AN AXIOMATIC THEORY OF ORDINAL NUMBERS

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In this paper, a formal theory of ordinal numbers is developed on an axiomatic basis whose details are described in § 1. Our primitive notions are set, class, collection, a binary relation $\in$, and collection formation $\{\,|\,\}$. Sets and classes in our theory play similar roles as sets and classes respectively in Gödel [1] except the difference that an element of a class is a class but not necessarily a set. A new notion, introduced into our theory is that of collections. A collection relates to a class, just as a class relates to a set in von Neumann's theory. That is, a set is a class and a class is a collection but the converses are not generally the case. For example, all the natural numbers, all the real numbers etc. constitute sets, the ordinal numbers which are sets constitute a proper class, and the totality of ordinal numbers as well as that of all classes are proper collections. These relations are described by axiom group (A).

Let Γ be a term, Φ a formula and Θ, Σ variables. We take $\{\Gamma|\Phi\}_{\Theta,\Sigma}$ to be a collection, and regulate its use by axiom scheme (B): If Π is a class, a class variable, or a set variable, then $\Pi\in\{\Gamma|\Phi\}_{\Theta,\Sigma}.\equiv.\exists\Theta\Sigma(\Pi=\Gamma.\ \Phi)$. Since Φ is an arbitrary formula, Φ may contain bound collection variables. This fact renders impredicative character to our system. Such systems which are of impredicative character were already proposed by many authors, e.g. by Quine [3], Wang [6] and [7], Takeuti [4], and Kuroda [2]. In our system impredicativity occurs in the formulation of the axiom of foundation and also in the definition of the collection of all ordinal numbers. The 'axiom of foundation' (D) is $\mathscr{A}\notin\mathscr{D}es(\mathscr{A})$ where $\mathscr{D}es(\mathscr{A})$[1] denotes the collection of U such that $U\in X\in\cdots\in\mathscr{A}$, or precisely the term $\{U|\forall\mathscr{X}(\mathscr{A}\subseteq\mathscr{X}.\ \forall X(X\in\mathscr{X}.\supset.X\subseteq\mathscr{X}):\supset.\ U\in\mathscr{X})\}$ in which $\mathscr{X}$ is a collection variable, and U and X are class variables. The collection of all ordinal numbers, $\mathscr{O}$, is given by $\{U|\forall\mathscr{X}(0\in\mathscr{X}.\forall X(X\in X\supset X'\in\mathscr{X}).\forall X(X\in\mathscr{L}im.\,X\subseteq\mathscr{X}:\supset.X\in\mathscr{X}):\supset.U\in\mathscr{X})\}$

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1) The letters 'Des' refers to 'descending'. Another formulation of the restrictive axiom which may be adopted in our theory is $\exists U(U\in\mathscr{A})\rightarrow\neg\neg\exists U(U\in\mathscr{A}.\ U\cap\mathscr{A}=0)$, from which we can deduce 1.03–1.06.

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where 0 is the null set, X' is the successor of X and $\mathscr{L}im$ is the collection of limit classes.

Our axiom system consists of four groups (A), (B), (C) and (D). The axiom group (C) provides the existence of sets and classes. The axioms concerning the existence of sets are the same as the axiom group C in Gödel [1] with a slight difference in the formulation of the axiom of infinity. The axioms of sum, power and replacement for classes are obtained from the respective axioms for sets in replacing set variables by class variables. Furthermore, (C) contains the following axiom: $\exists X[0 \in X.\ \forall u(u \in X \supset u' \in X).\ \forall u(u \in \mathscr{L}im.\ u \subseteq X : \supset .\ u \in X)]$ which assures that the totality of ordinal numbers which are sets is a class.

Now we mention the basic feature of our theory. Although the usual two-valued logic, i.e. the classical logic is adopted as the logical frame of this theory, proofs of theorems are carried out as far as possible on the basis of formal laws of the intuitionistic logic. To clarify this, we attach a sign * to the number of a theorem if the theorem is proved beyond the scope of intuitionistic logic. In this paper, theorems with the sign * are proved by using the law of the excluded middle and on that occasion we have not been able to prove them by logics which are weaker than the classical logic but stronger than the intuitionistic logic, i.e. by intermediate logics (Umezawa [5]). For this purpose, a scrutiny of usual proofs is required and the different forms of some axioms have been adopted from the usual ones. For example, in order to prove the transfinite induction in full strength without relying upon the law of the excluded middle, the ordinal numbers are defined in the above mentioned way instead of adopting von Neumann's definition. Hence, the axiom scheme (B) is necessary to be able to use such a definition of ordinal numbers in our theory. It is proved within the classical logic that $\mathscr{O}$ is co-extensional with the totality of ordinal numbers in the sense of von Neumann i.e., with the collection of classes which are wellordered by the $\in$-relation and are identical with the segments of the collection generated by the classes (cf. p. 22, Gödel [1]). The axiom of foundation in von Neumann's form is avoided in this paper, since from $\exists X(X \in \mathscr{A}) \to \exists X(X \in \mathscr{A}.\ X \cap \mathscr{A} = 0)$ we obtain the law of the excluded middle by substituting for $\mathscr{A}$ the collection $\{U \mid (U = 0.\ \Phi) \vee U = \{0\}\}$ where Φ is an arbitrary formula. That is, our axiom system is so chosen that

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the law of the excluded middle might not be derived from it by the intuitionistic logic. However, the metamathematical question of proving this fact remains open. Even when a theorem is proved in this paper by the intuitionistic logic, we do not assert strictly that the theorem is proved by using only the intuitisnistic logic, because our system may contain the law of the excluded middle implicitly.

Next a comment about the notion of collections follows. Although this notion appears in each section, it is especially introduced for the purpose of defining in § 5 those symbols in a general and explicit way, which are introduced into a formal system usually in a recursive way. Namely, let $T`n$ be the class of $n$-tuples $\langle\alpha_1\cdots\alpha_n\rangle$ such that $\alpha_1, \ldots, \alpha_n$ are elements of the class $\Omega$ of all ordinal numbers which are sets, and let $T`\omega$ be the class of all such n-tuples. Then a function L is defined in such a way that $L`\langle\alpha_1\cdots\alpha_n\rangle = (\alpha_1\times\Omega\times\cdots\times\Omega)\cup\cdots\cup(\{\langle\alpha_1\cdots\alpha_i\rangle\}\times\alpha_{i+1}\times\Omega\times\cdots\times\Omega)\cup\cdots\cup(\{\langle\alpha_1\cdots\alpha_{n-1}\rangle\}\times\alpha_n)\subseteq T`n$. We can prove an induction of the form: $\forall u(u\in T`\omega.\ L`u\subseteq\mathscr{A}:\supset.\ u\in\mathscr{A})\rightarrow T`\omega\subseteq\mathscr{A}$ and by this induction we prove the existence of a recursive function which takes the form (cf. 5.47) $\cup\{F|\exists u(F\ \mathrm{Fnc}\ L`u.\ u\in T`\omega.\ \forall x[x\in L`u.\supset.F`x=\mathscr{G}`(L`x\upharpoonright F)])\}$, where $F$ Fnc $L`u$ means that F is a function over $L`u$ and $L`x\upharpoonright F$ denotes the F restricted on $L`x$. In general $L`x$ is a proper class and so is also $L`x\upharpoonright F$. So, in order that we define such a function for any free variable $\mathscr{G}$, we need a new type of variables which we call collection variables. The notion is also used in the definitions of T and L mentioned above.

