \mathbb{G} can be characterized by automatic normality of completely bounded right $T(L_2(\mathbb{G}))$ -module maps and completely bounded $L_\infty(\widehat{\mathbb{G}})$ -bimodule maps on $B(L_2(\mathbb{G}))$. Therefore, in the representation framework developed in [16, 22], compactness and discreteness can be characterized simultaneously via quantum group bimodule maps, and thus their duality is expressed through a simple duality between automatic normality properties over the “home space” $B(L_2(\mathbb{G}))$. We also characterize some commutation relations of completely bounded multipliers of $T(L_2(\mathbb{G}))$ over $B(L_2(\mathbb{G}))$. Several instances of the asymmetric behaviour of the algebra $(T(L_2(\mathbb{G})), \triangleright)$ are revealed in the paper.

2 Preliminaries

In this section, we recall some notation and results related to locally compact quantum groups. The reader is referred to Kustermans and Vaes [17, 18], Runde [25, 26], van Daele [28], and [11, 12, 14] for more information. Let $\mathbb{G} = (M, \Gamma, \varphi, \psi)$ be a von Neumann algebraic locally compact quantum group. Then the pre-adjoint of the co-multiplication Γ induces on M_* an associative completely contractive multiplication $\star: M_* \widehat{\otimes} M_* \rightarrow M_*$, where $\widehat{\otimes}$ is the operator space projective tensor product. In the case where M is $L_\infty(G)$ or $VN(G)$ with G a locally compact group, the algebra (M_*, \star) is the usual convolution group algebra $L_1(G)$, respectively, the Fourier algebra $A(G)$.

As for locally compact groups, the von Neumann algebra M , the convolution algebra (M_*, \star) , and the Hilbert space H_φ are denoted by $L_\infty(\mathbb{G})$, $L_1(\mathbb{G})$, and $L_2(\mathbb{G})$, respectively. Then $L_1(\mathbb{G})$ is a faithful completely contractive Banach algebra satisfying $\langle L_1(\mathbb{G}) \star L_1(\mathbb{G}) \rangle = L_1(\mathbb{G})$ (cf. [11, Fact 1] and [12, Proposition 1]). The multiplication on $L_1(\mathbb{G})$ induces canonically a completely contractive $L_1(\mathbb{G})$ -bimodule structure on $L_\infty(\mathbb{G})$ satisfying

$$x \star f = (f \otimes \iota)\Gamma(x) \quad \text{and} \quad f \star x = (\iota \otimes f)\Gamma(x) \quad (x \in L_\infty(\mathbb{G}), f \in L_1(\mathbb{G})).$$

The quantum group \mathbb{G} is said to be *co-amenable* if $L_1(\mathbb{G})$ has a bounded approximate identity.

Let $C_0(\mathbb{G})$ be the reduced C^* -algebra associated with \mathbb{G} (cf. [18]) and let $M(C_0(\mathbb{G}))$ be the multiplier algebra of $C_0(\mathbb{G})$. Then $C_0(\mathbb{G}) \subseteq M(C_0(\mathbb{G})) \subseteq L_\infty(\mathbb{G})$, and $C_0(\mathbb{G})$ is a w^* -dense C^* -subalgebra of $L_\infty(\mathbb{G})$. A quantum group \mathbb{G} is *compact* if $1 \in C_0(\mathbb{G})$, and is *discrete* if the dual quantum group $\widehat{\mathbb{G}}$ of \mathbb{G} is compact, which is equivalent to $L_1(\mathbb{G})$ being unital (cf. [8, 25]). The co-multiplication Γ maps $C_0(\mathbb{G})$ into the multiplier algebra $M(C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))$ of the minimal C^* -algebra tensor product $C_0(\mathbb{G}) \otimes C_0(\mathbb{G})$. Then $C_0(\mathbb{G})^*$ is a completely contractive Banach algebra under the multiplication (also denoted by \star) given by

$$\begin{aligned} \langle \mu \star \nu, x \rangle &= \langle \mu \otimes \nu, \Gamma(x) \rangle = \langle \mu, (\iota \otimes \nu)\Gamma(x) \rangle \\ &= \langle \nu, (\mu \otimes \iota)\Gamma(x) \rangle \quad (\mu, \nu \in C_0(\mathbb{G})^*, x \in C_0(\mathbb{G})), \end{aligned}$$

where $\mu \otimes \nu = \mu(\iota \otimes \nu) = \nu(\mu \otimes \iota) \in M(C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))^*$. We let

$$M(\mathbb{G}) = (C_0(\mathbb{G})^*, \star)$$

be the quantum measure algebra of \mathbb{G} . Then $M(\mathbb{G})$ is faithful (cf. [14, Proposition 2.2]), and is a dual Banach algebra in the sense of [24, Definition 1.1] that \star is separately w^* -continuous on $M(\mathbb{G}) = C_0(\mathbb{G})^*$. It is known that $L_1(\mathbb{G})$ is canonically identified with a closed two-sided ideal in $M(\mathbb{G})$ via $f \mapsto f|_{C_0(\mathbb{G})}$ (cf. [18, pp. 913–914]). If the quantum group is commutative (respectively, co-commutative), then $C_0(\mathbb{G}) = C_0(G)$ and $M(\mathbb{G}) = M(G)$ (respectively, $C_0(\mathbb{G}) = C_\lambda^*(G)$ and $M(\mathbb{G}) = B_\lambda(G)$) for some locally compact group G , where $C_\lambda^*(G)$ is the reduced group C^* -algebra of G and $B_\lambda(G)$ is the reduced Fourier–Stieltjes algebra of G .

It is known that there are two Banach algebra multiplications, \square and \diamond , on $L_1(\mathbb{G})^{**}$, each extending the multiplication \star on $L_1(\mathbb{G})$. For $m, n \in L_1(\mathbb{G})^{**}$ and $x \in L_\infty(\mathbb{G})$, by definition, the *left Arens product* $m\square n \in L_1(\mathbb{G})^{**}$ satisfies $\langle m\square n, x \rangle = \langle m, n\square x \rangle$, where $n\square x = (\iota \otimes n)\Gamma(x) \in L_\infty(\mathbb{G})$ is given by $\langle n\square x, f \rangle = \langle n, x\star f \rangle$ ($f \in L_1(\mathbb{G})$). Similarly, the *right Arens product* $m\diamond n \in L_1(\mathbb{G})^{**}$ satisfies $\langle x, m\diamond n \rangle = \langle x\diamond m, n \rangle$ with $x\diamond m = (m \otimes \iota)\Gamma(x) \in L_\infty(\mathbb{G})$ given by $\langle x\diamond m, f \rangle = \langle f\star x, m \rangle$ ($f \in L_1(\mathbb{G})$). Both Arens products are completely contractive multiplications on $L_1(\mathbb{G})^{**}$, and $L_1(\mathbb{G})$ is said to be *Arens regular* if \square and \diamond coincide on $L_1(\mathbb{G})^{**}$.

For an $L_1(\mathbb{G})$ -submodule X of $L_\infty(\mathbb{G})$ and for $x \in X$ and $m \in X^*$, one can naturally define $m\square x$ in $L_\infty(\mathbb{G})$. Then X is called *left introverted* in $L_\infty(\mathbb{G})$ if $X^*\square X \subseteq X$. In this case, the canonical quotient map $L_1(\mathbb{G})^{**} \rightarrow X^*$ yields a Banach algebra multiplication on X^* (also denoted by \square) such that

$$(2.1) \quad (X^*, \square) \cong (L_1(\mathbb{G})^{**}, \square)/X^\perp.$$

Right introverted subspaces of $L_\infty(\mathbb{G})$ are defined similarly.

According to [11, 26], the subspaces $LUC(\mathbb{G})$ and $RUC(\mathbb{G})$ of $L_\infty(\mathbb{G})$ are defined by

$$LUC(\mathbb{G}) = \langle L_\infty(\mathbb{G}) \star L_1(\mathbb{G}) \rangle \quad \text{and} \quad RUC(\mathbb{G}) = \langle L_1(\mathbb{G}) \star L_\infty(\mathbb{G}) \rangle.$$

Here, $\langle \cdot \rangle$ denotes the closed linear span. We note that for a general Banach algebra A , the spaces $UC_r(A) = \langle A^*A \rangle$ and $UC_l(A) = \langle AA^* \rangle$ were introduced by Lau [19] and were called the spaces of right and left uniformly continuous functionals on A , respectively. Therefore, we have that $LUC(\mathbb{G}) = UC_r(L_1(\mathbb{G}))$ and $RUC(\mathbb{G}) = UC_l(L_1(\mathbb{G}))$. It is clear that $LUC(\mathbb{G})$ is left introverted in $L_\infty(\mathbb{G})$ and $RUC(\mathbb{G})$ is right introverted in $L_\infty(\mathbb{G})$. They are the usual spaces $LUC(G)$ and $RUC(G)$ if $L_\infty(\mathbb{G}) = L_\infty(G)$ for a locally compact group G , where $LUC(G)$ (respectively, $RUC(G)$) is the space of bounded left (respectively, right) uniformly continuous functions on G . If $L_\infty(\mathbb{G}) = VN(G)$, then $LUC(\mathbb{G}) = RUC(\mathbb{G})$ is the space $UCB(\widehat{G})$ of uniformly continuous functionals on $A(G)$ (cf. [9]). In [26, Theorem 2.4], Runde showed that $LUC(\mathbb{G})$ and $RUC(\mathbb{G})$ are operator systems in $L_\infty(\mathbb{G})$ such that

$$C_0(\mathbb{G}) \subseteq LUC(\mathbb{G}) \cap RUC(\mathbb{G}) \subseteq LUC(\mathbb{G}) \cup RUC(\mathbb{G}) \subseteq M(C_0(\mathbb{G})).$$

It is known from [14] that $LUC(\mathbb{G})$ and $RUC(\mathbb{G})$ are C^* -subalgebras of $M(C_0(\mathbb{G}))$ if \mathbb{G} is semi-regular (see Section 3 for the definition of semi-regularity).

Let A be a Banach algebra. A bounded linear map μ on A is called a right multiplier of A if $\mu(ab) = a\mu(b)$ for all $a, b \in A$. We use $RM(A)$ to denote the right multiplier algebra of A (with opposite composition as the multiplication) and use $B_A(A^*)$ to denote the Banach algebra of bounded right A -module maps on A^* . Then $RM(A) \cong B_A^r(A^*)$ via $\mu \mapsto \mu^*$, where $B_A^r(A^*)$ is the Banach algebra consisting of w^* - w^* continuous maps in $B_A(A^*)$. When A is a completely contractive Banach algebra, $RM_{cb}(A)$ and $CB_A(A^*)$ will denote the algebras consisting of completely bounded

maps in $RM(A)$ and $B_A(A^*)$, respectively. Then we have $RM_{cb}(A^*) \cong CB_A(A^*)$. Similarly, the left (respectively, completely bounded left) multiplier algebra $LM(A)$ (respectively, $LM_{cb}(A)$) of A is defined, and we have the canonical anti-homomorphic embedding $LM(A) \rightarrow B(A^*)$ (respectively, $LM_{cb}(A^*) \rightarrow CB(A^*)$), $\mu \mapsto \mu^*$.

For $m \in LUC(\mathbb{G})^*$, let $m_L(x) = m \square x$ ($x \in L_\infty(\mathbb{G})$). Then $LUC(\mathbb{G})^* \rightarrow B_{L_1(\mathbb{G})}(L_\infty(\mathbb{G}))$, $m \mapsto m_L$ is an injective, contractive, and w^* - w^* continuous algebra homomorphism (cf. [14]). It is clear that we have the commutative diagram

$$(2.2) \quad \begin{array}{ccccc} M(\mathbb{G}) & \xrightarrow{m^r} & RM(L_1(\mathbb{G})) & \xrightarrow{\cong} & B_{L_1(\mathbb{G})}^\sigma(L_\infty(\mathbb{G})) \\ \pi \downarrow & & & & \downarrow \\ LUC(\mathbb{G})^* & \xrightarrow{n \mapsto n_L} & & & B_{L_1(\mathbb{G})}(L_\infty(\mathbb{G})) \end{array}$$

of algebra homomorphisms, where $m^r: M(\mathbb{G}) \rightarrow RM(L_1(\mathbb{G}))$ is the canonical embedding defined by $m^r(\mu)(f) = f \star \mu$ ($\mu \in M(\mathbb{G})$, $f \in L_1(\mathbb{G})$), \downarrow is the inclusion map, and $\pi: M(\mathbb{G}) \rightarrow LUC(\mathbb{G})^*$ is the completely isometric embedding given in [14, Proposition 6.1]. Also, the algebras $RM(L_1(\mathbb{G}))$, $B_{L_1(\mathbb{G})}^\sigma(L_\infty(\mathbb{G}))$, and $B_{L_1(\mathbb{G})}(L_\infty(\mathbb{G}))$ in (2.2) can be replaced by $RM_{cb}(L_1(\mathbb{G}))$, $CB_{L_1(\mathbb{G})}^\sigma(L_\infty(\mathbb{G}))$, and $CB_{L_1(\mathbb{G})}(L_\infty(\mathbb{G}))$, respectively; in this case, all the maps there will be completely contractive injections.

3 Convolutions on $T(L_2(\mathbb{G}))$, C^* -subalgebras of $B(L_2(\mathbb{G}))$, and Stone–von Neumann Type Theorems

Let \mathbb{G} be a locally compact quantum group. We recall that the right fundamental unitary V of \mathbb{G} induces on $B(L_2(\mathbb{G}))$ a co-associative co-multiplication

$$\Gamma^r: B(L_2(\mathbb{G})) \longrightarrow B(L_2(\mathbb{G})) \widehat{\otimes} B(L_2(\mathbb{G})), \quad x \longmapsto V(x \otimes 1)V^*$$

such that the restriction of Γ^r to $L_\infty(\mathbb{G})$ is just the co-multiplication Γ on $L_\infty(\mathbb{G})$. The pre-adjoint of Γ^r defines on $T(L_2(\mathbb{G}))$ an associative completely contractive multiplication

$$\triangleright: T(L_2(\mathbb{G})) \widehat{\otimes} T(L_2(\mathbb{G})) \longrightarrow T(L_2(\mathbb{G})), \quad \omega \otimes \gamma \longmapsto \omega \triangleright \gamma = (\omega \otimes \gamma) \circ \Gamma^r.$$

Analogously, the left fundamental unitary W of \mathbb{G} induces on $B(L_2(\mathbb{G}))$ a co-associative co-multiplication

$$\Gamma^l: B(L_2(\mathbb{G})) \longrightarrow B(L_2(\mathbb{G})) \widehat{\otimes} B(L_2(\mathbb{G})), \quad x \longmapsto W^*(1 \otimes x)W$$

such that the restriction of Γ^l to $L_\infty(\mathbb{G})$ is also equal to the co-multiplication Γ on $L_\infty(\mathbb{G})$. The pre-adjoint of Γ^l defines on $T(L_2(\mathbb{G}))$ another associative completely contractive multiplication

$$\triangleleft: T(L_2(\mathbb{G})) \widehat{\otimes} T(L_2(\mathbb{G})) \longrightarrow T(L_2(\mathbb{G})), \quad \omega \otimes \gamma \longmapsto \omega \triangleleft \gamma = (\omega \otimes \gamma) \circ \Gamma^l.$$

It is known from [14, Lemma 5.2] that we have the completely isometric algebra isomorphisms

$$(T(L_2(\mathbb{G})), \triangleright) / L_\infty(\mathbb{G})_\perp \cong (L_1(\mathbb{G}), \star) \cong (T(L_2(\mathbb{G})), \triangleleft) / L_\infty(\mathbb{G})_\perp$$

induced by the canonical complete quotient map $T(L_2(\mathbb{G})) \rightarrow L_1(\mathbb{G}), \omega \mapsto \omega|_{L_\infty(\mathbb{G})}$. Therefore, we can regard $(T(L_2(\mathbb{G})), \triangleright)$ and $(T(L_2(\mathbb{G})), \triangleleft)$ as the right and left lifting convolution algebras of $L_1(G)$ via V and W , respectively. Note that $(T(L_2(\mathbb{G})), \triangleright)$ is left faithful and $(T(L_2(\mathbb{G})), \triangleleft)$ is right faithful, but they are not faithful if \mathbb{G} is nontrivial. This shows in particular that each of \triangleright and \triangleleft cannot be commutative and the two convolutions are distinct if \mathbb{G} is nontrivial. However, we always have

$$(3.1) \quad \langle T(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G})) \rangle = T(L_2(\mathbb{G})) = \langle T(L_2(\mathbb{G})) \triangleleft T(L_2(\mathbb{G})) \rangle.$$

It is also known from [14, Proposition 5.3] that

$$(3.2) \quad \langle B(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G})) \rangle = LUC(\mathbb{G}) \text{ and } \langle T(L_2(\mathbb{G})) \triangleleft B(L_2(\mathbb{G})) \rangle = RUC(\mathbb{G}).$$

In particular, if $x \in L_\infty(\mathbb{G})$ and $\omega \in T(L_2(\mathbb{G}))$ with $f = \omega|_{L_\infty(\mathbb{G})}$, then we have

$$(3.3) \quad x \triangleright \omega = x \triangleleft \omega = x \star f \quad \text{and} \quad \omega \triangleleft x = \omega \triangleright x = f \star x.$$

In this paper, we often simply use $T(L_2(\mathbb{G}))$ for the algebra $(T(L_2(\mathbb{G})), \triangleright)$, and often consider only results for the right convolution \triangleright . The corresponding results with \triangleright replaced by \triangleleft also hold.