An important application of transfinite induction is the existence proof of the function on ordinal numbers, that is, the proof of $\forall\mathscr{G}\exists\mathscr{F}[\mathscr{F}\ \mathrm{Fnc}\ \mathscr{O}.\ \forall\mathfrak{a}[\mathscr{F}`\mathfrak{a}=\mathscr{G}`(\mathfrak{a}\upharpoonright\mathscr{F})]]$ (See Gödel [1]) where $\mathscr{G}$ and $\mathscr{F}$ are collection variables and $\mathfrak{a}$ is a variable ranging over $\mathscr{O}$. We have not been able to prove this theorem by the intuitionistic logic but can prove by the classical logic. The corresponding statement within the scope of intuitionistic logic is this: (1) $\mathrm{Funct}\ (\mathscr{R}(\mathscr{G})).\ \forall\mathfrak{a}[\mathfrak{a}\subseteq\mathscr{D}(\mathscr{R}(\mathscr{G})).\ \mathfrak{a}\upharpoonright\mathscr{R}(\mathscr{G})\in\mathscr{D}(\mathscr{G}):\supset.\ \mathscr{R}(\mathscr{G})`\mathfrak{a}=\mathscr{G}`(\mathfrak{a}\upharpoonright\mathscr{R}(\mathscr{G}))].\ [\forall\mathfrak{a}(\mathfrak{a}\subseteq\mathscr{D}(\mathscr{R}(\mathscr{G})).\supset.\mathfrak{a}\upharpoonright\mathscr{R}(\mathscr{G})\in\mathscr{D}(\mathscr{G})).\supset.\ \mathscr{D}(\mathscr{R}(\mathscr{G}))=\mathscr{O}]$ where Funct $(\mathscr{R}(\mathscr{G}))$ means that $\mathscr{R}(\mathscr{G})$ is a function and $\mathscr{D}(\mathscr{R}(\mathscr{G}))$ denotes the domain of $\mathscr{R}(\mathscr{G})$ and $\mathscr{R}(\mathscr{G})$ is the collection $\cup\{F|\exists\mathfrak{y}(F\ \mathrm{Fnc}\ \mathfrak{y}.\ \forall\mathfrak{a}[\mathfrak{a}\in\mathfrak{y}\supset F`\mathfrak{a}=\mathscr{G}`(\mathfrak{a}\upharpoonright F)])\}$ in which $\mathfrak{y}$ is a variable ranging over $\mathscr{O}$. The statement is rather complicated and lacks uniqueness assertion. However, the functions on ordinal numbers used in this paper satisfy the premises of the theorem (1) and hence we obtain (2) $\mathscr{R}(\mathscr{G})$ Fnc $\mathscr{O}$. $\forall\mathfrak{a}[\mathscr{R}(\mathscr{G})\text{ ' }\mathfrak{a} = \mathscr{G}\text{ ' }(\mathfrak{a} \upharpoonright \mathscr{R}(\mathscr{G}))]$. This is an example to show a significance of the law of the excluded middle, i.e. that in virtue of the law we may obtain in some cases a general theorem even when some particular cases of a theorem but not the theorem itself is proved without the law of the excluded middle. In fact, addition, product and exponentiation of ordinal numbers etc. can be written by the above formula (2). E.g. if $\mathscr{I}t(\mathscr{S}c, \mathfrak{a})$ is $\mathscr{R}(\mathscr{G})$ for an appropriate $\mathscr{G}$ (cf. § 4), then $\mathscr{I}t(\mathscr{S}c, \mathfrak{a})$ Fnc $\mathscr{O}$. $\mathfrak{a}+0=\mathfrak{a}$. $\mathfrak{a}+\mathfrak{b}=(\mathfrak{a}+\mathfrak{b})'$. $\mathfrak{a}+\mathfrak{l}=\cup\{\mathfrak{a}+\mathfrak{y}\mid\mathfrak{y}\in\mathfrak{l}\}$ where $\mathfrak{l}$ is a variable over $\mathscr{O}\cap\mathscr{L}im$.

For any arbitrary ordinal numbers, we define $\mathfrak{a}<\mathfrak{b}$ as usual to be $\exists\mathfrak{y}(\mathfrak{a}+\mathfrak{y}=\mathfrak{b}.\ \mathfrak{y}\neq 0)$. $\mathfrak{a}\in\mathfrak{b}.\equiv.\mathfrak{a}\subset\mathfrak{b}.\equiv.\mathfrak{a}<\mathfrak{b}$ is a theorem within the classical logic but only $\mathfrak{a}<\mathfrak{b}\rightarrow\mathfrak{a}\in\mathfrak{b}$ and $\mathfrak{a}\in\mathfrak{b}\rightarrow\mathfrak{a}\subset\mathfrak{b}$ are theorems within the intuitionistic logic. Further, the law of the excluded middle plays an important role in the proofs of theorems concerning connexity and wellorder. Let Connex$(\mathscr{A})$ denote $\forall XY(X, Y\in\mathscr{A}.\supset.X\in Y\vee X=Y\vee Y\in X)$. By the classical logic, we prove $\forall\mathfrak{a}$ Connex$(\mathfrak{a})$ where $\mathfrak{a}$ is a variable over $\mathscr{O}$. In our theory, wellorder is defined in two senses. That $\mathscr{A}$ is wellordered in the weak sense means (Connex $(\mathscr{A})$. $\forall\mathscr{X}[0\in\mathscr{X}.\ \forall X(X\in\mathscr{A}\cap\mathscr{X}\supset X'\in\mathscr{X}).\ \forall X(X\in\mathscr{A}\cap\mathscr{L}im.\ X\subseteq\mathscr{X}:\supset.\ X\in\mathscr{X}):\supset.\ \mathscr{A}\subseteq\mathscr{X}]$), which is denoted by Word$(\mathscr{A})$, and that $\mathscr{A}$ is wellordered in the strong sense means Connex$(\mathscr{A})$. $\forall\mathscr{X}[\exists X(X\in\mathscr{X}).\ \mathscr{X}\subseteq\mathscr{A}:\supset.\ \exists Y(Y\in\mathscr{X}.\ Y\cap\mathscr{X}=0)]$. We need the law of the excluded middle to prove that, for an arbitrary ordinal number $\mathfrak{a}$, $\mathfrak{a}$ is wellordered in the weak sense and it is the same for wellorder in the strong sense.

Let $\mathfrak{a}$ and $\mathfrak{b}$ be elements of ω. Then, $\mathfrak{a}\in\mathfrak{b}.\equiv.\mathfrak{a}\subset\mathfrak{b}.\equiv.\mathfrak{a}<\mathfrak{b}$ is proved by the intuitionistic logic and the same for Connex(ω) and Word(ω). But we can hardly expect to prove by the intuitionistic logic that the ordinal number ω, and a fortiori an ordinal number greater than ω, is wellordered in the strong sense. Connex(ε_0) and Word(ε_0) are also proved by the intuitionistic logic where ε_0 is the first ε-number. Hence it seems that, for an ordinal number $\mathfrak{a}$ defined by some constructive way, we can prove Connex$(\mathfrak{a})$ and Word$(\mathfrak{a})$ by the intuitionistic iogic though $\forall\mathfrak{a}$ Connex$(\mathfrak{a})$ and $\forall\mathfrak{a}$ Word$(\mathfrak{a})$ are not obtained by the intuitionistic logic. The purpose of § 5 is to construct such a function.

Taking $\mathscr{G}$ appropriately, we obtain a recursive function F_1 (cf. 5.65-5.72) which has the properties: F_1 Fnc T‘ ω. F_1‘ $\langle 0 \cdots 0 \alpha_n \rangle = \alpha_n^{\alpha_n}$ $(0 < n)$. F_1‘ $\langle \alpha_1 \cdots \alpha_{k-1} \alpha'_k\ 0 \cdots 0\ \alpha_n \rangle = F_1$‘ $\langle \alpha_1 \cdots \alpha_{k-1} \alpha_k \alpha_n \cdots \alpha_n \rangle$ $(0 < k < n-1)$. F_1‘ $\langle \alpha_1 \cdots \alpha_{n-2} \alpha'_{n-1} \alpha_n \rangle = F_1$‘ $\langle \alpha_1 \cdots \alpha_{n-1} F_1$‘ $\langle \alpha_1 \cdots \alpha_n \rangle\rangle$ $(1 < n)$. F_1‘ $\langle \alpha_1 \cdots \alpha_{k-1} \alpha_k\ 0 \cdots 0 \alpha_n \rangle = \cup \{F_1$‘ $\langle \alpha_1 \cdots \alpha_{k-1} \theta\ 0 \cdots 0\ \alpha_n \rangle \mid \theta \in \alpha_k\}$ $(0 < k < n$, α_k is a limit number). F_1 satisfies the above mentioned requirement, since F_1‘ $\langle \alpha_1 \cdots \alpha_n \rangle$ is connected and wellordered in the weak sense if $\alpha_1 \cdots \alpha_n$ are all connected and wellordered in the weak sense respectively. The construction of F_1 and the proofs of the properties of F_1 are carried out within our system, using no metalogical numbers as the above n in n-tuple $\langle \alpha_1 \cdots \alpha_n \rangle$. No metalogical numbers are used not only in §5 but also throughout this paper.

Most proofs are omitted as far as they are analogus to those in the theory of ordinal numbers with set and class variables.

§1. Axiom system.

In this section we provide an axiomatic basis for our theory which contains not only axioms but also axiom schemes. Our primitive notions are set, class, collection, $\in$ and $\{\,|\,\}$ which appear in context as follows: Set(a), Cls(A), Coll(A), $A \in B$, $X \in \{U | \Phi\}$ etc. where the convention is made that $a, b, c, \ldots, h$; $u, v, \ldots, z$ are set variables, and that $A, B, C, \ldots, H$; $U, \ldots, Z$ are class variables, and that $\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots, \mathscr{H}$; $\mathscr{X}, \mathscr{Y}, \mathscr{Z}$ are collection variables.

Term, formula, free and bound occurrences are recursively defined as usual.

Hereafter it will be taken for granted in any context that Φ, Ψ are formulas and Γ, Δ are terms and Σ, Θ are variables. In some cases $\Phi \supset \Psi$ is replaced by $\Phi \to \Psi$ and $\neg(\Gamma \in \Delta)$ by $\Gamma \notin \Delta$ and $\neg(\Gamma = \Delta)$ by $\Gamma \neq \Delta$. We write $\{\Gamma | \Phi\}_\Theta$ in place of $\{\Gamma | \Phi\}_{\Theta,\Theta}$. When there is no fear of misunderstanding, we omit in $\{\Gamma | \Phi\}_\Theta$ and $\{\Gamma | \Phi\}_{\Theta,\Sigma}$ "Θ" or "Θ, Σ". Whenever a quantifier is followed by quantifiers of the same kind, then symbols $\forall$ or $\exists$ of quantifiers which follow are omitted.

We use a metamathematical notation $\stackrel{D}{=}$, which denotes that an expression on the left hand of this notation is an abbreviation for an expression on the right hand.

Def $\qquad \Gamma = \Delta \stackrel{D}{=} \forall U(U \in \Gamma \equiv U \in \Delta)$

where U occurs neither in Γ nor in Δ and $\equiv$ is defined in terms of "$\cdot$" and "$\supset$" as usual.

The unique existence $\exists!\Sigma\Phi(\Sigma)$ where $\Phi(\Sigma)$ is a formula, is also defined as usual.

Now the axiom system is divided into four parts.

(A)

1. $\mathrm{Cls}(a)$. 2. $\mathrm{Coll}(A)$. 3. $\mathrm{Coll}(\{\Gamma|\Phi\}_{\Theta,\Sigma})$. 4. $\mathscr{A}\in\mathscr{B}\rightarrow\mathrm{Cls}(\mathscr{A})$.

5. $A\in a\rightarrow\mathrm{Set}(A)$.

6. Let $\Phi(\mathscr{A})$ be a formula and $\Phi(\mathscr{B})$ be the result of substituting $\mathscr{B}$ for each free occurrence (if any) of $\mathscr{A}$ in $\Phi(\mathscr{A})$ where we assume that the substitution causes no confusion of bound variables. Then

$$\mathscr{A}=\mathscr{B}\rightarrow\Phi(\mathscr{A})\supset\Phi(\mathscr{B}).$$

(B) Let Π be a class or a class variable or a set variable and Γ be a term in which Θ and Σ occur free.

$$\Pi\in\{\Gamma|\Phi\}_{\Theta,\Sigma}\equiv\exists\Theta\Sigma(\Pi=\Gamma.\ \Phi).$$

Now let Π be a class variable and $\Xi(\Pi)$ be an expression consisting of Lation letters, which we denote by Ξ, followed by (Π) and denoting a formula in which Π is only one variable such that there are free occurrences in $\Xi(\Pi)$. Let Ξ^* be the result of replacing the first letter of Ξ by the corresponding script font. Then we define Ξ^* to be an abbreviation for $\{\Pi|\Xi(\Pi)\}_\Pi$.

Example. $\mathscr{S}et\overset{D}{=}\{A|\mathrm{Set}(A)\}_A$. $\mathscr{C}ls\overset{D}{=}\{A|\mathrm{Cls}(A)\}_A$, which represents the collection of all the classes. $\mathscr{L}im\overset{D}{=}\{A|\mathrm{Lim}(A)\}_A$. $\mathscr{A}\subseteq\mathscr{B}$, $\{\mathscr{A}\mathscr{B}\}$, $\{\mathscr{A}\}$, $\langle\mathscr{A}\mathscr{B}\rangle$, $\mathrm{Un}(\mathscr{A})$, 0, $\cup\mathscr{A}$, $\mathscr{A}\cup\mathscr{B}$, $\mathscr{A}'$ are defined as usual. $\mathscr{A}\subset\mathscr{B}$ is defined as $\mathscr{A}\subseteq\mathscr{B}.\ \exists U(U\in\mathscr{B}.\ U\notin\mathscr{A})$ and $\mathrm{Lim}(\mathscr{A})$ as $\mathscr{A}=\cup\mathscr{A}.\ \exists U(U\in\mathscr{A})$.