Comparing with (3.2), it is natural to study the restriction of the convolution $T(L_2(\mathbb{G}))$ -module action on $B(L_2(\mathbb{G}))$ to $K(L_2(\mathbb{G}))$. First, since

$$(3.4) \quad \langle \omega \triangleright \gamma, x \rangle = \langle \omega, \gamma \triangleright x \rangle = \langle \gamma, x \triangleright \omega \rangle \quad (\omega, \gamma \in T(L_2(\mathbb{G})), x \in K(L_2(\mathbb{G}))),$$

we conclude that

$$(3.5) \quad K(L_2(\mathbb{G})) \text{ is a right } T(L_2(\mathbb{G}))\text{-submodule of } B(L_2(\mathbb{G})) \Leftrightarrow$$

\triangleright is w^* -continuous on the right.

In fact, it is seen from (3.4) that \triangleright is w^* -continuous on the right if $K(L_2(\mathbb{G}))$ is a right $T(L_2(\mathbb{G}))$ -submodule of $B(L_2(\mathbb{G}))$. Conversely, if there are $x \in K(L_2(\mathbb{G}))$ and $\omega \in T(L_2(\mathbb{G}))$ such that $x \triangleright \omega \notin K(L_2(\mathbb{G}))$, then there exists $m \in B(L_2(\mathbb{G}))^*$ such that $m|_{K(L_2(\mathbb{G}))} = 0$ and $m(x \triangleright \omega) \neq 0$. In this situation, choosing a net (γ_i) in $T(L_2(\mathbb{G}))$ such that $\gamma_i \rightarrow m$ in the w^* -topology of $B(L_2(\mathbb{G}))^*$, we obtain that $\gamma_i \rightarrow 0$ in the w^* -topology of $T(L_2(\mathbb{G}))$, but $\langle \omega \triangleright \gamma_i, x \rangle = \langle \gamma_i, x \triangleright \omega \rangle \rightarrow \langle m, x \triangleright \omega \rangle \neq 0$. Therefore, \triangleright is not w^* -continuous on the right. Similarly, we can obtain

$$(3.6) \quad K(L_2(\mathbb{G})) \text{ is a left } T(L_2(\mathbb{G}))\text{-submodule of } B(L_2(\mathbb{G})) \Leftrightarrow$$

\triangleright is w^* -continuous on the left.

We consider now the restriction of the canonical *right* $T(L_2(\mathbb{G}))$ -module action on $B(L_2(\mathbb{G}))$ to $K(L_2(\mathbb{G}))$. It turns out that the reduced quantum group C^* -algebra $C_0(\mathbb{G})$ can be fully recovered from this restriction.

Theorem 3.1 *Let \mathbb{G} be a locally compact quantum group. Then we have*

$$(3.7) \quad \langle K(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G})) \rangle = C_0(\mathbb{G}).$$

Proof The inclusion “ \subseteq ” has been shown and used in [14, Section 7]. For completeness, we include here the full proof of (3.7), which is also needed in the proof of Proposition 3.3. For vectors ξ', ξ, η', η in $L_2(\mathbb{G})$, let $x_{\xi', \eta'} \in K(L_2(\mathbb{G}))$ be the rank one operator defined by $x_{\xi', \eta'}(\zeta) = \langle \zeta | \eta' \rangle \xi'$ ($\zeta \in L_2(\mathbb{G})$), and let $\omega_{\xi, \eta} \in T(L_2(\mathbb{G}))$ be the functional given by $\omega_{\xi, \eta}(x) = \langle x\xi | \eta \rangle$ ($x \in B(L_2(\mathbb{G}))$). Then we have

$$\begin{aligned} x_{\xi', \eta'} \triangleright \omega_{\xi, \eta} &= (\omega_{\xi, \eta} \otimes \iota)(V(x_{\xi', \eta'} \otimes 1)V^*) = ((\omega_{\xi', \eta'} \otimes \iota)V)((\omega_{\xi, \eta} \otimes \iota)V^*) \\ &= ((\omega_{\xi', \eta'} \otimes \iota)V)((\omega_{\eta', \xi} \otimes \iota)V)^*. \end{aligned}$$

Since $C_0(\mathbb{G}) = \overline{\text{span}}^{\|\cdot\|} \{(\omega \otimes \iota)(V) : \omega \in T(L_2(\mathbb{G}))\}$ and $C_0(\mathbb{G}) = C_0(\mathbb{G})C_0(\mathbb{G})$, it follows from the above that $\langle K(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G})) \rangle = C_0(\mathbb{G})$. ■

To study the canonical left $T(L_2(\mathbb{G}))$ -module action on $B(L_2(\mathbb{G}))$, we need to recall from [1–3] some definitions and properties of multiplicative unitaries and locally compact quantum groups. For a multiplicative unitary \mathcal{V} on $L_2(\mathbb{G}) \otimes L_2(\mathbb{G})$, let

$$(3.8) \quad \begin{aligned} C(\mathcal{V}) &= \{(\iota \otimes \omega)(\Sigma\mathcal{V}) : \omega \in T(L_2(\mathbb{G}))\}, \\ C'(\mathcal{V}) &= \{(\omega \otimes \iota)(\Sigma\mathcal{V}) : \omega \in T(L_2(\mathbb{G}))\}. \end{aligned}$$

Here, Σ is the flip map $(\xi, \eta) \mapsto (\eta, \xi)$ on $L_2(\mathbb{G}) \otimes L_2(\mathbb{G})$. Then \mathcal{V} is called *regular* (respectively, *semi-regular*) if $K(L_2(\mathbb{G})) = \langle C(\mathcal{V}) \rangle$ (respectively, $K(L_2(\mathbb{G})) \subseteq \langle C(\mathcal{V}) \rangle$), and *bi-regular* (respectively, *bi-semi-regular*) if $K(L_2(\mathbb{G})) = \langle C(\mathcal{V}) \rangle = \langle C'(\mathcal{V}) \rangle$ (respectively, $K(L_2(\mathbb{G})) \subseteq \langle C(\mathcal{V}) \rangle \cap \langle C'(\mathcal{V}) \rangle$). It is known from [2, Proposition 3.2] that

$$\mathcal{V} \text{ is regular} \Leftrightarrow K(L_2(\mathbb{G}) \otimes L_2(\mathbb{G})) = \overline{\text{span}}^{\|\cdot\|} \{(a \otimes 1)\mathcal{V}(1 \otimes b) : a, b \in K(L_2(\mathbb{G}))\}.$$

The proof of [2, Proposition 3.2] also shows that we have the corresponding equivalences for bi-regularity, semi-regularity, and bi-semi-regularity.

Proposition 3.2 *Let \mathbb{G} be a locally compact quantum group, and let W and V be the left and right fundamental unitaries of \mathbb{G} , respectively. Then we have*

$$\begin{aligned} K(L_2(\mathbb{G})) = \langle C(V) \rangle &\Leftrightarrow K(L_2(\mathbb{G})) = \langle C'(V) \rangle \Leftrightarrow K(L_2(\mathbb{G})) = \langle C(W) \rangle \\ &\Leftrightarrow K(L_2(\mathbb{G})) = \langle C'(W) \rangle \end{aligned}$$

and

$$\begin{aligned} K(L_2(\mathbb{G})) \subseteq \langle C(V) \rangle &\Leftrightarrow K(L_2(\mathbb{G})) \subseteq \langle C'(V) \rangle \Leftrightarrow K(L_2(\mathbb{G})) \subseteq \langle C(W) \rangle \\ &\Leftrightarrow K(L_2(\mathbb{G})) \subseteq \langle C'(W) \rangle. \end{aligned}$$

Proof Let J and \widehat{J} be the modular conjugations of the left Haar weights on \mathbb{G} and $\widehat{\mathbb{G}}$, respectively. Then $U = \widehat{J}J$ is a unitary operator on $L_2(\mathbb{G})$ such that

$$V = \Sigma(1 \otimes U)W(1 \otimes U^*)\Sigma.$$

It is then easy to show from definition (3.8) that

$$(3.9) \quad \langle C(W) \rangle = U^* \langle C'(V) \rangle \quad \text{and} \quad \langle C(V) \rangle = \langle C'(W) \rangle U^*.$$

Let $\widetilde{R}: B(L_2(\mathbb{G})) \rightarrow B(L_2(\mathbb{G}))$ be the $*$ -anti-automorphism $x \mapsto \widehat{J}x^*\widehat{J}$. Then \widetilde{R} satisfies the generalized antipode relation

$$(3.10) \quad (\widetilde{R} \otimes \widetilde{R}) \circ \Gamma^l = \Sigma(\Gamma^r \circ \widetilde{R})\Sigma,$$

and $\widetilde{R}|_{L_\infty(\mathbb{G})}$ is the unitary antipode R of \mathbb{G} . It is clear that we have

$$(3.11) \quad \begin{aligned} \widetilde{R}(K(L_2(\mathbb{G}))) &= K(L_2(\mathbb{G})), & \widetilde{R}(C_0(\mathbb{G})) &= C_0(\mathbb{G}), & \text{and} \\ \widetilde{R}(C_0(\widehat{\mathbb{G}}')) &= C_0(\widehat{\mathbb{G}}'). \end{aligned}$$

For $\omega \in T(L_2(\mathbb{G}))$, since $V^* = \Sigma\widehat{J} \otimes \widehat{J}W(\widehat{J} \otimes \widehat{J})\Sigma$, we have

$$\begin{aligned} ((\iota \otimes \omega)(\Sigma V))^* &= (\iota \otimes \omega^*)(V^*\Sigma) = (\iota \otimes \omega^*)(\widehat{J} \otimes \widehat{J})\Sigma W(\widehat{J} \otimes \widehat{J}) \\ &= (\iota \otimes \omega^*)(\widetilde{R} \otimes \widetilde{R})(\Sigma W)^*, \end{aligned}$$

where $\omega^* \in T(L_2(\mathbb{G}))$ is given by $\langle \omega^*, x \rangle = \overline{\langle \omega, x^* \rangle}$. This shows that $(\iota \otimes \omega)(\Sigma V) = (\iota \otimes \omega)(\widetilde{R} \otimes \widetilde{R})(\Sigma W)$, or equivalently, $\widetilde{R}((\iota \otimes \omega)(\Sigma V)) = (\iota \otimes (\omega \circ \widetilde{R}))(\Sigma W)$. Similarly, we have $\widetilde{R}((\omega \otimes \iota)(\Sigma V)) = ((\omega \circ \widetilde{R}) \otimes \iota)(\Sigma W)$. Thus we obtain

$$(3.12) \quad \widetilde{R}(\langle C(V) \rangle) = \langle C(W) \rangle \quad \text{and} \quad \widetilde{R}(\langle C'(V) \rangle) = \langle C'(W) \rangle.$$

Therefore, the proposition follows immediately by combining (3.9), (3.11), and (3.12). ■

For convenience, a locally compact quantum group \mathbb{G} is said to be *regular* (respectively, *semi-regular*) if one of the equivalent equalities (respectively, inclusions) in Proposition 3.2 holds. Therefore, \mathbb{G} is regular (respectively, semi-regular) if and only if it is bi-regular (respectively, bi-semi-regular). Note that the left fundamental unitary \widehat{W} of $\widehat{\mathbb{G}}$ is $\Sigma W^*\Sigma$, and the right fundamental unitary \widehat{V}' of $\widehat{\mathbb{G}}'$ is $\Sigma V^*\Sigma$. So, we have

$$C(\widehat{W}) = C'(W)^* \quad \text{and} \quad C(\widehat{V}') = C'(V)^*.$$

Therefore, by Proposition 3.2, we also have that \mathbb{G} is regular if and only if $\widehat{\mathbb{G}}$ (respectively, $\widehat{\mathbb{G}}'$) is regular, and \mathbb{G} is semi-regular if and only if $\widehat{\mathbb{G}}$ (respectively, $\widehat{\mathbb{G}}'$) is semi-regular. All Kac algebras are regular and so are all compact quantum groups and discrete quantum groups. However, as shown in [3], there do exist locally compact quantum groups that are not even semi-regular.

It is known from [3, Corollary 2.7] that $\langle C(W) \rangle$ and $\langle C(V) \rangle$ are both C^* -subalgebras of $B(L_2(\mathbb{G}))$. In fact, it can be seen from the proof of [3, Proposition 2.6] that we have

$$(3.13) \quad U^* \langle C(V) \rangle U = \langle C_0(\mathbb{G})C_0(\widehat{\mathbb{G}}') \rangle.$$

We note that the spaces S and \widehat{S} in [3, Proposition 2.6] are the C^* -algebras $C_0(\mathbb{G})$ and $C_0(\widehat{\mathbb{G}}')$, respectively. Combining (3.11), (3.12), and (3.13), we obtain

$$(3.14) \quad U^* \langle C(W) \rangle U = \langle C_0(\mathbb{G})C_0(\widehat{\mathbb{G}}) \rangle,$$

noticing that $U^* = J\widehat{J}$ and $\widetilde{R}(U) = U$. We point out that the unitary operator U used in [3] is equal to $\widehat{J}\widetilde{J}$ rather than $\widehat{J}J$ as taken in this paper. However, it is known that $\widehat{J}\widetilde{J} = \nu^{\frac{i}{4}}\widetilde{J}\widehat{J}$ (or $U = \nu^{\frac{i}{4}}U^*$) for some positive number ν (cf. [16, (2.4)]). Therefore, (3.13) and (3.14) hold for both choices of U .

For the sake of simplicity, we let

$$(3.15) \quad \begin{aligned} K_{\triangleleft}(L_2(\mathbb{G})) &= \langle K(L_2(\mathbb{G})) \triangleleft T(L_2(\mathbb{G})) \rangle, \\ {}_{\triangleright}K(L_2(\mathbb{G})) &= \langle T(L_2(\mathbb{G})) \triangleright K(L_2(\mathbb{G})) \rangle. \end{aligned}$$

It is seen from (3.10) that

$$(3.16) \quad \begin{aligned} \widetilde{R}(x \triangleleft \omega) &= (\omega \circ \widetilde{R}) \triangleright \widetilde{R}(x), \\ \widetilde{R}(x \triangleright \omega) &= (\omega \circ \widetilde{R}) \triangleleft \widetilde{R}(x) \quad (x \in B(L_2(\mathbb{G})), \omega \in T(L_2(\mathbb{G}))). \end{aligned}$$

Therefore, we have

$$\widetilde{R}(K_{\triangleleft}(L_2(\mathbb{G}))) = {}_{\triangleright}K(L_2(\mathbb{G})) \quad \text{and} \quad \widetilde{R}({}_{\triangleright}K(L_2(\mathbb{G}))) = K_{\triangleleft}(L_2(\mathbb{G})).$$

Corresponding to (3.7), by combining (3.11) and (3.16) and applying \widetilde{R} to (3.7), we also have

$$(3.17) \quad \langle T(L_2(\mathbb{G})) \triangleleft K(L_2(\mathbb{G})) \rangle = C_0(\mathbb{G}).$$

In the proof of the second equality below, we use an argument contained in the proof of [3, Proposition 5.7]. For convenience, we give the details of the calculation in the following proof.