(C)

1. $\exists x[0\in x.\ \forall u(u\in x\supset u'\in x)]$.

2. $\exists X[0\in X.\ \forall u(u\in X\supset u'\in X).\ \forall u(u\in\mathscr{L}im.\ u\subseteq X:\supset.\ u\in X)]$.

3. $\forall ab\exists x\forall u[u\in x.\equiv.u=a\vee u=b]$.

4. $\forall AB\exists X\forall U[U\in X.\equiv.U=A\vee U=B]$.

5. $\forall x\exists y\forall uv[u\in v\in x\supset u\in y]$. 6. $\forall X\exists Y\forall UV[U\in V\in X\supset U\in Y]$.

7. $\forall x\exists y\forall u[u\subseteq x.\supset.u\in y]$. 8. $\forall X\exists Y\forall U[U\subseteq X.\supset.U\in Y]$.

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9. $\forall x\mathscr{A}[\mathrm{Un}(\mathscr{A}) \supset \exists y \forall u[u \in y \equiv \exists v(v \in x.\ \langle vu\rangle \in \mathscr{A})]]$.

10. $\forall X\mathscr{A}[\mathrm{Un}(\mathscr{A}) \supset \exists Y \forall U[U \in Y \equiv \exists V(V \in X.\ \langle VU\rangle \in \mathscr{A})]]$.

Def $\mathscr{D}es(\mathscr{A}) \overset{D}{=} \{U \mid \forall \mathscr{X}(\mathscr{A} \subseteq \mathscr{X}.\ \forall X(X \in \mathscr{X} .\supset. X \subseteq \mathscr{X}) : \supset.\ U \in \mathscr{X})\}$

1.01. $\mathscr{A} \subseteq \mathscr{D}es(\mathscr{A})$. 1.02. $X \in \mathscr{D}es(\mathscr{A}) \rightarrow X \subseteq \mathscr{D}es(\mathscr{A})$.

(D) $\mathscr{A} \notin \mathscr{D}es(\mathscr{A})$.

1.03. $a \notin a$. 1.04. $A \notin A$. 1.05. $\neg(a \in b \in a)$. 1.06. $\neg(A \in B \in A)$.
1.07. $\neg(A \in B \in C \in A)$.

§2. Sets and classes and collections.

2.01. $\{ab\} \in \mathscr{S}et$. 2.02. $\langle ab\rangle \in \mathscr{S}et$. 2.03. $\{AB\} \in \mathscr{C}ls$.
2.04. $\langle AB\rangle \in \mathscr{C}ls$. 2.05. $\langle AB\rangle = \langle CD\rangle \rightarrow A = C.\ B = D$.

$\mathscr{A} \cap \mathscr{B}$, $\mathscr{D}(\mathscr{A})$, $\mathscr{W}(\mathscr{A})$, $-\mathscr{A}$, $\mathscr{A} \times \mathscr{B}$, $\mathscr{B} \upharpoonright \mathscr{A}$, $\mathscr{A}``\mathscr{B}$, $\mathscr{A}^{-1}$ are defined as usual. $\mathscr{I}d$ is the collection $\{\langle YZ\rangle \mid Y = Z\}$.

2.06. $(\mathscr{A} \cup \mathscr{B}) \times \mathscr{C} = (\mathscr{A} \times \mathscr{C}) \cup (\mathscr{B} \times \mathscr{C})$.
2.07. $\mathscr{A} \subseteq \mathscr{B} \rightarrow \mathscr{A} \upharpoonright (\mathscr{B} \upharpoonright \mathscr{C}) = \mathscr{A} \upharpoonright \mathscr{C}$.
2.08. $\mathrm{Un}(\mathscr{A}).\ \mathscr{A}``a \subseteq \mathscr{S}et \rightarrow \mathscr{A}``a \in \mathscr{S}et$.
2.09. $\mathrm{Un}(\mathscr{A}) \rightarrow \mathscr{A}``X \in \mathscr{C}ls$. 2.10. $a \cap \mathscr{A} \in \mathscr{S}et$.
2.11. $X \cap \mathscr{A} \in \mathscr{C}ls$. 2.12. $\mathscr{A} \subseteq a \rightarrow \mathscr{A} \in \mathscr{S}et$. 2.13. $0 \in \mathscr{S}et$.
2.14. $\mathscr{A} \subseteq X \rightarrow \mathscr{A} \in \mathscr{C}ls$.

Def $\mathscr{F}`\mathscr{A} \overset{D}{=} \{U \mid \exists X(U \in X.\ \langle \mathscr{A}X\rangle \in \mathscr{F})$.
$\forall YZ(\langle \mathscr{A}Y\rangle, \langle \mathscr{A}Z\rangle \in \mathscr{F} : \supset.\ Y = Z)\}$.

2.15. $\mathscr{A} = \mathscr{B} \rightarrow \mathscr{F}`\mathscr{A} = \mathscr{F}`\mathscr{B}$. 2.16. $A \in \mathscr{D}(\mathscr{F}) \rightarrow \mathscr{F}`A \in \mathscr{C}ls$.

Proof. By $A \in \mathscr{D}(\mathscr{F})$, there is a class B such that $\langle AB\rangle \in \mathscr{F}$. Let $U \in \mathscr{F}`A$. Then there is a class $C$ such that $U \in C.\ \langle AC\rangle \in \mathscr{F}$. By $\forall YZ(\langle AY\rangle, \langle AZ\rangle \in \mathscr{F} : \supset.\ Y = Z)$, $B = C$. Hence $U \in B$. So we obtain $\mathscr{F}`A \subseteq B$. Accordingly, by 2.14, $\mathscr{F}`A \in \mathrm{Cls}$.

2.17. $\exists!X(\langle AX\rangle \in \mathscr{F}) \rightarrow \langle A, \mathscr{F}`A\rangle \in \mathscr{F}$.

Proof. By the premise, $\exists X(\langle AX\rangle \in \mathscr{F})$. Let $\langle AB\rangle \in \mathscr{F}$. In an analogous fashion to the proof of 2.16, we obtain $\mathscr{F}`A \subseteq B$. Now let $U \in B$. Then $U \in B.\ \langle AB\rangle \in \mathscr{F}$. Hence $\exists X(U \in X.\ \langle AX\rangle \in \mathscr{F})$. So we obtain $U \in \mathscr{F}`A$, using the premise. Hence $B \subseteq \mathscr{F}`A$. Accordingly, $\mathscr{F}`A = B$ and hence

$\langle A, \mathscr{F}``A\rangle \in F$.

2.18. $\neg\exists!X(\langle AX\rangle \in \mathscr{F}) \to \mathscr{F}`A = 0$.

2.19. $A \in \mathscr{D}(\mathscr{F}) \cup -\mathscr{D}(\mathscr{F}) \to \mathscr{F}`A \in \mathscr{C}\text{ls}$.

Def $\mathscr{P}ow(\mathscr{A}) \overset{D}{=} \{U \mid U \subseteq \mathscr{A}\}$.

2.20. $\mathscr{P}ow(a) \in \mathscr{S}et$. 2.21. $\mathscr{P}ow(X) \in \mathscr{C}ls$. 2.22. $\cup a \in \mathscr{S}et$.

2.23. $\cup X \in \mathscr{C}ls$. 2.24. $a \cup b \in \mathscr{S}et$. 2.25. $a' \in \mathscr{S}et$.

2.26. $X \cup Y \in \mathscr{C}ls$. 2.27. $X' \in \mathscr{C}ls$. 2.28. $X' = Y' \to X = Y$.

Def $\mathscr{P}_1 \overset{D}{=} \{\langle XY\rangle \mid \exists AB(X = \langle AB\rangle.\ Y = A) \vee (\neg\exists AB(X = \langle AB\rangle).\ Y = X)\}$.

Def $\mathscr{P}_2 \overset{D}{=} \{\langle XY\rangle \mid \exists AB(X = \langle AB\rangle.\ Y = B) \vee (\neg\exists AB(X = \langle AB\rangle).\ Y = X)\}$.

Def $\mathscr{P}_3 \overset{D}{=} \{\langle XY\rangle \mid \exists AB(X = \langle AB\rangle.\ Y = \langle BA\rangle)\}$.

Def $\text{Rel}(\mathscr{A}) \overset{D}{=} \mathscr{A} \subseteq \mathscr{C}ls \times \mathscr{C}ls$. Def $\text{Funct}(\mathscr{A}) \overset{D}{=} \text{Un}(\mathscr{A}).\ \text{Rel}(\mathscr{A})$.

Def $\mathscr{F}$ Fnc $\mathscr{D} \overset{D}{=} \text{Funct}(\mathscr{F}).\ \mathscr{D}(\mathscr{F}) = \mathscr{D}$.

Def $\mathscr{D}o \overset{D}{=} \{\langle XY\rangle \mid Y = \mathscr{D}(X)\}$. Def $\mathscr{W} \overset{D}{=} \{\langle XY\rangle \mid Y = \mathscr{W}(X)\}$.

2.29. $X \in \mathscr{D}(\mathscr{A} \upharpoonright \mathscr{F}).\ \text{Funct}(\mathscr{F}) \to (\mathscr{A} \upharpoonright \mathscr{F})`X = \mathscr{F}`X \in \mathscr{C}ls$.

2.30. $\mathscr{D}(X) \subseteq \mathscr{P}_1``X$. 2.31. $\mathscr{W}(X) \subseteq \mathscr{P}_2``X$. 2.32. $\mathscr{A}^{-1} = \mathscr{P}_3``\mathscr{A}$.

2.33. $\mathscr{D}(a) \in \mathscr{S}et$.

Proof. First we prove $\text{Un}(\mathscr{P}_1)$. Let $\langle XY\rangle, \langle XZ\rangle \in \mathscr{P}_1$. Assume $\exists AB(X = \langle AB\rangle.\ Y = A)$. Then $\exists AB(X = \langle AB\rangle.\ Z = A)$. Hence, by 2.05, $Y = Z$. Assume $\neg\exists AB(X = \langle AB\rangle).\ Y = X$. Then $\neg\exists AB(X = \langle AB\rangle).\ Z = X$. Hence $Y = Z$. Accordingly we obtain $\text{Un}(\mathscr{P}_1)$. Now let $U \in \mathscr{P}_1``a$. Then there is an X such that $\langle XU\rangle \in \mathscr{P}_1.\ X \in a$. Assume $\exists AB(X = \langle AB\rangle.\ U = A)$. Let $X = \langle AB\rangle.\ U = A$. By $X \in a$ and $A\,5$, $\langle AB\rangle \in \mathscr{S}et$. $A \in \{A\} \in \langle AB\rangle$ and hence, by $A\,5$, $A \in \mathscr{S}et$. So $U \in \mathscr{S}et$. Assume $\neg\exists AB(X = \langle AB\rangle).\ U = X$. By $X \in a$ and $A\,5$, $U \in \mathscr{S}et$. Hence $\mathscr{P}_1``a \subseteq \mathscr{S}et$ and so, by 2.08, $\mathscr{P}_1``a \in \mathscr{S}et$. By 2.30, $\mathscr{D}o(a) \subseteq \mathscr{P}_1``a$ and hence, by 2.12, $\mathscr{D}o(a) \in \mathscr{S}et$.

2.34. $\mathscr{D}(X) \in \mathscr{C}ls$. 2.35. $\mathscr{D}o`X = \mathscr{D}(X)$. 2.36. $\mathscr{D}o`a \in \mathscr{S}et$.

2.37. $\mathscr{D}o`X \in \mathscr{C}ls$. 2.38. $\mathscr{W}(a) \in \mathscr{S}et$. 2.39. $\mathscr{W}(X) \in \mathscr{C}ls$.

2.40. $\mathscr{W}`X = \mathscr{W}(X)$. 2.41. $\mathscr{W}`a \in \mathscr{S}et$. 2.42. $\mathscr{W}`X \in \mathscr{C}ls$.

2.43. $a \times b \in \mathscr{S}et$. 2.44. $A \times B \in \mathscr{C}ls$.

2.45. $A \times B \in \mathscr{C}ls \to A, B \in \mathscr{C}ls$.

2.46. $(\cup\{\mathscr{F}`U \mid U \in \mathscr{A}\}) \times B = \cup\{\mathscr{F}`U \times B \mid U \in \mathscr{A}\}$

2.47. $\mathscr{F}$ Fnc $a.\ \mathscr{F}``a \subseteq \mathscr{S}et \to \mathscr{F} \in \mathscr{S}et$.

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2.48. $\mathrm{Funct}(\mathscr{F})$. $\mathscr{D}(\mathscr{F}) \in \mathscr{C}\mathit{ls} \to \mathscr{F} \in \mathscr{C}\mathit{ls}$.

2.49. $\mathrm{Un}(\mathscr{F})$. $\mathscr{F}\text{“}a \subseteq \mathscr{S}\mathit{et} \to a \upharpoonright \mathscr{F} \in \mathscr{S}\mathit{et}$.

2.50. $\mathrm{Un}(\mathscr{F}) \to X \upharpoonright \mathscr{F} \in \mathscr{C}\mathit{ls}$.

Def $\mathrm{Pro\,Cls}(A) = \neg\mathrm{Set}(A)$. Def $\mathrm{ProColl}(\mathscr{A}) = \neg\mathrm{Cls}(\mathscr{A})$.

2.51. $\mathrm{ProColl}(\mathscr{C}\mathit{ls})$. 2.52. $\mathscr{F}$ Fnc $\mathscr{D}$. $\mathrm{ProColl}(\mathscr{D}) \to \mathrm{ProColl}(\mathscr{F})$.

2.53. F Fnc D. $D \in \mathscr{P}\mathit{roCls} \to F \in \mathscr{P}\mathit{roCls}$.

§ 3. Ordinal numbers.

Def $\mathscr{O} \overset{D}{=} \{U \mid \forall \mathscr{X}(0 \in \mathscr{X}.\ \forall X(X \in \mathscr{X} \supset X' \in \mathscr{X}).$
$\forall X(X \in \mathscr{L}\mathit{im}.\ X \subseteq \mathscr{X} :\supset.\ X \in \mathscr{X}) :\supset.\ U \in \mathscr{X})\}$.