Proposition 3.3 *Let \mathbb{G} be a locally compact quantum group. Then we have*

$${}_{\triangleright}K(L_2(\mathbb{G})) = U^* \langle C(W) \rangle U \quad \text{and} \quad K_{\triangleleft}(L_2(\mathbb{G})) = U^* \langle C(V) \rangle U.$$

Therefore, ${}_{\triangleright}K(L_2(\mathbb{G})) = \langle C_0(\mathbb{G})C_0(\widehat{\mathbb{G}}) \rangle$ and $K_{\triangleleft}(L_2(\mathbb{G})) = \langle C_0(\mathbb{G})C_0(\widehat{\mathbb{G}}') \rangle$ are C^* -subalgebras of $B(L_2(\mathbb{G}))$.

Proof Clearly, the first equality follows from the second one, since $\widetilde{R}(K_{\triangleleft}(L_2(\mathbb{G}))) = \triangleright K(L_2(\mathbb{G}))$ and

$$\widetilde{R}(U^* \langle C(V) \rangle U) = U \widetilde{R}(\langle C(V) \rangle) U^* = U \langle C(W) \rangle U^* = U^* \langle C(W) \rangle U,$$

noticing that $\widetilde{R}(U) = U$, $\widetilde{R}(\langle C(V) \rangle) = \langle C(W) \rangle$, and $U = v^{\frac{i}{4}} U^*$.

To show the second equality, let $x \in K(L_2(\mathbb{G}))$ and $\omega \in T(L_2(\mathbb{G}))$. By the definition of the convolution \triangleleft , we have

$$x \triangleleft \omega = (\omega \otimes \iota) \Gamma^l(x) = (\omega \otimes \iota) W^*(1 \otimes x) W.$$

Since $W = \Sigma(U^* \otimes 1) V(U \otimes 1) \Sigma$, we obtain

$$\begin{aligned} x \triangleleft \omega &= (\omega \otimes \iota) \Sigma(U^* \otimes 1) V^*(U \otimes 1) \Sigma(1 \otimes x) \Sigma(U^* \otimes 1) V(U \otimes 1) \Sigma \\ &= (\omega \otimes \iota) \Sigma(U^* \otimes 1) V^* \Sigma(1 \otimes U)(1 \otimes x)(1 \otimes U^*) \Sigma V(U \otimes 1) \Sigma \\ &= (\iota \otimes \omega)(U^* \otimes 1) V^* \Sigma(1 \otimes y) \Sigma V(U \otimes 1) \\ &= U^* ((\iota \otimes \omega) V^* \Sigma(1 \otimes y) \Sigma V) U, \end{aligned}$$

where $y = UxU^* \in K(L_2(\mathbb{G}))$. For vectors ξ', ξ, η', η in $L_2(\mathbb{G})$, following the notation used in the proof of Theorem 3.1, we have

$$(\iota \otimes \omega_{\xi, \eta}) (V^* \Sigma(1 \otimes x_{\xi', \eta'}) \Sigma V) = ((\iota \otimes \omega_{\xi', \eta}) V^* \Sigma) ((\iota \otimes \omega_{\xi, \eta'}) \Sigma V).$$

Therefore, we have

$$K_{\triangleleft}(L_2(\mathbb{G})) = \langle K(L_2(\mathbb{G})) \triangleleft T(L_2(\mathbb{G})) \rangle = U^* \langle C(V)^* C(V) \rangle U = U^* \langle C(V) \rangle U.$$

The final assertion holds by (3.13), (3.14), and the fact that $B(L_2(\mathbb{G}))$ is the w^* -closed linear span of $L_\infty(\mathbb{G})L_\infty(\widehat{\mathbb{G}})$ (respectively, $L_\infty(\mathbb{G})L_\infty(\widehat{\mathbb{G}}')$). ■

Proposition 3.3 shows that the C^* -algebras $\triangleright K(L_2(\mathbb{G}))$ and $K_{\triangleleft}(L_2(\mathbb{G}))$ encode the following cross relations between the pairs $\{V, \triangleright\}$ and $\{W, \triangleleft\}$. The left module action of the right convolution \triangleright induced by the right fundamental unitary V is related to the space $C(W)$ defined by the left fundamental unitary W , and the right module action of the left convolution \triangleleft induced by the left fundamental unitary W is related to the space $C(V)$ defined by the right fundamental unitary V . We shall further link the C^* -algebras $\triangleright K(L_2(\mathbb{G}))$ and $K_{\triangleleft}(L_2(\mathbb{G}))$ to reduced crossed products.

Let G be a locally compact group. The Stone–von Neumann theorem says that

$$C_0(G) \rtimes_r G \cong K(L_2(G)).$$

More precisely, if $M: a \mapsto M(a)$ is the canonical representation of $C_0(G)$ on $L_2(G)$ and λ is the left regular representation of G on $L_2(G)$, then (M, λ) is a covariant representation of $(C_0(G), G, \tau)$ (that is, $M({}_s a) = \lambda(s^{-1})M(a)\lambda(s)$ for all $a \in C_0(G)$ and $s \in G$, where ${}_s a$ denotes the left translate of a by s), and the map

$$M \rtimes \lambda: C_0(G) \rtimes_r G \longrightarrow B(L_2(G))$$

determined by $f \in C_c(G, C_0(G)) \mapsto \int_G M(f(s))\lambda(s)ds$ is a faithful irreducible representation of $C_0(G) \rtimes_r G$ on $L_2(\mathbb{G})$ with range $K(L_2(G))$. It seems that the name “Stone–von Neumann theorem” can be traced back to the title of the important paper [20] of Mackey. The reader is referred to [23] for information on some history of the Stone–von Neumann theorem. We shall present quantum group versions of this theorem via the convolutions \triangleleft and \triangleright .

We recall from [27] that a *continuous left action* of a locally compact quantum group \mathbb{G} on a C^* -algebra B is a nondegenerate $*$ -homomorphism

$$\alpha: B \longrightarrow M(C_0(\mathbb{G}) \otimes B)$$

satisfying

$$(1 \otimes \alpha)\alpha = (\Gamma \otimes \iota)\alpha \quad \text{and} \quad \langle \alpha(B)(C_0(\mathbb{G}) \otimes 1) \rangle = C_0(\mathbb{G}) \otimes B.$$

In this case, $\langle \alpha(B)(C_0(\widehat{\mathbb{G}}) \otimes 1) \rangle$ is a C^* -subalgebra of $M(K(L_2(\mathbb{G})) \otimes B)$. The reduced crossed product $C_0(\widehat{\mathbb{G}}) \rtimes B$ is defined by

$$C_0(\widehat{\mathbb{G}}) \rtimes B = \langle \alpha(B)(C_0(\widehat{\mathbb{G}}) \otimes 1) \rangle.$$

In particular, taking $(B, \alpha) = (C_0(\mathbb{G}), \Gamma)$, we obtain

$$C_0(\widehat{\mathbb{G}}) \rtimes C_0(\mathbb{G}) \subseteq M(K(L_2(\mathbb{G})) \otimes C_0(\mathbb{G})).$$

Analogously, a *continuous right action* $\beta: B \rightarrow M(B \otimes C_0(\mathbb{G}))$ can be considered. The reduced crossed product $B \rtimes_r C_0(\widehat{\mathbb{G}}')$ is defined to be the C^* -subalgebra $\langle \beta(B)(1 \otimes C_0(\widehat{\mathbb{G}}')) \rangle$ of $M(B \otimes K(L_2(\mathbb{G})))$, and, in particular, we have

$$C_0(\mathbb{G}) \rtimes_r C_0(\widehat{\mathbb{G}}') \subseteq M(C_0(\mathbb{G}) \otimes K(L_2(\mathbb{G}))).$$

The C^* -algebra identification $C_0(\mathbb{G}) \rtimes_r C_0(\widehat{\mathbb{G}}') \cong \langle C(V) \rangle$ proved in [3, Proposition 2.6] can be viewed as a quantum group version of the classical Stone–von Neumann theorem, noticing that $\langle C(V) \rangle = K(L_2(G))$ if \mathbb{G} is the commutative quantum group $L_\infty(G)$ over a locally compact group G . As shown below, in terms of the convolutions \triangleright and \triangleleft , a pair of Stone-von Neumann type theorems over general locally compact quantum groups \mathbb{G} can be obtained with the C^* -algebra $K(L_2(\mathbb{G}))$ present explicitly.

Theorem 3.4 *Let \mathbb{G} be a locally compact quantum group. Then we have*

$$(3.18) \quad C_0(\widehat{\mathbb{G}}) \rtimes C_0(\mathbb{G}) \cong \triangleright K(L_2(\mathbb{G})) \quad \text{and} \quad C_0(\mathbb{G}) \rtimes_r C_0(\widehat{\mathbb{G}}') \cong K_\triangleleft(L_2(\mathbb{G})).$$

Proof To show the first isomorphism, let $a \in C_0(\mathbb{G})$ and $\widehat{a} \in C_0(\widehat{\mathbb{G}})$. Since $V \in L_\infty(\widehat{\mathbb{G}}') \otimes L_\infty(\mathbb{G})$, we have

$$\Gamma(a)(\widehat{a} \otimes 1) = V(a \otimes 1)V^*(\widehat{a} \otimes 1) = V(a \otimes 1)(\widehat{a} \otimes 1)V^* = V(a\widehat{a} \otimes 1)V^*.$$

By Proposition 3.3 and the definition of the reduced crossed product $C_0(\widehat{\mathbb{G}}) \rtimes_r C_0(\mathbb{G})$, we obtain

$$C_0(\widehat{\mathbb{G}}) \rtimes_r C_0(\mathbb{G}) \cong \langle C_0(\mathbb{G})C_0(\widehat{\mathbb{G}}) \rangle = {}_{\triangleright}K(L_2(\mathbb{G})).$$

The second isomorphism follows from Proposition 3.3 and the identification $\langle C(V) \rangle \cong C_0(\mathbb{G}) \rtimes_r C_0(\widehat{\mathbb{G}}')$ (cf. [3, Proposition 2.6]). ■

Remark 3.5 Clearly, we have $C_r \rtimes C_0(\mathbb{G}) \cong C_0(\mathbb{G}) \cong C_0(\mathbb{G}) \rtimes_r \mathbb{C}$. By (3.7) and (3.17), it is interesting to see that when we switch the order of $T(L_2(\mathbb{G}))$ and $K(L_2(\mathbb{G}))$ appearing in ${}_{\triangleright}K(L_2(\mathbb{G}))$ and $K_{\triangleleft}(L_2(\mathbb{G}))$ in (3.18), we should replace $C_0(\widehat{\mathbb{G}})$ and $C_0(\widehat{\mathbb{G}}')$ there by the trivial quantum group \mathbb{C} . That is, we have

$$\begin{aligned} C_r \rtimes C_0(\mathbb{G}) &\cong \langle K(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G})) \rangle, \\ C_0(\mathbb{G}) \rtimes_r \mathbb{C} &\cong \langle T(L_2(\mathbb{G})) \triangleleft K(L_2(\mathbb{G})) \rangle. \end{aligned}$$

The following corollary is an immediate consequence of Proposition 3.3 and Theorem 3.4.

Corollary 3.6 *Let \mathbb{G} be a locally compact quantum group. Then*

- (i) $K(L_2(\mathbb{G})) = {}_{\triangleright}K(L_2(\mathbb{G})) \Leftrightarrow \mathbb{G}$ is regular $\Leftrightarrow K(L_2(\mathbb{G})) = K_{\triangleleft}(L_2(\mathbb{G}))$;
- (ii) $K(L_2(\mathbb{G})) \subseteq {}_{\triangleright}K(L_2(\mathbb{G})) \Leftrightarrow \mathbb{G}$ is semi-regular $\Leftrightarrow K(L_2(\mathbb{G})) \subseteq K_{\triangleleft}(L_2(\mathbb{G}))$.

Therefore, $C_0(\widehat{\mathbb{G}}) \rtimes_r C_0(\mathbb{G}) \cong K(L_2(\mathbb{G})) \cong C_0(\mathbb{G}) \rtimes_r C_0(\widehat{\mathbb{G}}')$ if \mathbb{G} is regular.

Since $T(L_2(\mathbb{G})) = K(L_2(\mathbb{G}))^*$, it is natural to consider when the convolution \triangleright on $T(L_2(\mathbb{G}))$ is separately w^* -continuous. We shall characterize this property in the following theorem. To state the theorem, we need also consider the restriction map

$$\chi: T(L_2(\mathbb{G})) \longrightarrow M(\mathbb{G}), \omega \longmapsto \omega|_{C_0(\mathbb{G})}.$$

By definition, we have $\chi = i \circ \pi_{\triangleright}$, where $\pi_{\triangleright}: (T(L_2(\mathbb{G})), \triangleright) \rightarrow L_1(\mathbb{G})$ is the canonical quotient map and $i: L_1(\mathbb{G}) \rightarrow M(\mathbb{G})$ is the inclusion map. Therefore, we have

$$(3.19) \quad \chi: (T(L_2(\mathbb{G})), \triangleright) \longrightarrow M(\mathbb{G}), \omega \mapsto \omega|_{C_0(\mathbb{G})} \text{ is an algebra homomorphism}$$

and

$$(3.20) \quad \chi: T(L_2(\mathbb{G})) \longrightarrow M(\mathbb{G}) \text{ is surjective} \Leftrightarrow L_1(\mathbb{G}) = M(\mathbb{G}) \text{ canonically.}$$

Clearly, $\chi^*|_{C_0(\mathbb{G})}: C_0(\mathbb{G}) \rightarrow B(L_2(\mathbb{G}))$ is just the canonical inclusion map.

Theorem 3.7 *Let \mathbb{G} be a locally compact quantum group. Then the following statements are equivalent:*

- (i) the convolution \triangleright on $T(L_2(\mathbb{G}))$ is separately w^* -continuous;
- (ii) the convolution \triangleright on $T(L_2(\mathbb{G}))$ is w^* -continuous on the right;
- (iii) the map $\chi: T(L_2(\mathbb{G})) \rightarrow M(\mathbb{G}), \omega \mapsto \omega|_{C_0(\mathbb{G})}$ is w^* - w^* continuous;
- (iv) $C_0(\mathbb{G}) \subseteq K(L_2(\mathbb{G}))$ canonically;
- (v) \mathbb{G} is discrete.

In this case, $\chi: (T(L_2(\mathbb{G})), \triangleright) \rightarrow M(\mathbb{G})$ is a surjective and w^* - w^* continuous algebra homomorphism.

Proof (i) \Rightarrow (ii). This is trivial.

(ii) \Leftrightarrow (iv). This is immediate by (3.5) and Theorem 3.1.

(iii) \Leftrightarrow (iv). This is true, since $\chi^*|_{C_0(\mathbb{G})}: C_0(\mathbb{G}) \rightarrow B(L_2(\mathbb{G}))$ is the canonical inclusion map, and hence $\chi: T(L_2(\mathbb{G})) \rightarrow M(\mathbb{G})$ is w^* - w^* continuous if and only if $C_0(\mathbb{G}) = \chi^*(C_0(\mathbb{G})) \subseteq K(L_2(\mathbb{G}))$.

(iv) \Rightarrow (i). Suppose that $C_0(\mathbb{G}) \subseteq K(L_2(\mathbb{G}))$ canonically. Since (iv) \Leftrightarrow (ii) as shown above, we only have to prove that the convolution \triangleright is w^* -continuous on the left. Since $K(L_2(\mathbb{G}))$ is an ideal in $B(L_2(\mathbb{G}))$, by Proposition 3.3, we obtain

$$\triangleright K(L_2(\mathbb{G})) = \langle C_0(\mathbb{G})C_0(\widehat{\mathbb{G}}) \rangle \subseteq K(L_2(\mathbb{G})).$$

Therefore, \triangleright is w^* -continuous on the left by (3.6).