Def $\mathscr{O}(\mathscr{A}) \overset{D}{=} \mathscr{A} \in \mathscr{O} \vee \mathscr{A} = \mathscr{O}$.

Elements of $\mathscr{O}$ are called *ordinal numbers* and $\mathscr{A}$ such that $\mathscr{O}(\mathscr{A})$ is called *ordinal*.

Def $\mathrm{Induc}(\mathscr{A}) \overset{D}{=} \forall \mathscr{X}[0 \in \mathscr{X}.\ \forall X(X \in \mathscr{A} \cap \mathscr{X} \supset X' \in \mathscr{X}).$
$\forall X(X \in \mathscr{A} \cap \mathscr{L}\mathit{im}. X \subseteq \mathscr{X} :\supset.\ X \in \mathscr{X}) :\supset.\ \mathscr{A} \subseteq \mathscr{X}]$.

Def $\mathrm{Comp}(\mathscr{A}) \overset{D}{=} \cup \mathscr{A} \subseteq \mathscr{A}$.

$\mathscr{A}$ such that $\mathrm{Induc}(\mathscr{A})$ or $\mathrm{Comp}(\mathscr{A})$ is called *inductive* or *complete*. Now we make the convention that $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}, \ldots, \mathfrak{h}$; $\mathfrak{u}, \ldots, \mathfrak{z}$ are variables whose range is $\mathscr{O}$, and that $\mathfrak{l}$ is a variable whose range is $\mathscr{O} \cap \mathscr{L}\mathit{im}$.

3.01. $0 \in \mathscr{O}$. 3.02. $\mathfrak{a}' \in \mathscr{O}$. 3.03. $A \in \mathscr{L}\mathit{im}. A \subseteq \mathscr{O} \to A \in \mathscr{O}$.

3.04. $\mathrm{Induc}(\mathscr{O})$.

3.05. There holds one and only one of $\mathfrak{a} = 0$, $\exists \mathfrak{b}(\mathfrak{a} = \mathfrak{b}')$ and $\mathfrak{a} \in \mathscr{L}\mathit{im}$.

3.06. $\mathscr{O}(\mathscr{A}) \to \mathrm{Comp}(\mathscr{A})$. 3.07. $\mathfrak{a} \in \mathfrak{b} \to \mathfrak{a} \subset \mathfrak{b}$.

3.08. $\mathscr{O}(\mathscr{A}) \to \mathrm{Induc}(\mathscr{A})$.

3.09. $\mathrm{Induc}(\mathscr{A}) \to \forall \mathscr{X}[\forall X(X \in \mathscr{A}.\ X \subseteq \mathscr{X} :\supset.\ X \in \mathscr{X}) .\supset. \mathscr{A} \subseteq \mathscr{X}]$.

3.10. $\mathscr{O}(\mathscr{A}) \to \forall \mathscr{X}[\forall X(X \in A.\ X \subseteq \mathscr{X} :\supset.\ X \in \mathscr{X}) .\supset. \mathscr{A} \subseteq \mathscr{X}]$.

3.11. If $U \in \mathscr{A}$. $\mathrm{Induc}(\mathscr{A})$, then holds one and only one of $U = 0$, $\exists X(U = X'.\ X \in \mathscr{A})$ and $U \in \mathscr{L}\mathit{im}$.

3.12. $\mathfrak{a} = 0 \vee 0 \in \mathfrak{a}$. 3.13. $0 \in \mathfrak{l}$. 3.14. $\mathfrak{a} \in \mathfrak{b} \equiv \mathfrak{a}' \in \mathfrak{b}'$.

Def $\mathscr{C}\mathit{ompar} \overset{D}{=} \{\langle XY \rangle \mid X \in Y \vee X = Y \vee Y \in X\}$.

Def $\mathrm{Connex}(\mathscr{A}) \overset{D}{=} \forall XY(X, Y \in \mathscr{A} \supset \langle XY \rangle \in \mathscr{C}\mathit{ompar})$.

Def $\mathscr{A} \underline{\in} \mathscr{B} \overset{D}{=} \mathscr{A} \in \mathscr{B} \vee \mathscr{A} = \mathscr{B}$.

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3.15. $\langle \mathfrak{a}\mathfrak{b}\rangle \in \mathscr{C}\textit{ompar} \equiv \langle \mathfrak{a}\mathfrak{b}'\rangle \in \mathscr{C}\textit{ompar} \equiv \langle \mathfrak{a}'\mathfrak{b}'\rangle \in \mathscr{C}\textit{ompar}$.

3.16. $\mathscr{O}(\mathscr{A}) \rightarrow \mathrm{Lim}(\mathscr{A}) \equiv [\forall X(X \in \mathscr{A} \supset X' \in \mathscr{A}).\ \exists U(U \in \mathscr{A})]$.

3.17. $\mathrm{Lim}(\mathscr{O})$. 3.18. $\mathrm{ProColl}(\mathscr{O})$.

*3.19. $\mathrm{Induc}(\mathscr{A}) \rightarrow \forall \mathscr{X}[\exists X(X \in \mathscr{X} \cap \mathscr{A}) \supset \exists Y(Y \in \mathscr{X} \cap \mathscr{A}.\ Y \cap \mathscr{X} = 0)]$.

Proof. Put $\mathscr{B} \overset{D}{=} \{X | X \in \mathscr{X} \supset \exists Y(Y \in \mathscr{X} \cap \mathscr{A}. Y \cap \mathscr{X} = 0)\}$. Assume $X \in \mathscr{A}$. $X \subseteq \mathscr{B}$. Let $X \in \mathscr{X}$. If $X \cap \mathscr{X} = 0$, then $\exists Y(Y \in \mathscr{X} \cap \mathscr{A}.\ Y \cap \mathscr{X} = 0)$. Hence we assume $X \cap \mathscr{X} \neq 0$. Take $Y \in X \cap \mathscr{X}$. By $X \subseteq \mathscr{B}$, $Y \in \mathscr{B}$. Hence, by $Y \in \mathscr{X}$, $\exists Y(Y \in \mathscr{X} \cap \mathscr{A}.\ Y \cap \mathscr{X} = 0)$. So $X \in \mathscr{X} \supset \exists Y(Y \in \mathscr{X} \cap \mathscr{A}.\ Y \cup \mathscr{X} = 0)$. That is, $X \in \mathscr{B}$. Accordingly $\forall X(X \in \mathscr{A}.\ X \subseteq \mathscr{B} : \supset.\ X \in \mathscr{B})$. By $\mathrm{Induc}(\mathscr{A})$ and 3.09, $\mathscr{A} \subseteq \mathscr{B}$. Hence, by the rules of inference, $\exists X(X \in \mathscr{X} \cap \mathscr{A}) \supset \exists Y(Y \in \mathscr{X} \cup \mathscr{A}.\ Y \cap \mathscr{X} = 0)$. Since $\mathscr{X}$ is arbitrary, we obtain the theorem.

*3.20. $\mathrm{Induc}(\mathscr{A}) \rightarrow \forall \mathscr{X}[\exists X(X \in \mathscr{X}).\ \mathscr{X} \subseteq \mathscr{A} : \supset.\ \exists Y(Y \in \mathscr{X}.\ Y \cap \mathscr{X} = 0)]$.

3.21. $\langle \mathfrak{a}\mathfrak{b}\rangle \in \mathscr{C}\textit{ompar} \rightarrow \mathfrak{a} \in \mathfrak{b} \equiv \mathfrak{a} \subset \mathfrak{b}$.

*3.22. $\mathscr{O}(\mathscr{A}) \rightarrow \mathrm{Connex}(\mathscr{A})$.

Proof. We prove $\mathrm{Connex}(\mathscr{O})$. Then, by this and $\mathrm{Comp}(\mathscr{O})$, $\mathscr{A} \in \mathscr{O} \rightarrow \mathrm{Connex}(\mathscr{A})$. Put $\mathscr{B} \overset{D}{=} \{\mathfrak{b} | \forall \mathfrak{a}(\langle \mathfrak{a}\mathfrak{b}\rangle \in \mathscr{C}\textit{ompar})\}$. Assume $\mathfrak{b} \subseteq \mathscr{B}$. If $\mathfrak{a} \in \mathfrak{b}$, then $\langle \mathfrak{a}\mathfrak{b}\rangle \in \mathscr{C}\textit{ompar}$. Then we may assume $\mathfrak{a} \notin \mathfrak{b}$. Let $\mathfrak{c} \in \mathfrak{b}$. Then $\mathfrak{c} \in \mathscr{B}$ and hence $\langle \mathfrak{a}\mathfrak{c}\rangle \in \mathscr{C}\textit{ompar}$. If $\mathfrak{a} \leqq \mathfrak{c}$, then $\mathfrak{a} \in \mathfrak{b}$, which is contrary to $\mathfrak{a} \notin \mathfrak{b}$. Hence $\mathfrak{c} \in \mathfrak{a}$. From this, $\mathfrak{b} \subseteq \mathfrak{a}$ and hence $\mathfrak{b} \subset \mathfrak{a} \vee \mathfrak{b} = \mathfrak{a}$. If $\mathfrak{b} = \mathfrak{a}$, then $\langle \mathfrak{a}\mathfrak{b}\rangle \in \mathscr{C}\textit{ompar}$. Assume $\mathfrak{b} \subset \mathfrak{a}$. Then $\exists X(X \in \mathfrak{a} \cap -\mathfrak{b})$. Hence, by *3.20, $\exists Y(Y \in \mathfrak{a} \cap -\mathfrak{b}.\ Y \cap (\mathfrak{a} \cap -\mathfrak{b}) = 0)$. Let $Y \in \mathfrak{a} \cap -\mathfrak{b}$. $Y \cap (\mathfrak{a} \cap -\mathfrak{b}) = 0$. We prove $\mathfrak{b} = Y$. $Y \in \mathscr{O}$. Let $\mathfrak{c} \in \mathfrak{b}$. Then $\mathfrak{c} \in \mathscr{B}$ and hence $\langle \mathfrak{c}Y\rangle \in \mathscr{C}\textit{ompar}$. If $Y \leqq \mathfrak{c}$, then $Y \in \mathfrak{b}$, which is contrary to $Y \in (\mathfrak{a} \cap -\mathfrak{b})$. Hence $\mathfrak{c} \in Y$. Since $\mathfrak{c}$ is arbitrary, $\mathfrak{b} \subseteq Y$. Conversely, let $\mathfrak{c} \in Y$. By $Y \in \mathfrak{a} \cap -\mathfrak{b}$, $\mathfrak{c} \in \mathfrak{a}$. By $Y \cap (\mathfrak{a} \cap -\mathfrak{b}) = 0$, $\mathfrak{c} \notin \mathfrak{a} \cap -\mathfrak{b}$. So $\mathfrak{c} \in \mathfrak{b}$. Hence $Y \subseteq \mathfrak{b}$. Accordingly $\mathfrak{b} = Y \in \mathfrak{a}$. Hence $\langle \mathfrak{a}\mathfrak{b}\rangle \in \mathscr{C}\textit{ompar}$. Since $\mathfrak{a}$ is arbitrary, $\forall \mathfrak{a}(\langle \mathfrak{a}\mathfrak{b}\rangle \in \mathscr{C}\textit{ompar})$ and hence $\mathfrak{b} \in \mathscr{B}$. So $\forall \mathfrak{b}(\mathfrak{b} \subseteq \mathscr{B} . \supset . \mathfrak{b} \in \mathscr{B})$ and, by 3.09, $\mathscr{O} \subseteq \mathscr{B}$. Namely, $\forall \mathfrak{a}\mathfrak{b}(\langle \mathfrak{a}\mathfrak{b}\rangle \in \mathscr{C}\textit{ompar})$, which implies $\mathrm{Connex}(\mathscr{O})$.

*3.23. $\mathfrak{a} \in \mathfrak{b} \equiv \mathfrak{a} \subset \mathfrak{b}$.

Def $\mathrm{Word}(\mathscr{A}) \overset{D}{=} \mathrm{Connex}(\mathscr{A}).\ \mathrm{Induc}(\mathscr{A})$.

Def $\mathrm{Word}^*(\mathscr{A}) \overset{D}{=} \mathrm{Connex}(\mathscr{A}).\ \forall \mathscr{X}[\exists X(X \in \mathscr{X}).\ \mathscr{X} \subseteq \mathscr{A}$

$: \supset.\ \exists Y(Y \in \mathscr{X}.\ Y \cap \mathscr{X} = 0)]$

Def $\mathscr{O}_1(\mathscr{A}) \overset{D}{=} \mathrm{Word}(\mathscr{A}).\ \mathrm{Comp}(\mathscr{A})$. Def $\mathscr{C}l \overset{D}{=} \{\langle XY\rangle | Y \in X\}$.

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Def $\mathscr{S}eg_{\mathscr{X}}(\mathscr{A}, U) \overset{D}{=} \mathscr{A} \cap (\mathscr{X}``\{U\})$.

Def $\mathscr{O}_N(\mathscr{A}) \overset{D}{=} \text{Word}^*(\mathscr{A}).\ \forall U(U \in \mathscr{A} .\supset. U = \mathscr{S}eg_{\mathscr{E}l}(\mathscr{A}, U))$.

$\mathscr{A}$ is called ***well-ordered by*** $\in$ ***in the weak sense*** or ***in the strong sense*** when Word($\mathscr{A}$) or Word*($\mathscr{A}$). $\mathscr{O}_I(\mathscr{A})$ is a definition of ordinals where Induc($\mathscr{A}$) is a component which features the definition. $\mathscr{O}_N(\mathscr{A})$ is a definition of ordinals corresponds to von Neumann's way. Next we prove the equivalences of these definitions, using essentially the law of the excluded middle.

*3.24. $\mathscr{O}(\mathscr{A}) \rightarrow \mathscr{O}_I(\mathscr{A})$. *3.25. Word($\mathscr{A}$) → Word*($\mathscr{A}$).

3.26. Comp($\mathscr{A}$) $\equiv \forall U(U \in \mathscr{A} .\supset. U = \mathscr{S}eg_{\mathscr{E}l}(\mathscr{A}, U))$.

*3.27. $\mathscr{O}_I(\mathscr{A}) \rightarrow \mathscr{O}_N(\mathscr{A})$.