(v) \Rightarrow (iv). This is clear.

(iv) \Rightarrow (v). Suppose that $C_0(\mathbb{G}) \subseteq K(L_2(\mathbb{G}))$ canonically. Then the double adjoint of this inclusion map defines an injective and w^* - w^* continuous algebra homomorphism $\theta: C_0(\mathbb{G})^{**} \rightarrow B(L_2(\mathbb{G}))$.

Assume that $a \in C_0(\mathbb{G})$ and $m \in C_0(\mathbb{G})^{**}$ such that $a \cdot m \notin C_0(\mathbb{G})$, or equivalently, $a \cdot \theta(m) \notin C_0(\mathbb{G})$. Then $a \cdot \theta(m) \in K(L_2(\mathbb{G})) \setminus C_0(\mathbb{G})$, since $C_0(\mathbb{G}) \subseteq K(L_2(\mathbb{G}))$. Thus there exists $\omega \in T(L_2(\mathbb{G}))$ such that $\omega|_{C_0(\mathbb{G})} = 0$, but $\langle \omega, a \cdot \theta(m) \rangle \neq 0$. Choosing a net (b_i) in $C_0(\mathbb{G})$ such that $b_i \rightarrow m$ in the w^* -topology of $C_0(\mathbb{G})^{**}$, we have that $b_i = \theta(b_i) \rightarrow \theta(m)$ and hence $ab_i \rightarrow a \cdot \theta(m)$ in the w^* -topology on $B(L_2(\mathbb{G}))$. Then we have $\langle \omega, a \cdot \theta(m) \rangle = \lim_i \langle \omega, ab_i \rangle = 0$, a contradiction. Therefore, $a \cdot m \in C_0(\mathbb{G})$.

Similarly, we can show that $m \cdot a \in C_0(\mathbb{G})$ for all $a \in C_0(\mathbb{G})$ and $m \in C_0(\mathbb{G})^{**}$. It follows that $C_0(\mathbb{G})$ is an ideal in $C_0(\mathbb{G})^{**}$, and hence \mathbb{G} is discrete (cf. [25, Theorem 4.4]).

The final assertion follows from (3.19) and (3.20). ■

4 Completely Bounded Right Multipliers and Invariance Problems

Let \mathbb{G} be a locally compact quantum group. As in Section 3, we often simply use $T(L_2(\mathbb{G}))$ for the algebra $(T(L_2(\mathbb{G})), \triangleright)$ and $B_{T(L_2(\mathbb{G}))}(B(L_2(\mathbb{G})))$ for the space of bounded right $(T(L_2(\mathbb{G})), \triangleright)$ -module maps on $B(L_2(\mathbb{G}))$. In this section, we study the completely bounded right multiplier algebra $RM_{cb}(T(L_2(\mathbb{G})))$ of $T(L_2(\mathbb{G}))$, and relate it to the representation theorem (1.1) of Junge–Neufang–Ruan.

Let $\mu \in RM(T(L_2(\mathbb{G})))$. Then $\mu^* \in B_{T(L_2(\mathbb{G}))}^\sigma(B(L_2(\mathbb{G})))$, and we have

$$\mu^*(\langle B(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G})) \rangle) \subseteq \langle B(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G})) \rangle;$$

that is, $\mu^*(LUC(\mathbb{G})) \subseteq LUC(\mathbb{G})$ (cf. (3.2)). This shows that

$$\mu^*(L_\infty(\mathbb{G})) \subseteq L_\infty(\mathbb{G}),$$

since $LUC(\mathbb{G})$ is w^* -dense in $L_\infty(\mathbb{G})$. Furthermore, it follows from (3.3) that $\mu^*|_{L_\infty(\mathbb{G})} \in B_{L_1(\mathbb{G})}^\sigma(L_\infty(\mathbb{G}))$. Let $\tilde{\mu} = (\mu^*|_{L_\infty(\mathbb{G})})_*$. Then we have that $\tilde{\mu} \in RM(L_1(\mathbb{G}))$

and $\|\tilde{\mu}\| \leq \|\mu\|$. Due to (3.1) and (3.2), we derive that the map

$$(4.1) \quad \Pi: RM(T(L_2(\mathbb{G}))) \longrightarrow RM(L_1(\mathbb{G})), \quad \mu \longmapsto \tilde{\mu}$$

is an injective and contractive algebra homomorphism, mapping $RM_{cb}(T(L_2(\mathbb{G})))$ into $RM_{cb}(L_1(\mathbb{G}))$. Equivalently, the algebra homomorphism

$$B_{T(L_2(\mathbb{G}))}^\sigma(B(L_2(\mathbb{G}))) \longrightarrow B_{L_1(\mathbb{G})}^\sigma(L_\infty(\mathbb{G})), \quad \Psi \longmapsto \Psi|_{L_\infty(\mathbb{G})}$$

is injective and contractive, mapping $CB_{T(L_2(\mathbb{G}))}^\sigma(B(L_2(\mathbb{G})))$ into $CB_{L_1(\mathbb{G})}^\sigma(L_\infty(\mathbb{G}))$. When linking to the commutative diagram (2.2), we also have $\tilde{\mu} = (\mu^*|_{LUC(\mathbb{G})})^*|_{L_1(\mathbb{G})}$.

Using the arguments of [16] for the establishment of the representation Θ^r in (1.1), we will see that the map Π in fact maps $RM_{cb}(T(L_2(\mathbb{G})))$ onto $RM_{cb}(L_1(\mathbb{G}))$. We shall still use Π to denote the restriction of the map in (4.1) to $RM_{cb}(T(L_2(\mathbb{G})))$. In the case where \mathbb{G} is the commutative quantum group $L_\infty(G)$ over a locally compact group G , the equality in (4.2) below can also be obtained from [21, Proposition 2.3.1(i), Satz 5.1.2, and Satz 5.2.3]. It is an open question whether we have $\Pi(RM(T(L_2(\mathbb{G})))) = RM(L_1(\mathbb{G}))$ in general.

Theorem 4.1 *Let \mathbb{G} be a locally compact quantum group. Then we have the commutative diagram*

$$(4.2) \quad \begin{array}{ccc} RM_{cb}(T(L_2(\mathbb{G}))) & \xrightarrow{\Pi} & RM_{cb}(L_1(\mathbb{G})) \\ \mu \longmapsto \mu^* \downarrow & & \downarrow \Theta^r \\ CB_{T(L_2(\mathbb{G}))}^\sigma(B(L_2(\mathbb{G}))) & \xlongequal{\quad} & CB_{L_\infty(\mathbb{G})}^{\sigma, L_\infty(\mathbb{G})}(B(L_2(\mathbb{G}))) \end{array}$$

of completely isometric algebra isomorphisms.

Therefore, the bicommutant relation $RM_{cb}(T(L_2(\mathbb{G})))^{cc} = RM_{cb}(T(L_2(\mathbb{G})))$ holds in $CB(B(L_2(\mathbb{G})))$.

Proof For a normal completely bounded operator Φ on $B(L_2(\mathbb{G}))$, by [16, Proposition 4.3] and its proof, $\Phi \in CB_{L_\infty(\mathbb{G})}^{\sigma, L_\infty(\mathbb{G})}(B(L_2(\mathbb{G})))$ if and only if we have

$$(4.3) \quad (\iota \otimes \Phi)(V(x \otimes 1)V^*) = V(\Phi(x) \otimes 1)V^* \quad (x \in B(L_2(\mathbb{G}))).$$

Note that (4.3) is equivalent to $\Phi \in CB_{T(L_2(\mathbb{G}))}^\sigma(B(L_2(\mathbb{G})))$. So the equality in (4.2) holds. On the other hand, let

$$\mu \in RM_{cb}(T(L_2(\mathbb{G}))) \quad \text{and} \quad \nu = \Pi(\mu) \in RM_{cb}(L_1(\mathbb{G})).$$

By the proof of [16, Proposition 4.3] once again, $\Theta^r(\nu)$ is the unique element of $CB_{L_\infty(\mathbb{G})}^{\sigma, L_\infty(\mathbb{G})}(B(L_2(\mathbb{G})))$ whose restriction to $L_\infty(\mathbb{G})$ is ν^* . It follows that we have $\Theta^r(\nu) = \mu^*$, since $\mu^*|_{L_\infty(\mathbb{G})} = \nu^*$. Therefore, the diagram (4.2) commutes; in particular, we obtain that the map $\Pi: RM_{cb}(T(L_2(\mathbb{G}))) \rightarrow RM_{cb}(L_1(\mathbb{G}))$ is a completely isometric algebra isomorphism, since the two column maps are completely isometric algebra isomorphisms.

The final assertion follows from (4.2) and [16, Corollary 5.3]. ■

For the completely isometric algebra isomorphism

$$\Theta^r : RM_{cb}(L_1(\mathbb{G})) \longrightarrow CB_{L_\infty(\widehat{\mathbb{G}})}^{\sigma, L_\infty(\mathbb{G})}(B(L_2(\mathbb{G}))) = CB_{T(L_2(\mathbb{G}))}^\sigma(B(L_2(\mathbb{G}))),$$

it is natural to ask the following two questions, which have motivated our study in Section 3.

- (i) When is $K(L_2(\mathbb{G}))$ invariant under $\Theta^r(f)$ for each $f \in L_1(\mathbb{G})$?
- (ii) When is $K(L_2(\mathbb{G}))$ invariant under $\Theta^r(\mu)$ for each $\mu \in RM_{cb}(L_1(\mathbb{G}))$?

If $L_\infty(\mathbb{G}) = L_\infty(G)$, then $RM_{cb}(L_1(\mathbb{G})) \cong M(\mathbb{G})$, and hence $K(L_2(\mathbb{G}))$ is invariant under $\Theta^r(M(\mathbb{G}))$ (cf. [5, p. 397]). For $L_\infty(\mathbb{G}) = VN(G)$, we also have that $K(L_2(\mathbb{G}))$ is invariant under $\Theta^r(RM_{cb}(L_1(\mathbb{G})))$: in this case, as shown in [6, Satz 2.1 and Folgerung 2.2], we even have that each T in $CB_{L_\infty(\widehat{\mathbb{G}})}^\sigma(B(L_2(\mathbb{G})))$ leaves $K(L_2(\mathbb{G}))$ invariant.

In general, for a subspace X of $B(L_2(\mathbb{G}))$, we use $CB_{T(L_2(\mathbb{G}))}^{\sigma, X}(B(L_2(\mathbb{G})))$ to denote the algebra of operators in $CB_{T(L_2(\mathbb{G}))}^\sigma(B(L_2(\mathbb{G})))$ that map X into X . Then we have

$$CB_{T(L_2(\mathbb{G}))}^{\sigma, K(L_2(\mathbb{G}))}(B(L_2(\mathbb{G}))) = \{S \in CB_{T(L_2(\mathbb{G}))}^\sigma(B(L_2(\mathbb{G}))) : S = F^{**} \text{ for some } F \in CB(K(L_2(\mathbb{G})))\}.$$

It can be seen from [16, Proposition 4.3] (the uniqueness part) that

$$(4.4) \quad \Theta^r(f)(x) = \omega \triangleright x \quad \text{for all } f \in L_1(\mathbb{G}) \text{ and } x \in B(L_2(\mathbb{G})),$$

where $\omega \in T(L_2(\mathbb{G}))$ with $\omega|_{L_\infty(\mathbb{G})} = f$. Thus each $\Theta^r(f)$ maps $K(L_2(\mathbb{G}))$ into ${}_\triangleright K(L_2(\mathbb{G}))$, and we have

$$(4.5) \quad \Theta^r(L_1(\mathbb{G})) \subseteq CB_{T(L_2(\mathbb{G}))}^{\sigma, K(L_2(\mathbb{G}))}(B(L_2(\mathbb{G}))) \Leftrightarrow {}_\triangleright K(L_2(\mathbb{G})) \subseteq K(L_2(\mathbb{G})).$$

Since ${}_\triangleright K(L_2(\mathbb{G})) = \langle C_0(\mathbb{G})C_0(\widehat{\mathbb{G}}) \rangle$ (Proposition 3.3) and $\Theta^r(RM_{cb}(L_1(\mathbb{G}))) = CB_{L_\infty(\widehat{\mathbb{G}})}^{\sigma, L_\infty(\mathbb{G})}(B(L_2(\mathbb{G})))$ satisfying $\Theta^r(RM_{cb}(L_1(\mathbb{G})))C_0(\mathbb{G}) \subseteq C_0(\mathbb{G})$ (cf. [16, Proposition 4.1]), we obtain that

$$(4.6) \quad {}_\triangleright K(L_2(\mathbb{G})) \text{ is always invariant under } \Theta^r(RM_{cb}(L_1(\mathbb{G}))).$$

Also, note that the C^* -algebra ${}_\triangleright K(L_2(\mathbb{G})) = \langle C_0(\mathbb{G})C_0(\widehat{\mathbb{G}}) \rangle$ acts irreducibly on $L_2(\mathbb{G})$. Thus we have

$${}_\triangleright K(L_2(\mathbb{G})) \cap K(L_2(\mathbb{G})) \neq \{0\} \Leftrightarrow K(L_2(\mathbb{G})) \subseteq {}_\triangleright K(L_2(\mathbb{G}))$$

(i.e., \mathbb{G} is semi-regular by Corollary 3.6). In particular, we obtain from (4.4) that

$$\Theta^r(L_1(\mathbb{G})) \cap CB_{T(L_2(\mathbb{G}))}^{\sigma, K(L_2(\mathbb{G}))}(B(L_2(\mathbb{G}))) \neq \{0\} \implies \mathbb{G} \text{ is semi-regular.}$$

We conclude that ${}_\triangleright K(L_2(\mathbb{G})) = K(L_2(\mathbb{G}))$ whenever ${}_\triangleright K(L_2(\mathbb{G})) \subseteq K(L_2(\mathbb{G}))$. Equivalently, by Corollary 3.6, we have

$$(4.7) \quad {}_\triangleright K(L_2(\mathbb{G})) \subseteq K(L_2(\mathbb{G})) \Leftrightarrow \mathbb{G} \text{ is regular.}$$

With all of these preparations, we are now ready to answer the above questions (i) and (ii).

Theorem 4.2 *Let \mathbb{G} be a locally compact quantum group. Then the following statements are equivalent:*

- (i) *each map in $\Theta^r(L_1(\mathbb{G}))$ leaves $K(L_2(\mathbb{G}))$ invariant, that is, we have*

$$\Theta^r(L_1(\mathbb{G})) \subseteq CB_{T(L_2(\mathbb{G}))}^{\sigma, K(L_2(\mathbb{G}))}(B(L_2(\mathbb{G})));$$

- (ii) *each map in $\Theta^r(RM_{cb}(L_1(\mathbb{G})))$ leaves $K(L_2(\mathbb{G}))$ invariant, that is, we have*

$$\Theta^r(RM_{cb}(L_1(\mathbb{G}))) = CB_{T(L_2(\mathbb{G}))}^{\sigma, K(L_2(\mathbb{G}))}(B(L_2(\mathbb{G})));$$

- (iii) \mathbb{G} *is regular.*

Proof (i) \Leftrightarrow (iii). This follows from (4.5) and (4.7).

(iii) \Rightarrow (ii). This holds by (4.6) and Corollary 3.6.

(ii) \Rightarrow (i). This is trivial. ■

As mentioned before, [16, Proposition 4.1] shows that

$$\Theta^r(RM_{cb}(L_1(\mathbb{G})))C_0(\mathbb{G}) \subseteq C_0(\mathbb{G}).$$

Therefore, combining (3.7), (4.2), (4.6), and Proposition 3.3, we have the following interesting formulation of the representation theorem (1.1) of Junge–Neufang–Ruan, in which the right convolution $T(L_2(\mathbb{G}))$ -module action and the duality between $C_0(\mathbb{G})$ and $C_0(\widehat{\mathbb{G}})$ are both encoded and reflected each other.

Proposition 4.3 *Let \mathbb{G} be a locally compact quantum group. Then we have*

$$\Theta^r(RM_{cb}(L_1(\mathbb{G}))) = CB_{T(L_2(\mathbb{G}))}^{\sigma, \triangleright K(L_2(\mathbb{G}))}(B(L_2(\mathbb{G}))) = CB_{C_0(\widehat{\mathbb{G}})}^{\sigma, C_0(\mathbb{G})}(B(L_2(\mathbb{G}))),$$

where the pairs $\{T(L_2(\mathbb{G})), \triangleright K(L_2(\mathbb{G}))\}$ and $\{C_0(\widehat{\mathbb{G}}), C_0(\mathbb{G})\}$ are related by

$$\triangleright K(L_2(\mathbb{G})) = \langle C_0(\widehat{\mathbb{G}})C_0(\mathbb{G}) \rangle \quad \text{and} \quad C_0(\mathbb{G}) = \langle T(L_2(\mathbb{G})) \triangleright \triangleright K(L_2(\mathbb{G})) \rangle.$$

Note that $\langle T(L_2(\mathbb{G})) \triangleright C_0(\mathbb{G}) \rangle = C_0(\mathbb{G})$, since $\langle T(L_2(\mathbb{G})) \triangleright C_0(\mathbb{G}) \rangle = \langle L_1(\mathbb{G}) \star C_0(\mathbb{G}) \rangle = C_0(\mathbb{G})$ (cf. [14, Proposition 2.2]). We also have $\langle T(L_2(\mathbb{G})) \triangleright C_0(\widehat{\mathbb{G}}) \rangle = C_0(\widehat{\mathbb{G}})$, since $\omega \triangleright \widehat{a} = \langle \omega, 1 \rangle \widehat{a}$ for all $\omega \in T(L_2(\mathbb{G}))$ and $\widehat{a} \in C_0(\widehat{\mathbb{G}})$. The corollary below shows that the “fixed point” relation $\langle T(L_2(\mathbb{G})) \triangleright Y \rangle = Y$ holds for $Y = C_0(\widehat{\mathbb{G}}')$ exactly when the right convolution \triangleright is w^* -continuous on the left, which is equivalent to \mathbb{G} being regular.

Corollary 4.4 *Let \mathbb{G} be a locally compact quantum group. Then the following statements are equivalent:*

- (i) *the convolution \triangleright on $T(L_2(\mathbb{G}))$ is w^* -continuous on the left;*
- (ii) $\langle T(L_2(\mathbb{G})) \triangleright C_0(\widehat{\mathbb{G}}') \rangle = C_0(\widehat{\mathbb{G}}')$;
- (iii) \mathbb{G} *is regular.*

Proof (i) \Leftrightarrow (iii). This follows from (3.6) and (4.7).

(ii) \Leftrightarrow (iii). Recall that the right fundamental unitary of $\widehat{\mathbb{G}}'$ is the operator $\widehat{V}' = \Sigma V^* \Sigma$. Then for $\omega \in T(L_2(\mathbb{G}))$ and $\widehat{a}' \in C_0(\widehat{\mathbb{G}}')$, we have

$$(4.8) \quad (\omega \otimes \iota)(\widehat{V}')^*(1 \otimes \widehat{a}')\widehat{V}' = (\iota \otimes \omega)V(\widehat{a}' \otimes 1)V^* = (\iota \otimes \omega)\Gamma^r(\widehat{a}') = \omega \triangleright \widehat{a}'.$$

According to [3, Proposition 5.6], we have

$$\overline{\text{span}}^{\|\cdot\|} \{(\omega \otimes \iota)(\widehat{V}')^*(1 \otimes C_0(\widehat{\mathbb{G}}'))\widehat{V}' : \omega \in T(L_2(\mathbb{G}))\} = C_0(\widehat{\mathbb{G}}') \Leftrightarrow \widehat{\mathbb{G}}' \text{ is regular.}$$

It follows from (4.8) that $\langle T(L_2(\mathbb{G})) \triangleright C_0(\widehat{\mathbb{G}}') \rangle = C_0(\widehat{\mathbb{G}}')$ if and only if $\widehat{\mathbb{G}}'$ is regular, which is true if and only if \mathbb{G} is regular (cf. Section 3). ■

5 \mathbb{G} -continuous Operators

For a locally compact quantum group \mathbb{G} , the two convolutions \triangleleft and \triangleright on $T(L_2(\mathbb{G}))$ induce the pairs $\{RUC(\mathbb{G}), LUC(\mathbb{G})\}$ and $\{K_{\triangleleft}(L_2(\mathbb{G})), \triangleright K(L_2(\mathbb{G}))\}$ of subspaces of $B(L_2(\mathbb{G}))$ (cf. (3.2) and (3.15)). These convolution $T(L_2(\mathbb{G}))$ -module actions on $B(L_2(\mathbb{G}))$ provide us with another natural pair of subspaces of $B(L_2(\mathbb{G}))$ defined by

$$X_{\triangleleft}(L_2(\mathbb{G})) = \langle B(L_2(\mathbb{G})) \triangleleft T(L_2(\mathbb{G})) \rangle \quad \text{and} \quad \triangleright X(L_2(\mathbb{G})) = \langle T(L_2(\mathbb{G})) \triangleright B(L_2(\mathbb{G})) \rangle.$$

By (3.3), we have

$$\begin{aligned} \langle L_{\infty}(\mathbb{G}) \triangleleft T(L_2(\mathbb{G})) \rangle &= \langle L_{\infty}(\mathbb{G}) \triangleright T(L_2(\mathbb{G})) \rangle = LUC(\mathbb{G}), \\ \langle T(L_2(\mathbb{G})) \triangleright L_{\infty}(\mathbb{G}) \rangle &= \langle T(L_2(\mathbb{G})) \triangleleft L_{\infty}(\mathbb{G}) \rangle = RUC(\mathbb{G}). \end{aligned}$$

Also, since $V \in L_{\infty}(\widehat{\mathbb{G}}') \bar{\otimes} L_{\infty}(\mathbb{G})$ and $W \in L_{\infty}(\mathbb{G}) \bar{\otimes} L_{\infty}(\widehat{\mathbb{G}}')$, it is easy to see that for $\widehat{x} \in L_{\infty}(\widehat{\mathbb{G}}')$ and $\widehat{x}' \in L_{\infty}(\widehat{\mathbb{G}}')$, we have

$$\omega \triangleright \widehat{x} = \langle \omega, 1 \rangle \widehat{x} \quad \text{and} \quad \widehat{x}' \triangleleft \omega = \langle \omega, 1 \rangle \widehat{x}' \quad (\omega \in T(L_2(\mathbb{G}))).$$

It follows that

$$\langle T(L_2(\mathbb{G})) \triangleright L_{\infty}(\widehat{\mathbb{G}}') \rangle = L_{\infty}(\widehat{\mathbb{G}}') \quad \text{and} \quad \langle L_{\infty}(\widehat{\mathbb{G}}') \triangleleft T(L_2(\mathbb{G})) \rangle = L_{\infty}(\widehat{\mathbb{G}}').$$

Therefore, we have

$$K_{\triangleleft}(L_2(\mathbb{G})) \cup LUC(\mathbb{G}) \cup L_{\infty}(\widehat{\mathbb{G}}') \subseteq X_{\triangleleft}(L_2(\mathbb{G}))$$

and

$$(5.1) \quad \triangleright K(L_2(\mathbb{G})) \cup RUC(\mathbb{G}) \cup L_{\infty}(\widehat{\mathbb{G}}') \subseteq \triangleright X(L_2(\mathbb{G})),$$

which indicate in particular that the spaces $X_{\triangleleft}(L_2(\mathbb{G}))$ and ${}_{\triangleright}X(L_2(\mathbb{G}))$ are quite large and complicated. It is clear that $X_{\triangleleft}(L_2(\mathbb{G}))$ and ${}_{\triangleright}X(L_2(\mathbb{G}))$ are both operator systems in $B(L_2(\mathbb{G}))$, and by (3.16), we have

$$(5.2) \quad \widetilde{R}(X_{\triangleleft}(L_2(\mathbb{G}))) = {}_{\triangleright}X(L_2(\mathbb{G})).$$

For any unitary representation π on a locally compact group G , Bekka defined the C^* -algebra $X(H_\pi)$ of G -continuous operators on $L_2(G)$ (cf. [5, Definition 3.1]). In [21], Neufang introduced a convolution $*$ on the space $\mathcal{N}(L_p(G))$ of nuclear operators on $L_p(G)$ ($1 < p < \infty$). It turns out that when $L_\infty(\mathbb{G}) = L_\infty(G)$, we have $(T(L_2(G)), \triangleleft) = (\mathcal{N}(L_2(G)), *)^{\text{op}}$. In this case, $X_{\triangleleft}(L_2(G)) = X(H_\lambda)$, where λ is the left regular representation of G (cf. [21, Satz 5.4.11]), and then $X_{\triangleleft}(L_2(G))$ and hence ${}_{\triangleright}X(L_2(G))$ (cf. (5.2)) are both C^* -subalgebras of $B(L_2(G))$.

For general locally compact quantum groups \mathbb{G} , we call $X_{\triangleleft}(L_2(\mathbb{G}))$ and ${}_{\triangleright}X(L_2(\mathbb{G}))$ the spaces of *left* and *right* \mathbb{G} -continuous operators on $L_2(\mathbb{G})$, respectively. We show below that ${}_{\triangleright}X(L_2(\mathbb{G}))$ and $X_{\triangleleft}(L_2(\mathbb{G}))$ can be obtained respectively from a left $L_1(\mathbb{G})$ -module structure and a right $L_1(\mathbb{G})$ -module structure on $B(L_2(\mathbb{G}))$. To see this, let $x \in B(L_2(\mathbb{G}))$, $f \in L_1(\mathbb{G})$, and $\omega \in T(L_2(\mathbb{G}))$ such that $\omega|_{L_\infty(\mathbb{G})} = f$. Then for all $\gamma \in T(L_2(\mathbb{G}))$, we have

$$\langle \gamma, \omega \triangleright x \rangle = \langle \omega, x \triangleright \gamma \rangle = \langle f, x \triangleright \gamma \rangle \quad \text{and} \quad \langle \gamma, x \triangleleft \omega \rangle = \langle \omega, \gamma \triangleleft x \rangle = \langle f, \gamma \triangleleft x \rangle,$$

since $x \triangleright \gamma$ and $\gamma \triangleleft x$ are in $L_\infty(\mathbb{G})$ (cf. (3.2)). Let

$$(5.3) \quad f \otimes x = \omega \triangleright x \quad \text{and} \quad x \otimes f = x \triangleleft \omega.$$

Then $f \otimes x$ and $x \otimes f$ are well defined elements of $B(L_2(\mathbb{G}))$ satisfying

$$\|f \otimes x\| \leq \|f\| \|x\| \quad \text{and} \quad \|x \otimes f\| \leq \|x\| \|f\|.$$

It is also easy to see that

$$\begin{aligned} f \otimes (g \otimes x) &= (f \star g) \otimes x, \\ (x \otimes f) \otimes g &= x \otimes (f \star g) \quad (f, g \in L_1(\mathbb{G}), x \in B(L_2(\mathbb{G}))). \end{aligned}$$

Therefore, $B(L_2(\mathbb{G}))$ is both a left Banach $L_1(\mathbb{G})$ -module and a right Banach $L_1(\mathbb{G})$ -module, which extend the canonical $L_1(\mathbb{G})$ -bimodule structure on $L_\infty(\mathbb{G})$ (cf. (3.3)). However, these two one-sided $L_1(\mathbb{G})$ -module actions on $B(L_2(\mathbb{G}))$ do not commute in general (that is, $B(L_2(\mathbb{G}))$ is not an $L_1(\mathbb{G})$ -bimodule), since we do not have that $(\omega \triangleright x) \triangleleft \gamma = \omega \triangleright (x \triangleleft \gamma)$ for all $\omega, \gamma \in T(L_2(\mathbb{G}))$ and $x \in B(L_2(\mathbb{G}))$.

Proposition 5.1 *Let \mathbb{G} be a locally compact quantum group. Then the following assertions hold:*

- (i) ${}_{\triangleright}X(L_2(\mathbb{G})) = \langle L_1(\mathbb{G}) \otimes B(L_2(\mathbb{G})) \rangle$ and $X_{\triangleleft}(L_2(\mathbb{G})) = \langle B(L_2(\mathbb{G})) \otimes L_1(\mathbb{G}) \rangle$.
- (ii) *If \mathbb{G} is co-amenable, then we have*
 - (a) ${}_{\triangleright}X(L_2(\mathbb{G})) = L_1(\mathbb{G}) \otimes B(L_2(\mathbb{G})) = T(L_2(\mathbb{G})) \triangleright B(L_2(\mathbb{G}))$;

- (b) $X_{\triangleleft}(L_2(\mathbb{G})) = B(L_2(\mathbb{G})) \otimes L_1(\mathbb{G}) = B(L_2(\mathbb{G})) \triangleleft T(L_2(\mathbb{G}));$
- (c) $L_{\infty}(\mathbb{G}) \cap \triangleright X(L_2(\mathbb{G})) = RUC(\mathbb{G})$ and $L_{\infty}(\mathbb{G}) \cap X_{\triangleleft}(L_2(\mathbb{G})) = LUC(\mathbb{G}).$

(iii) If \mathbb{G} is discrete, then $\triangleright X(L_2(\mathbb{G})) = B(L_2(\mathbb{G})) = X_{\triangleleft}(L_2(\mathbb{G})).$

Proof We consider only the space $\triangleright X(L_2(\mathbb{G}));$ the case of $X_{\triangleleft}(L_2(\mathbb{G}))$ can be shown similarly.

(i) By definition and (5.3), we have

$$(5.4) \quad L_1(\mathbb{G}) \otimes B(L_2(\mathbb{G})) = T(L_2(\mathbb{G})) \triangleright B(L_2(\mathbb{G})),$$

and hence $\triangleright X(L_2(\mathbb{G})) = \langle L_1(\mathbb{G}) \otimes B(L_2(\mathbb{G})) \rangle.$

(ii) Suppose that \mathbb{G} is co-amenable (that is, $L_1(\mathbb{G})$ has a BAI). Then by (i), (5.4), and the Cohen factorization theorem, we have $\triangleright X(L_2(\mathbb{G})) = L_1(\mathbb{G}) \otimes B(L_2(\mathbb{G})) = T(L_2(\mathbb{G})) \triangleright B(L_2(\mathbb{G})).$

We always have $RUC(\mathbb{G}) \subseteq L_{\infty}(\mathbb{G}) \cap \triangleright X(L_2(\mathbb{G})).$ Conversely, let $x \in L_{\infty}(\mathbb{G}) \cap \triangleright X(L_2(\mathbb{G})).$ By the preceding equality, we have $x = f \otimes y$ for some $f \in L_1(\mathbb{G})$ and $y \in B(L_2(\mathbb{G})).$ Let (f_{α}) be a bounded approximate identity of $L_1(\mathbb{G}).$ Then we obtain

$$x = f \otimes y = \lim_{\alpha} (f_{\alpha} \star f) \otimes y = \lim_{\alpha} f_{\alpha} \otimes (f \otimes y) = \lim_{\alpha} f_{\alpha} \otimes x = \lim_{\alpha} f_{\alpha} \star x \in RUC(\mathbb{G}).$$

Therefore, $L_{\infty}(\mathbb{G}) \cap \triangleright X(L_2(\mathbb{G})) = RUC(\mathbb{G}).$

(iii) Suppose that \mathbb{G} is discrete with f_0 the identity of $L_1(\mathbb{G}).$ Let $\omega_0 \in T(L_2(\mathbb{G}))$ be such that $\omega_0|_{L_{\infty}(\mathbb{G})} = f_0.$ Then ω_0 is a right identity of $(T(L_2(\mathbb{G})), \triangleright).$ In this situation, we have $\omega_0 \triangleright x = x$ for all $x \in B(L_2(\mathbb{G})).$ Therefore, $\triangleright X(L_2(\mathbb{G})) = B(L_2(\mathbb{G})).$ ■

We recall that the class of Banach algebras of *type (M)* was introduced in [12]. Roughly speaking, a Banach algebra A is of *type (M)* if an algebraic form of the Kakutani–Kodaira theorem on locally compact groups holds for A (see [12] for the precise definition). It is known from [12] that every $L_1(G)$ is in this class, and so is $A(G)$ if G is amenable. Also, any separable quantum group algebra $L_1(\mathbb{G})$ with \mathbb{G} co-amenable is of *type (M).* The reader is referred to [12] for more information on this class of Banach algebras. As shown below, the converse of Proposition 5.1(iii) holds if $L_1(\mathbb{G})$ is of *type (M).*

Corollary 5.2 *Let \mathbb{G} be a locally compact quantum group with $L_1(\mathbb{G})$ of type (M). Then the following statements are equivalent:*

- (i) $\triangleright X(L_2(\mathbb{G})) = B(L_2(\mathbb{G}));$
- (ii) $L_{\infty}(\mathbb{G}) \subseteq \triangleright X(L_2(\mathbb{G}));$
- (iii) \mathbb{G} is discrete.

Furthermore, we have

$$(5.5) \quad \Theta^r(L_1(\mathbb{G})) = CB_{T(L_2(\mathbb{G}))}^{\sigma, (B(L_2(\mathbb{G})), \triangleright X(L_2(\mathbb{G})))} (B(L_2(\mathbb{G}))),$$

the latter is the space of operators in $CB_{T(L_2(\mathbb{G}))}^{\sigma} (B(L_2(\mathbb{G})))$ that maps $B(L_2(\mathbb{G}))$ into $\triangleright X(L_2(\mathbb{G})).$

Proof It is clear that we have (iii) \Rightarrow (i) \Rightarrow (ii) (cf. Proposition 5.1(iii)). To show (ii) \Rightarrow (iii), suppose that $L_\infty(\mathbb{G}) \subseteq \triangleright X(L_2(\mathbb{G}))$. By Proposition 5.1(ii), we obtain that $L_\infty(\mathbb{G}) \subseteq RUC(\mathbb{G})$; that is, $L_\infty(\mathbb{G}) = RUC(\mathbb{G})$. Since $L_1(\mathbb{G})$ is of type (M) , by [11, Theorem 22], the quantum group \mathbb{G} is discrete.

To see (5.5), we first have $\Theta^r(L_1(\mathbb{G})) \subseteq CB_{T(L_2(\mathbb{G}))}^{\sigma, (B(L_2(\mathbb{G})), \triangleright X(L_2(\mathbb{G})))}(B(L_2(\mathbb{G})))$ by (4.4). Conversely, let $\mu \in RM_{cb}(L_1(\mathbb{G}))$ be such that $\Theta^r(\mu)(B(L_2(\mathbb{G}))) \subseteq \triangleright X(L_2(\mathbb{G}))$. Then by Proposition 5.1(ii), we obtain

$$\Theta^r(\mu)(L_\infty(\mathbb{G})) \subseteq L_\infty(\mathbb{G}) \cap \triangleright X(L_2(\mathbb{G})) = RUC(\mathbb{G}).$$

According to [12, Theorem 14], we have that $\mu \in L_1(\mathbb{G})$. Therefore, equality (5.5) holds. ■

Remark 5.3 (i) Note that the convolution algebra $(T(L_2(\mathbb{G})), \triangleright)$ does not have a left approximate identity unless \mathbb{G} is trivial. On the other hand, it was shown in [14, Proposition 5.4] that the quantum group \mathbb{G} is co-amenable if and only if $(T(L_2(\mathbb{G})), \triangleright)$ has a bounded *right* approximate identity; in this situation, we still have the left factorization $\triangleright X(L_2(\mathbb{G})) = T(L_2(\mathbb{G})) \triangleright B(L_2(\mathbb{G}))$ in Proposition 5.1(ii).

(ii) Recall that $\triangleright K(L_2(\mathbb{G})) = \langle C_0(\mathbb{G})C_0(\widehat{\mathbb{G}}) \rangle$ (cf. Proposition 3.3), and hence we have

$$\overline{\triangleright K(L_2(\mathbb{G}))}^{w*} = \overline{\langle L_\infty(\mathbb{G})L_\infty(\widehat{\mathbb{G}}) \rangle}^{w*} = B(L_2(\mathbb{G})).$$

However, Corollary 5.2 together with Corollary 3.6 shows that, even for the commutative compact quantum group $L_\infty(\mathbb{G}) = L_\infty(\mathbb{T})$, we do not have $\triangleright X(L_2(\mathbb{G})) = \langle L_\infty(\mathbb{G})L_\infty(\widehat{\mathbb{G}}) \rangle$ nor $\triangleright X(L_2(\mathbb{G})) = M(\triangleright K(L_2(\mathbb{G})))$.

Since $LUC(\mathbb{G}) = \langle B(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G})) \rangle$ is left introverted in $(T(L_2(\mathbb{G})), \triangleright)^*$, by the definition of Arens products, the adjoint of the identity map $\langle B(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G})) \rangle \rightarrow LUC(\mathbb{G})$ in (3.2) induces a canonical isometric algebra isomorphism

$$(5.6) \quad \langle B(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G})) \rangle^* \cong LUC(\mathbb{G})^*.$$

Therefore, the Banach algebra structure on $LUC(\mathbb{G})^*$ obtained from $L_1(\mathbb{G})^{**}$ is the same as the one obtained from $T(L_2(\mathbb{G}))^{**}$. That is, via (2.1), we have the isometric algebra isomorphisms

$$(L_1(\mathbb{G})^{**}, \square) / LUC(\mathbb{G})^\perp \cong LUC(\mathbb{G})^* \cong ((T(L_2(\mathbb{G})), \triangleright)^{**}, \square) / LUC(\mathbb{G})^\perp.$$

Similarly, $RUC(\mathbb{G}) = \langle T(L_2(\mathbb{G})) \triangleleft B(L_2(\mathbb{G})) \rangle$ is right introverted in $(T(L_2(\mathbb{G})), \triangleleft)^*$, and we also have a canonical isometric algebra isomorphism

$$\langle T(L_2(\mathbb{G})) \triangleleft B(L_2(\mathbb{G})) \rangle^* \cong RUC(\mathbb{G})^*.$$

For $x \in B(L_2(\mathbb{G}))$ and $\omega, \gamma \in T(L_2(\mathbb{G}))$ with $f = \gamma|_{L_\infty(\mathbb{G})} \in L_1(\mathbb{G})$, by (3.2) and (3.3), we have that $(x \triangleright \omega) \triangleright \gamma = (x \triangleright \omega) \star f$ and hence

$$(x \triangleright \omega) \triangleright \gamma \diamond m = ((x \triangleright \omega) \star f) \diamond m$$

for all $m \in \langle B(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G})) \rangle^* \cong LUC(\mathbb{G})^*$.

Therefore, we have the following proposition.

Proposition 5.4 *Let \mathbb{G} be a locally compact quantum group. Then $LUC(\mathbb{G})$ is two-sided introverted in $L_\infty(\mathbb{G})$ if and only if $LUC(\mathbb{G})$ is two-sided introverted in $B(L_2(\mathbb{G}))$.*

It is known that for $L_\infty(\mathbb{G}) = L_\infty(G)$, the equality $LUC(\mathbb{G}) = RUC(\mathbb{G})$ holds exactly when the locally compact group G is SIN, where ‘‘SIN’’ stands for ‘‘small invariant neighborhood’’. A locally compact quantum group \mathbb{G} is therefrom called SIN if $LUC(\mathbb{G}) = RUC(\mathbb{G})$ (cf. [11]). Compact quantum groups, discrete quantum groups, and commutative quantum groups are clearly SIN. When the quantum group \mathbb{G} is SIN, the space $LUC(\mathbb{G})$ is two-sided introverted in $L_\infty(\mathbb{G})$, and the converse holds if the quantum group is commutative (cf. [11, Proposition 11]). Due to (5.1) and the fact that $L_\infty(\mathbb{G}) \cap L_\infty(\widehat{\mathbb{G}}) = \mathbb{C}1$, we cannot have $\langle B(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G})) \rangle = \langle T(L_2(\mathbb{G})) \triangleright B(L_2(\mathbb{G})) \rangle$ if \mathbb{G} is nontrivial. However, we show below that \mathbb{G} must be SIN if $\langle B(L_2(\mathbb{G})) \triangleleft T(L_2(\mathbb{G})) \rangle = \langle T(L_2(\mathbb{G})) \triangleright B(L_2(\mathbb{G})) \rangle$.

Proposition 5.5 *Let \mathbb{G} be a locally compact quantum group such that ${}_{\triangleright}X(L_2(\mathbb{G})) = X_{\triangleleft}(L_2(\mathbb{G}))$. Then \mathbb{G} is SIN.*

Proof Let $x \in B(L_2(\mathbb{G}))$ and $\omega_1, \omega_2, \omega_3 \in T(L_2(\mathbb{G}))$. Since $x \triangleright \omega_1 \in L_\infty(\mathbb{G})$, by (3.3), we have

$$(x \triangleright \omega_1) \triangleright \omega_2 = (x \triangleright \omega_1) \triangleleft \omega_2 \in X_{\triangleleft}(L_2(\mathbb{G})) = {}_{\triangleright}X(L_2(\mathbb{G})).$$

Then $(x \triangleright \omega_1) \triangleright \omega_2$ is a norm limit of a sequence of elements of the form $\sum_{i=1}^n \gamma_i \triangleright y_i$, where $\gamma_i \in T(L_2(\mathbb{G}))$ and $y_i \in B(L_2(\mathbb{G}))$. Thus $x \triangleright (\omega_1 \triangleright \omega_2 \triangleright \omega_3) = ((x \triangleright \omega_1) \triangleright \omega_2) \triangleright \omega_3$ is a norm limit of a sequence of elements of the form $\sum_{i=1}^n (\gamma_i \triangleright y_i) \triangleright \omega_3$. Note that $(\gamma_i \triangleright y_i) \triangleright \omega_3 = \gamma_i \triangleright (y_i \triangleright \omega_3) = \gamma_i \triangleleft (y_i \triangleright \omega_3)$, since $y_i \triangleright \omega_3 \in L_\infty(\mathbb{G})$. It follows that $(\gamma_i \triangleright y_i) \triangleright \omega_3 \in RUC(\mathbb{G})$. Hence, we obtain that $x \triangleright (\omega_1 \triangleright \omega_2 \triangleright \omega_3) \in RUC(\mathbb{G})$. Therefore, due to (3.1) and (3.2), $LUC(\mathbb{G}) \subseteq RUC(\mathbb{G})$. Similarly, we can show that $RUC(\mathbb{G}) \subseteq LUC(\mathbb{G})$. Consequently, \mathbb{G} is an SIN quantum group. ■

For semi-regular locally compact quantum groups \mathbb{G} , corresponding to the fact that $LUC(\mathbb{G})$ and $RUC(\mathbb{G})$ are C^* -subalgebras of $L_\infty(\mathbb{G})$ shown in [14], we prove below that the much larger spaces $X_{\triangleleft}(L_2(\mathbb{G}))$ and ${}_{\triangleright}X(L_2(\mathbb{G}))$ are indeed C^* -subalgebras of $B(L_2(\mathbb{G}))$. We start with a lemma on the right fundamental unitary V of \mathbb{G} .

Let $U = \widehat{J}J$ be the unitary operator on $L_2(\mathbb{G})$ as used in Section 3. For any multiplicative unitary \mathcal{V} on $L_2(\mathbb{G}) \otimes L_2(\mathbb{G})$, let

$$\mathcal{V}^U = (1 \otimes U)\mathcal{V}(1 \otimes U^*) \quad \text{and} \quad \mathcal{V}^{U \otimes U} = (U \otimes U)\mathcal{V}(U^* \otimes U^*).$$

Lemma 5.6 *Let \mathbb{G} be a locally compact group. Then we have $V_{23}V_{13}V_{12}^U = V_{12}^UV_{23}$.*

Proof We prove the lemma through four steps. First, by applying $(1 \otimes U \otimes U)(\cdot)(1 \otimes U^* \otimes U^*)$ to the pentagonal equation $W_{12}W_{13}W_{23} = W_{23}W_{12}$, we have

$$(5.7) \quad W_{12}^U W_{13}^U W_{23}^{U \otimes U} = W_{23}^{U \otimes U} W_{12}^U.$$

Notice that $V = \Sigma(1 \otimes U)W(1 \otimes U^*)\Sigma = \Sigma W^U \Sigma$. Then, by applying $\Sigma_{12}(\cdot)\Sigma_{12}$ to (5.7), we obtain

$$(5.8) \quad V_{12}W_{23}^U(U \otimes 1 \otimes U)W_{13}(U^* \otimes 1 \otimes U^*) = (U \otimes 1 \otimes U)W_{13}(U^* \otimes 1 \otimes U^*)V_{12}.$$

Next, we apply $\Sigma_{23}(\cdot)\Sigma_{23}$ to (5.8) and obtain

$$(5.9) \quad V_{13}V_{23}(U \otimes U \otimes 1)W_{12}(U^* \otimes U^* \otimes 1) = (U \otimes U \otimes 1)W_{12}(U^* \otimes U^* \otimes 1)V_{13}.$$

Finally, we conclude that $V_{23}V_{13}V_{12}^U = V_{12}^U V_{23}$ by applying $\Sigma_{12}(\cdot)\Sigma_{12}$ to (5.9). ■

Theorem 5.7 *Let \mathbb{G} be a locally compact quantum group. If \mathbb{G} is semi-regular, then $X_{\natural}(L_2(\mathbb{G}))$ and ${}_bX(L_2(\mathbb{G}))$ are unital C^* -subalgebras of $B(L_2(\mathbb{G}))$.*

Proof Since $X_{\natural}(L_2(\mathbb{G}))$ and ${}_bX(L_2(\mathbb{G}))$ are both operator systems on $L_2(\mathbb{G})$, due to (5.2), we only have to prove that ${}_bX(L_2(\mathbb{G}))$ is a subalgebra of $B(L_2(\mathbb{G}))$.

To show this, let $x, y \in B(L_2(\mathbb{G}))$ and $\xi_1, \eta_1, \xi_2, \eta_2 \in L_2(\mathbb{G})$. We want to prove that

$$(5.10) \quad (\omega_{\xi_1, \eta_1} \triangleright x)(\omega_{\xi_2, \eta_2} \triangleright y) = (\iota \otimes \eta_1^*)V(x \otimes 1)V^*(\iota \otimes \xi_1)(\iota \otimes \eta_2^*)V(y \otimes 1)V^*(\iota \otimes \xi_2)$$

is contained in ${}_bX(L_2(\mathbb{G}))$. Notice that the middle term $(\iota \otimes \xi_1)(\iota \otimes \eta_2^*)$ in (5.10) corresponds to the operator $1 \otimes x_{\xi_1, \eta_2}$. Since \mathbb{G} is semi-regular, we have

$$K(L_2(\mathbb{G})) \subseteq \langle C(V) \rangle = \langle (\iota \otimes T(L_2(\mathbb{G}))) (\Sigma V) \rangle.$$

We also note that $\langle C(V) \rangle$ is the closed linear span of operators of the form $(\iota \otimes \eta^*)V(\xi \otimes \iota)$, and we have $x_{\xi_1, \eta_2} = U^*x_{U\xi_1, \eta_2}$. We can thus replace the middle term $(\iota \otimes \xi_1)(\iota \otimes \eta_2^*) = 1 \otimes x_{\xi_1, \eta_2}$ in (5.10) by an operator of the form $1 \otimes (\iota \otimes \eta^*)(U^* \otimes 1)V(\xi \otimes \iota)$. Hence, it is sufficient to show that

$$(5.11) \quad (\iota \otimes \eta_1^*)V(x \otimes 1)V^*[1 \otimes (\iota \otimes \eta^*)(U^* \otimes 1)V(\xi \otimes \iota)]V(y \otimes 1)V^*(\iota \otimes \xi_2)$$

is contained in ${}_bX(L_2(\mathbb{G}))$.

For this purpose, by applying Lemma 5.6 twice, we can write (5.11) as

$$\begin{aligned} & (\iota \otimes \eta_1^* \otimes \eta^*)V_{12}(x \otimes 1 \otimes 1)V_{12}^*(1 \otimes U^* \otimes 1)V_{23}V_{13}(y \otimes 1 \otimes 1)V_{13}^*(\iota \otimes \xi \otimes \xi_2) \\ &= (\iota \otimes \eta_1^* \otimes \eta^*)V_{12}(x \otimes 1 \otimes 1)(1 \otimes U^* \otimes 1)[(V_{12}^U)^*V_{23}V_{13}](y \otimes 1 \otimes 1)V_{13}^*(\iota \otimes \xi \otimes \xi_2) \\ &= (\iota \otimes \eta_1^* \otimes \eta^*)(1 \otimes U^* \otimes 1)V_{12}^U(x \otimes 1 \otimes 1)[V_{23}(V_{12}^U)^*](y \otimes 1 \otimes 1)V_{13}^*(\iota \otimes \xi \otimes \xi_2) \\ &= (\iota \otimes \eta_1^* \otimes \eta^*)(1 \otimes U^* \otimes 1)[V_{12}^U V_{23}](x \otimes 1 \otimes 1)(V_{12}^U)^*(y \otimes 1 \otimes 1)V_{13}^*(\iota \otimes \xi \otimes \xi_2) \\ &= (\iota \otimes \eta_1^* \otimes \eta^*)(1 \otimes U^* \otimes 1)[V_{23}V_{13}V_{12}^U](x \otimes 1 \otimes 1)(V_{12}^U)^*(y \otimes 1 \otimes 1)V_{13}^*(\iota \otimes \xi \otimes \xi_2). \end{aligned}$$

Note that the linear functional $(\eta_1^* \otimes \eta^*)(U^* \otimes 1)V$ on $L_2(\mathbb{G}) \otimes L_2(\mathbb{G})$ can be approximated in norm by linear combinations of functionals of the form $\tilde{\eta}_1^* \otimes \tilde{\eta}_2^*$. Therefore, it is sufficient to show that

$$(\iota \otimes \tilde{\eta}_1^* \otimes \tilde{\eta}_2^*)[V_{13}V_{12}^U](x \otimes 1 \otimes 1)(V_{12}^U)^*(y \otimes 1 \otimes 1)V_{13}^*(\iota \otimes \xi \otimes \xi_2)$$

is contained in ${}_bX(L_2(\mathbb{G}))$. This is indeed true, since we have

$$\begin{aligned} & (\iota \otimes \tilde{\eta}_1^* \otimes \tilde{\eta}_2^*)[V_{13}V_{12}^U](x \otimes 1 \otimes 1)(V_{12}^U)^*(y \otimes 1 \otimes 1)V_{13}^*(\iota \otimes \xi \otimes \xi_2) \\ &= (\iota \otimes (U^*\tilde{\eta}_1)^* \otimes \tilde{\eta}_2^*)V_{13}V_{12}(x \otimes 1 \otimes 1)V_{12}^*(y \otimes 1 \otimes 1)V_{13}^*(\iota \otimes U^*\xi \otimes \xi_2) \\ &= (\iota \otimes \tilde{\eta}_2^*)V((\omega_{U^*\xi, U^*\tilde{\eta}_1} \triangleright x)y \otimes 1)V^*(\iota \otimes \xi_2) \\ &= \omega_{\xi_2, \tilde{\eta}_2} \triangleright ((\omega_{U^*\xi, U^*\tilde{\eta}_1} \triangleright x)y). \end{aligned}$$

That is, we obtain

$$(\iota \otimes \tilde{\eta}_1^* \otimes \tilde{\eta}_2^*)[V_{13}V_{12}^U](x \otimes 1 \otimes 1)(V_{12}^U)^*(y \otimes 1 \otimes 1)V_{13}^*(\iota \otimes \xi \otimes \xi_2) = \omega_{\xi_2, \tilde{\eta}_2} \triangleright ((\omega_{U^*\xi, U^*\tilde{\eta}_1} \triangleright x)y) \in \triangleright X(L_2(\mathbb{G})).$$

This completes the proof. ■

6 Automatic Normality and Commutation Relations

As in the previous sections, for a locally compact quantum group \mathbb{G} , we often use $T(L_2(\mathbb{G}))$ for the algebra $(T(L_2(\mathbb{G})), \triangleright)$ and $B_{T(L_2(\mathbb{G}))}(B(L_2(\mathbb{G})))$ for the space of bounded right $(T(L_2(\mathbb{G})), \triangleright)$ -module maps on $B(L_2(\mathbb{G}))$. Recall from (5.6) that the adjoint of the identity map $(B(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G}))) \rightarrow LUC(\mathbb{G})$ in (3.2) induces a canonical isometric algebra isomorphism

$$\langle B(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G})) \rangle^* \cong LUC(\mathbb{G})^*.$$

It is known from Section 2 that the map

$$\Phi_L: LUC(\mathbb{G})^* \rightarrow CB_{L_1(\mathbb{G})}(L_\infty(\mathbb{G})), m \mapsto m_L$$

is an injective and completely contractive algebra homomorphism, where $m_L \in CB_{L_1(\mathbb{G})}(L_\infty(\mathbb{G}))$ satisfies $\langle m_L(x), f \rangle = \langle m, x \star f \rangle$ ($x \in L_\infty(\mathbb{G}), f \in L_1(\mathbb{G})$). Due to (5.6), we also obtain an injective and completely contractive algebra homomorphism

$$\widetilde{\Phi}_L: LUC(\mathbb{G})^* \rightarrow CB_{T(L_2(\mathbb{G}))}(B(L_2(\mathbb{G})))$$

satisfying $\widetilde{\Phi}_L(m)|_{L_\infty(\mathbb{G})} = \Phi_L(m)$. Therefore, we have the commutative diagram

$$(6.1) \quad \begin{array}{ccc} \langle B(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G})) \rangle^* & \xrightarrow{\cong} & LUC(\mathbb{G})^* \\ \widetilde{\Phi}_L \downarrow & & \downarrow \Phi_L \\ CB_{T(L_2(\mathbb{G}))}(B(L_2(\mathbb{G}))) & \xrightarrow{F \mapsto F|_{L_\infty(\mathbb{G})}} & CB_{L_1(\mathbb{G})}(L_\infty(\mathbb{G})). \end{array}$$

We also have

$$\widetilde{\Phi}_L(LUC(\mathbb{G})^*) \subseteq CB_{L_\infty(\mathbb{G})}^{L_\infty(\mathbb{G})}(B(L_2(\mathbb{G}))),$$

and

$$\begin{aligned} LUC(\mathbb{G})^* &\cong CB_{L_1(\mathbb{G})}(L_\infty(\mathbb{G})) \\ &\cong CB_{T(L_2(\mathbb{G}))}(B(L_2(\mathbb{G}))) \text{ canonically} \Leftrightarrow \mathbb{G} \text{ is co-amenable.} \end{aligned}$$

(See [14, Sections 6 and 7] for a detailed discussion.)

It is seen from [15, Theorem 3.1], Theorem 4.1, and Theorem 6.2 that

$$CB_{L_1(\mathbb{G})}(L_\infty(\mathbb{G})) \cong CB_{T(L_2(\mathbb{G}))}(B(L_2(\mathbb{G}))) = CB_{L_\infty(\widehat{\mathbb{G}})}^{L_\infty(\mathbb{G})}(B(L_2(\mathbb{G})))$$

if \mathbb{G} is compact. However, we do not know whether this equality holds for, say, $L_\infty(\mathbb{G}) = l_\infty(\mathbb{Z})$, and whether we have the inclusion

$$CB_{T(L_2(\mathbb{G}))}(B(L_2(\mathbb{G}))) \subseteq CB_{L_\infty(\widehat{\mathbb{G}})}^{L_\infty(\mathbb{G})}(B(L_2(\mathbb{G})))$$

for general locally compact quantum groups \mathbb{G} (cf. [14, Remark 7.4]).

We shall characterize compactness of \mathbb{G} in terms of automatic normality of both completely bounded right $T(L_2(\mathbb{G}))$ -module maps and completely bounded $L_\infty(\widehat{\mathbb{G}})$ -bimodule maps on $B(L_2(\mathbb{G}))$. Let us first prove the following proposition, which is the (noncommutative) quantum group version of [21, Satz 5.4.5] on $L_\infty(G)$ and a trace class lifting of [25, Theorem 3.8] for $L_1(\mathbb{G})$.

Proposition 6.1 *Let \mathbb{G} be a locally compact quantum group. Then we have*

$$\mathbb{G} \text{ is compact} \iff T(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G}))^{**} \subseteq T(L_2(\mathbb{G})).$$

Proof Let $\pi_\triangleright: T(L_2(\mathbb{G})) \rightarrow L_1(\mathbb{G})$ be the canonical quotient map. Then, when $T(L_2(\mathbb{G}))^{**}$ and $L_1(\mathbb{G})^{**}$ are equipped with their left Arens products, the map

$$(\pi_\triangleright)^{**}: T(L_2(\mathbb{G}))^{**} \rightarrow L_1(\mathbb{G})^{**}$$

is a surjective algebra homomorphism. Thus we derive that

$$T(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G}))^{**} \subseteq T(L_2(\mathbb{G})) \implies L_1(\mathbb{G}) \star L_1(\mathbb{G})^{**} \subseteq L_1(\mathbb{G}).$$

On the other hand, we conclude from (3.2) that

$$L_1(\mathbb{G}) \star L_1(\mathbb{G})^{**} \subseteq L_1(\mathbb{G}) \implies (T(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G}))) \triangleright T(L_2(\mathbb{G}))^{**} \subseteq T(L_2(\mathbb{G})).$$

Since $\langle T(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G})) \rangle = T(L_2(\mathbb{G}))$ (cf. (3.1)), we obtain that $T(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G}))^{**} \subseteq T(L_2(\mathbb{G}))$ if and only if $L_1(\mathbb{G}) \star L_1(\mathbb{G})^{**} \subseteq L_1(\mathbb{G})$, which proves the proposition by [25, Theorem 3.8]. ■

Theorem 6.2 *Let \mathbb{G} be a locally compact quantum group. Then the following statements are equivalent:*

- (i) \mathbb{G} is compact;
- (ii) $CB_{T(L_2(\mathbb{G}))}(B(L_2(\mathbb{G}))) = CB_{T(L_2(\mathbb{G}))}^\sigma(B(L_2(\mathbb{G})))$;
- (iii) $CB_{L_\infty(\widehat{\mathbb{G}})}(B(L_2(\mathbb{G}))) = CB_{L_\infty(\widehat{\mathbb{G}})}^\sigma(B(L_2(\mathbb{G})))$;
- (iv) $CB_{L_\infty(\widehat{\mathbb{G}})}^{L_\infty(\mathbb{G})}(B(L_2(\mathbb{G}))) = CB_{L_\infty(\widehat{\mathbb{G}})}^{\sigma, L_\infty(\mathbb{G})}(B(L_2(\mathbb{G})))$;
- (v) the commutation relation

$$LM_{cb}(T(L_2(\mathbb{G})))^c = RM_{cb}(T(L_2(\mathbb{G})))$$

holds in $CB(B(L_2(\mathbb{G})))$.

Therefore, we have

$$CB_{T(L_2(\mathbb{G}))}(B(L_2(\mathbb{G}))) = CB_{L_\infty(\widehat{\mathbb{G}})}^{L_\infty(\mathbb{G})}(B(L_2(\mathbb{G})))$$

if \mathbb{G} is compact.

Proof (i) \Rightarrow (ii). We suppose that $T(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G}))^{**} \subseteq T(L_2(\mathbb{G}))$ (cf. Proposition 6.1). Let $\Psi \in CB_{T(L_2(\mathbb{G}))}(B(L_2(\mathbb{G})))$. Then we have

$$\Psi^*(\omega \triangleright \gamma) = \omega \triangleright \Psi^*(\gamma) \in T(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G}))^{**} \subseteq T(L_2(\mathbb{G}))$$

for all $\omega, \gamma \in T(L_2(\mathbb{G}))$, and thus $\Psi^*(T(L_2(\mathbb{G}))) \subseteq T(L_2(\mathbb{G}))$ (cf. (3.1)). Therefore, $\Psi \in CB_{T(L_2(\mathbb{G}))}^\sigma(B(L_2(\mathbb{G})))$.

(ii) \Rightarrow (i). Suppose that the equality in (ii) holds. Then, by (6.1) and Theorem 4.1, we have $LUC(\mathbb{G})^* \subseteq RM_{cb}(L_1(\mathbb{G}))$. Therefore, \mathbb{G} is compact by [15, Theorem 3.1].

(i) \Rightarrow (iii). Suppose that \mathbb{G} is compact. Then $\widehat{\mathbb{G}}$ is discrete, and thus both $L_\infty(\widehat{\mathbb{G}})$ and $L_\infty(\widehat{\mathbb{G}})'$ are finite atomic von Neumann algebras standardly represented on $L_2(\mathbb{G})$. By [10, Lemma 3.5], which in fact holds for all Hilbert spaces as observed in [22, p. 1136], we have $CB_{L_\infty(\widehat{\mathbb{G}})}(B(L_2(\mathbb{G}))) = CB_{L_\infty(\widehat{\mathbb{G}})}^\sigma(B(L_2(\mathbb{G})))$.

(iii) \Rightarrow (iv). This is trivial.

(iv) \Rightarrow (i). This follows from the same argument as used in the proof of (ii) \Rightarrow (i).

(ii) \Leftrightarrow (v). This is true, since under the canonical embeddings

$$RM_{cb}(T(L_2(\mathbb{G}))) \rightarrow CB(B(L_2(\mathbb{G}))) \quad \text{and} \quad LM_{cb}(T(L_2(\mathbb{G}))) \rightarrow CB(B(L_2(\mathbb{G})))$$

(cf. Section 2), we have

$$(6.2) \quad \begin{aligned} CB_{T(L_2(\mathbb{G}))}^\sigma(B(L_2(\mathbb{G}))) &= RM_{cb}(T(L_2(\mathbb{G}))), \\ CB_{T(L_2(\mathbb{G}))}(B(L_2(\mathbb{G}))) &= LM_{cb}(T(L_2(\mathbb{G})))^c. \end{aligned}$$

The final assertion holds by Theorem 4.1 and the equivalence between (i), (ii), and (iv). ■

The corollary below follows immediately from Theorem 6.2.

Corollary 6.3 *Let \mathbb{G} be a locally compact quantum group. Then the following statements are equivalent:*

- (i) \mathbb{G} is discrete;
- (ii) $CB_{L_\infty(\mathbb{G})}(B(L_2(\mathbb{G}))) = CB_{L_\infty(\mathbb{G})}^\sigma(B(L_2(\mathbb{G})))$;
- (iii) $CB_{L_\infty(\mathbb{G})}^{L_\infty(\widehat{\mathbb{G}})}(B(L_2(\mathbb{G}))) = CB_{L_\infty(\mathbb{G})}^{\sigma, L_\infty(\widehat{\mathbb{G}})}(B(L_2(\mathbb{G})))$.

To investigate when we have the commutation relation

$$RM_{cb}(T(L_2(\mathbb{G})))^c = LM_{cb}(T(L_2(\mathbb{G})))$$

in $CB(B(L_2(\mathbb{G})))$ (comparing with Theorem 6.2(v)), we use the $*$ -anti-automorphism $\tilde{R}: x \mapsto \widehat{J}x^*\widehat{J}$ on $B(L_2(\mathbb{G}))$ (cf. Section 3) to define the map

$$\Upsilon: CB(B(L_2(\mathbb{G}))) \longrightarrow CB(B(L_2(\mathbb{G}))), \quad \Psi \longmapsto \tilde{R} \circ \Psi \circ \tilde{R}.$$

Then Υ is an isometric algebra isomorphism on $CB(B(L_2(\mathbb{G})))$ satisfying $\Upsilon^2 = \text{id}$. As noted in [14, Section 4], we have

$$\Upsilon(CB^\sigma(B(L_2(\mathbb{G})))) = CB^\sigma(B(L_2(\mathbb{G}))).$$

For $A = (T(L_2(\mathbb{G})), \triangleright)$ or $(T(L_2(\mathbb{G})), \triangleleft)$, let ${}_A CB(B(L_2(\mathbb{G})))$ denote the subalgebra of $CB(B(L_2(\mathbb{G})))$ consisting of left A -module maps in $CB(B(L_2(\mathbb{G})))$. Under the canonical isomorphic and anti-isomorphic identifications

$$(6.3) \quad RM_{cb}(A) \cong CB_A^\sigma(B(L_2(\mathbb{G}))) \quad \text{and} \quad LM_{cb}(A) \cong {}_A CB^\sigma(B(L_2(\mathbb{G}))),$$

we can derive from (3.16) that

$$(6.4) \quad \begin{aligned} \Upsilon(RM_{cb}(T(L_2(\mathbb{G})), \triangleright)) &= LM_{cb}(T(L_2(\mathbb{G})), \triangleleft), \\ \Upsilon(LM_{cb}(T(L_2(\mathbb{G})), \triangleright)) &= RM_{cb}(T(L_2(\mathbb{G})), \triangleleft). \end{aligned}$$

It is easy to see that for any $Y \subseteq CB(B(L_2(\mathbb{G})))$, we have that

$$(6.5) \quad \Upsilon(Y^c) = \Upsilon(Y)^c \text{ holds in } CB(B(L_2(\mathbb{G}))).$$

Therefore, combining (6.4) and (6.5) with Theorem 6.2, we obtain

$$\mathbb{G} \text{ is compact} \Leftrightarrow RM_{cb}(T(L_2(\mathbb{G})), \triangleleft)^c = LM_{cb}(T(L_2(\mathbb{G})), \triangleleft) \text{ holds in } CB(B(L_2(\mathbb{G}))).$$

It is clear from (6.4) and (6.5) that

$$\begin{aligned} LM_{cb}(T(L_2(\mathbb{G})), \triangleleft)^c &= RM_{cb}(T(L_2(\mathbb{G})), \triangleleft) \\ \Leftrightarrow RM_{cb}(T(L_2(\mathbb{G})), \triangleright)^c &= LM_{cb}(T(L_2(\mathbb{G})), \triangleright). \end{aligned}$$

We show below that these two commutation relations are in fact equivalent to $T(L_2(\mathbb{G}))$ being a *left* ideal in $(T(L_2(\mathbb{G})), \triangleright)^{**}$, which is stronger than \mathbb{G} being compact; the latter has been shown to be equivalent to $T(L_2(\mathbb{G}))$ being a *right* ideal in $(T(L_2(\mathbb{G})), \triangleright)^{**}$ (cf. Proposition 6.1 and Theorem 6.2). Moreover, when linking to the representation

$$\Theta^r : RM_{cb}(L_1(\mathbb{G})) \longrightarrow CB_{L_\infty(\widehat{\mathbb{G}})}^{\sigma, L_\infty(\mathbb{G})}(B(L_2(\mathbb{G}))),$$

we will see that the commutation relation $RM_{cb}(T(L_2(\mathbb{G})), \triangleright)^c = LM_{cb}(T(L_2(\mathbb{G})), \triangleright)$ is much stronger than $\Theta^r(L_1(\mathbb{G}))(K(L_2(\mathbb{G}))) \subseteq K(L_2(\mathbb{G}))$, which is equivalent to \mathbb{G} being regular as shown in Theorem 4.2.

The Banach algebra $(T(L_2(\mathbb{G})), \triangleright)$ below is again simply denoted by $T(L_2(\mathbb{G}))$. Recall that a locally compact quantum group \mathbb{G} is said to be *finite* if $L_\infty(\mathbb{G})$ is finite dimensional (cf. [15]).

Theorem 6.4 *Let \mathbb{G} be a locally compact quantum group. Then the following statements are equivalent:*

- (i) *the commutation relation*

$$RM_{cb}(T(L_2(\mathbb{G})))^c = LM_{cb}(T(L_2(\mathbb{G})))$$

holds in $CB(B(L_2(\mathbb{G})))$;

- (ii) $T(L_2(\mathbb{G}))CB(B(L_2(\mathbb{G}))) = T(L_2(\mathbb{G}))CB^\sigma(B(L_2(\mathbb{G})))$;
- (iii) $T(L_2(\mathbb{G}))^{**} \triangleright T(L_2(\mathbb{G})) \subseteq T(L_2(\mathbb{G}))$;
- (iv) $\Theta^r(L_1(\mathbb{G}))^*(T(L_2(\mathbb{G}))^{**}) \subseteq T(L_2(\mathbb{G}))$;
- (v) \mathbb{G} is finite.

Proof Note that, similar to (6.2), under the identifications given in (6.3), we have

$$RM_{cb}(T(L_2(\mathbb{G})))^c = {}_{T(L_2(\mathbb{G}))}CB(B(L_2(\mathbb{G}))),$$

$$LM_{cb}(T(L_2(\mathbb{G}))) = {}_{T(L_2(\mathbb{G}))}CB^\sigma(B(L_2(\mathbb{G}))).$$

Then it is easy to see that (i)–(iii) are equivalent, noticing that

$$\langle T(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G})) \rangle = T(L_2(\mathbb{G})).$$

The equivalence (iii) \Leftrightarrow (iv) holds by the construction of the representation Θ^r (cf. (4.4)).

To complete the proof, we need only show that (i) \Rightarrow (v). Suppose that (i) holds. Then, by Theorem 4.1, we obtain that $LM_{cb}(T(L_2(\mathbb{G})))^c = RM_{cb}(T(L_2(\mathbb{G})))^{cc} = RM_{cb}(T(L_2(\mathbb{G})))$. It follows from Theorem 6.2 that \mathbb{G} is compact.

We claim that ${}_{\triangleright}K(L_2(\mathbb{G})) = {}_{\triangleright}X(L_2(\mathbb{G}))$. Otherwise, there is $m \in B(L_2(\mathbb{G}))^*$ such that $m|_{{}_{\triangleright}K(L_2(\mathbb{G}))} = 0$, but $m|_{{}_{\triangleright}X(L_2(\mathbb{G}))} \neq 0$. Clearly, we have $m_R \in {}_{T(L_2(\mathbb{G}))}CB(B(L_2(\mathbb{G})))$, where $m_R \in CB(B(L_2(\mathbb{G})))$ is given by $\langle m_R(x), \omega \rangle = \langle \omega \triangleright x, m \rangle$ ($x \in B(L_2(\mathbb{G})), \omega \in T(L_2(\mathbb{G}))$). By the assumption of (i) (\Leftrightarrow (ii)), we have $m_R \in {}_{T(L_2(\mathbb{G}))}CB^\sigma(B(L_2(\mathbb{G})))$. Then $m_R = \mu^*$ for some $\mu \in LM_{cb}(T(L_2(\mathbb{G})))$ (cf. (6.3)). For all $a \in K(L_2(\mathbb{G}))$ and $\omega \in T(L_2(\mathbb{G}))$, since $\omega \triangleright a \in {}_{\triangleright}K(L_2(\mathbb{G}))$, we have

$$\langle \mu(\omega), a \rangle = \langle \omega, m_R(a) \rangle = \langle \omega \triangleright a, m \rangle = 0.$$

Then $\mu = 0$, and thus $m_R = \mu^* = 0$; that is, $m|_{{}_{\triangleright}X(L_2(\mathbb{G}))} = 0$, a contradiction.

Therefore, ${}_{\triangleright}K(L_2(\mathbb{G})) = {}_{\triangleright}X(L_2(\mathbb{G}))$. In particular, we have $1 \in {}_{\triangleright}K(L_2(\mathbb{G}))$. Since \mathbb{G} is compact and thus regular, we obtain that $1 \in {}_{\triangleright}K(L_2(\mathbb{G})) = K(L_2(\mathbb{G}))$ by Corollary 3.6. It follows that $\dim(L_2(\mathbb{G})) < \infty$, and hence \mathbb{G} is finite. ■

Note that the canonical map $T(L_2(\mathbb{G})) \rightarrow RM_{cb}(T(L_2(\mathbb{G})))$ is not injective if \mathbb{G} is nontrivial. On the other hand, since the convolution \triangleright is left faithful, we have the natural embedding

$$T(L_2(\mathbb{G})) \longrightarrow LM_{cb}(T(L_2(\mathbb{G}))), \quad \omega \longmapsto \ell_\omega,$$

where $\ell_\omega \in LM_{cb}(T(L_2(\mathbb{G})))$ is given by $\ell_\omega(\gamma) = \omega \triangleright \gamma$. Under the canonical anti-homomorphic embedding $LM_{cb}(T(L_2(\mathbb{G}))) \rightarrow CB(B(L_2(\mathbb{G})))$, we have

$$LM_{cb}(T(L_2(\mathbb{G})))^c \subseteq T(L_2(\mathbb{G}))^c = CB_{T(L_2(\mathbb{G}))}(B(L_2(\mathbb{G}))).$$

Since $\langle T(L_2(\mathbb{G})) \triangleright T(L_2(\mathbb{G})) \rangle = T(L_2(\mathbb{G}))$ (cf. (3.1)), as shown in [13, Proposition 4.1], we always have

$$(6.6) \quad \begin{aligned} LM_{cb}(T(L_2(\mathbb{G})))^c &= T(L_2(\mathbb{G}))^c \text{ and hence} \\ LM_{cb}(T(L_2(\mathbb{G})))^{cc} &= T(L_2(\mathbb{G}))^{cc} \text{ in } CB(B(L_2(\mathbb{G}))). \end{aligned}$$

On the other hand, we recall from Theorem 4.1 that the double commutation relation $RM_{cb}(T(L_2(\mathbb{G})))^{cc} = RM_{cb}(T(L_2(\mathbb{G})))$ holds in $CB(B(L_2(\mathbb{G})))$ for all quantum groups \mathbb{G} . However, as seen below, the situation will be completely different if the above algebras $LM_{cb}(T(L_2(\mathbb{G})))$ and $RM_{cb}(T(L_2(\mathbb{G})))$ are replaced by $RM_{cb}(T(L_2(\mathbb{G})))$ and $LM_{cb}(T(L_2(\mathbb{G})))$, respectively.

Corollary 6.5 *Let \mathbb{G} be a locally compact quantum group. Then*

- (i) $RM_{cb}(T(L_2(\mathbb{G})))^{cc} = T(L_2(\mathbb{G}))^{cc}$ holds in $CB(B(L_2(\mathbb{G})))$ if and only if \mathbb{G} is trivial;
- (ii) $LM_{cb}(T(L_2(\mathbb{G}))) \not\subseteq LM_{cb}(T(L_2(\mathbb{G})))^{cc}$ if \mathbb{G} is compact and infinite.

Proof (i) We just need to prove the “only if” part. Suppose that $RM_{cb}(T(L_2(\mathbb{G})))^{cc} = T(L_2(\mathbb{G}))^{cc}$. Then by Theorem 4.1 and (6.6), we have

$$\begin{aligned} LM_{cb}(T(L_2(\mathbb{G}))) &\subseteq LM_{cb}(T(L_2(\mathbb{G})))^{cc} = RM_{cb}(T(L_2(\mathbb{G})))^{cc} \\ &= RM_{cb}(T(L_2(\mathbb{G}))) \subseteq LM_{cb}(T(L_2(\mathbb{G})))^c. \end{aligned}$$

This implies that $(T(L_2(\mathbb{G})), \triangleright)$ is commutative, or equivalently, that \mathbb{G} is trivial.

(ii) If \mathbb{G} is compact and infinite, then $LM_{cb}(T(L_2(\mathbb{G}))) \not\subseteq RM_{cb}(T(L_2(\mathbb{G})))^c = LM_{cb}(T(L_2(\mathbb{G})))^{cc}$ by Theorems 6.2 and 6.4. ■

Remark 6.6 It is interesting to compare and consider the following double commutation relations in $CB(B(L_2(\mathbb{G})))$ and $CB(L_\infty(\mathbb{G}))$, respectively. On the one hand, under the canonical embeddings

$$L_1(\mathbb{G}) \subseteq M(\mathbb{G}) \hookrightarrow RM_{cb}(L_1(\mathbb{G})) \hookrightarrow CB(B(L_2(\mathbb{G}))),$$

we have

$$(6.7) \quad L_1(\mathbb{G})^{cc} \subseteq M(\mathbb{G})^{cc} \subseteq RM_{cb}(L_1(\mathbb{G}))^{cc} = RM_{cb}(L_1(\mathbb{G})) \text{ in } CB(B(L_2(\mathbb{G}))),$$

where the last equality holds due to [16, Corollary 5.3]. Therefore, we obtain

$$L_1(\mathbb{G})^{cc} = L_1(\mathbb{G}) \Leftrightarrow \mathbb{G} \text{ is discrete}$$

and

$$(6.8) \quad M(\mathbb{G})^{cc} = M(\mathbb{G}) \Leftrightarrow \mathbb{G} \text{ is co-amenable} \Leftrightarrow L_1(\mathbb{G})^{cc} \subseteq M(\mathbb{G}),$$

since $\text{id} \in L_1(\mathbb{G})^{cc}$, and $M(\mathbb{G}) = RM_{cb}(L_1(\mathbb{G}))$ if and only if \mathbb{G} is co-amenable, which is true if and only if $M(\mathbb{G})$ is unital (cf. [4, Theorem 5.1] and [14, Proposition 3.1]). In this setting, $M(\mathbb{G})^{cc}$ provides us with a new algebra between $M(\mathbb{G})$ and $RM_{cb}(L_1(\mathbb{G}))$. It is not clear whether the equality $M(\mathbb{G})^{cc} = RM_{cb}(L_1(\mathbb{G}))$ also characterizes co-amenability of \mathbb{G} (cf. (6.7) and (6.8)). Moreover, it is open even for nondiscrete locally compact groups G whether we have $M(G) \subseteq L_1(G)^{cc}$ (cf. [22, Remark 5.6] and (6.8)). Therefore, for general quantum groups \mathbb{G} , we do not know exactly when $L_1(\mathbb{G})^{cc} = M(\mathbb{G})^{cc}$ holds (cf. (6.7)).

On the other hand, under the canonical embeddings

$$L_1(\mathbb{G}) \subseteq M(\mathbb{G}) \hookrightarrow RM_{cb}(L_1(\mathbb{G})) \hookrightarrow CB(L_\infty(\mathbb{G})),$$

by [15, Theorem 3.11 and Remark 3.16], we always have

$$L_1(\mathbb{G})^{cc} = M(\mathbb{G})^{cc} = RM_{cb}(L_1(\mathbb{G}))^{cc} \text{ in } CB(L_\infty(\mathbb{G})),$$

and in the case where $L_1(\mathbb{G})$ is separable, we have

$$M(\mathbb{G})^{cc} = M(\mathbb{G}) \Leftrightarrow \mathbb{G} \text{ is co-amenable and } L_1(\mathbb{G}) \text{ is SAI}$$

and

$$L_1(\mathbb{G})^{cc} = L_1(\mathbb{G}) \Leftrightarrow \mathbb{G} \text{ is discrete and } L_1(\mathbb{G}) \text{ is SAI.}$$

Here, SAI stands for “*strongly Arens irregular*”, a concept on Banach algebras introduced by Dales and Lau in [7]. Therefore, even for the co-commutative discrete quantum group $\mathbb{G} = VN(SU(3))$, though we have $L_1(\mathbb{G})^{cc} = L_1(\mathbb{G})$ in $CB(B(L_2(\mathbb{G})))$, we do not have $L_1(\mathbb{G})^{cc} = L_1(\mathbb{G})$ in $CB(L_\infty(\mathbb{G}))$, since $A(SU(3))$ is not SAI as shown by Losert. It is interesting to characterize when the relation

$$RM_{cb}(L_1(\mathbb{G}))^{cc} = RM_{cb}(L_1(\mathbb{G}))$$

holds in $CB(L_\infty(\mathbb{G}))$.

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