*3.28. $\mathscr{O}_N(\mathscr{A}).\ \mathscr{B} \subset \mathscr{A}.\ \text{Comp}(\mathscr{B}) \rightarrow \mathscr{B} \in \mathscr{A}$.

Proof. By $\mathscr{O}_N(\mathscr{A})$, Word*($\mathscr{A}$). By $\mathscr{B} \subset \mathscr{A}$, $\exists U(U \in \mathscr{A} \cap -\mathscr{B})$. Hence, by the definition of Word*($\mathscr{A}$), $\exists Y(Y \in \mathscr{A} \cap -\mathscr{B}.\ Y \cap (\mathscr{A} \cap -\mathscr{B}) = 0)$. Let $Y \in \mathscr{A} \cap -\mathscr{B}.\ Y \cap (\mathscr{A} \cap -\mathscr{B}) = 0$. We can prove $\mathscr{B} = Y$ in an analogous fashion to the proof of *3.22. $Y \in \mathscr{A}$. Hence $\mathscr{B} \in \mathscr{A}$.

*3.29. $\mathscr{A} \subseteq \mathscr{O}_N.\ \exists X(X \in \mathscr{A}) \rightarrow \exists Y(Y \in \mathscr{A}.\ Y \cap \mathscr{A} = 0)$.

Proof. Assume $\neg \exists Y(Y \in \mathscr{A}.\ Y \cap \mathscr{A} = 0)$. Put $\mathscr{B} \overset{D}{=} \{U | \forall X(X \in \mathscr{A} \supset U \in X)\}$. By the premise, there is an X such that $X \in \mathscr{A}$. Then $\mathscr{B} \subseteq X$. Hence, by 2.14, $\mathscr{B} \in \mathscr{C}ls$. Assume $\mathscr{B} = X$. Hence $\mathscr{B} \in A$. Let $\mathscr{B} \cap \mathscr{A} \neq 0$. Then there is a C such that $C \in \mathscr{B} \cap \mathscr{A}$. By the definition of $\mathscr{B}$ and $C \in \mathscr{A}$, we obtain $C \in C$, which is contrary to 1.04. Hence $\mathscr{B} \cap \mathscr{A} = 0$. Since $\mathscr{B}$ is a class, $\exists Y(Y \in \mathscr{A}.\ Y \cap \mathscr{A} = 0)$, which is contrary to the first assumption. Hence $\mathscr{B} \neq X$. By $\mathscr{B} \subseteq X$, $\mathscr{B} \subset X$. By $X \in \mathscr{A}$ and the premise, $X \in \mathscr{O}_N$. We prove $\mathscr{B} \in \mathscr{C}omp$. Let $D \in C \in \mathscr{B}$ and $Y \in \mathscr{A}$. By the definition of $\mathscr{B}$, $C \in Y$. By $Y \in \mathscr{A}$ and $\mathscr{A} \subseteq \mathscr{O}_N$, $Y \in \mathscr{O}_N$ and hence $Y \in \mathscr{C}omp$. So $D \in Y$ and hence $\forall Y(Y \in \mathscr{A} \supset D \in Y)$. Therefore $D \in \mathscr{B}$. That is $D \in C \in \mathscr{B} \supset D \in \mathscr{B}$. Since C, D are arbitrary, $\mathscr{B} \in \mathscr{C}omp$. Hence $\mathscr{O}_N(X).\ \mathscr{B} \subset X.\ \text{Comp}(\mathscr{B})$. By *3.28, $\mathscr{B} \in X$. Hence $\forall X(X \in \mathscr{A} \supset \mathscr{B} \in X)$ and so $\mathscr{B} \in \mathscr{B}$ which is a contradiction. Accordingly $\exists Y(Y \in \mathscr{A}.\ Y \cap \mathscr{A} = 0)$.

3.30. $\mathscr{O}_N(\mathscr{A}).\ U \in \mathscr{A} \rightarrow U \in \mathscr{O}_N$.

Proof. By $\mathscr{O}_N(\mathscr{A})$, Word*($\mathscr{A}$). Comp($\mathscr{A}$). By $U \in \mathscr{A}$, $U \in \mathscr{W}ord^*$. Let $X \in Y \in U$. By Comp($\mathscr{A}$), $X \in \mathscr{A}$. By Connex($\mathscr{A}$), $\langle UX \rangle \in \mathscr{C}ompar$. If $U \leqq X$, then $X \in Y \in U \in X$ or $X \in U \in X$, which is contrary to 1.07 or 1.06.

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Hence $X \in U$. Since X, Y are arbitrary, we obtain $U \in \mathscr{C}omp$. Thence $U \in \mathscr{O}_N$, by 3.26.

*3.31. $\mathscr{O}_N(\mathscr{A}) \to \mathscr{A} = 0 \vee \exists X(\mathscr{A} = X'.\ X \in \mathscr{O}_N) \vee \mathrm{Lim}(\mathscr{A})$.

Proof. Assume $\mathscr{A} \neq 0$. $\neg \exists X(\mathscr{A} = X'.\ X \in \mathscr{O}_N)$. Let $B \in \mathscr{A}$. By $\mathscr{O}_N(\mathscr{A})$ and 3.30, we obtain $B \in \mathscr{O}_N$. Also, by the premise, $\mathrm{Comp}(\mathscr{A})$ and hence $B' \subseteq \mathscr{A}$. Assume $B' = \mathscr{A}$. Then $\exists X(\mathscr{A} = X'.\ X \in \mathscr{O}_N)$, which is contrary to the assumption. Hence $B' \subset \mathscr{A}$. By $B \in \mathscr{O}_N$, $B \in \mathscr{C}omp$ and so $B' \in \mathscr{C}omp$. Hence $\mathscr{O}_N(\mathscr{A})$. $B' \subset \mathscr{A}$. $\mathrm{Comp}(B')$. By *3.28, we obtain $B' \in \mathscr{A}$. Hence $\exists X(B \in X \in \mathscr{A})$ which implies $B \in \cup \mathscr{A}$. Hence $\mathscr{A} \subseteq \cup \mathscr{A}$. So, in virtue of $\mathrm{Comp}(\mathscr{A})$, $\mathscr{A} = \cup \mathscr{A}$. $\mathscr{A} \neq 0$. Accordingly $\mathrm{Lim}(\mathscr{A})$.

*3.32. $\mathscr{O}_N(\mathscr{A}) \to \mathscr{O}(\mathscr{A})$.

Proof. If $\mathscr{A} = \mathscr{O}$, then $\mathscr{O}(\mathscr{A})$. Hence we may assume $\mathscr{A} \neq \mathscr{O}$. Let $U \in \mathscr{A}$. Assume $0 \in \mathscr{X}$. $\forall X(X \in \mathscr{X} \supset X' \in \mathscr{X})$. $\forall X(X \in \mathscr{L}im.\ X \subseteq \mathscr{X} :\supset.\ X \in \mathscr{X})$. We prove $U \in \mathscr{X}$. Put $\mathscr{B} \overset{D}{=} \{X \mid X \in \mathscr{A}.\ X \notin \mathscr{X}\}$. Then $\mathscr{B} \subseteq \mathscr{A}$. Hence, by 3.30, $\mathscr{B} \subseteq \mathscr{O}_N$. Assume $\exists X(X \in \mathscr{B})$. By *3.29, $\exists Y(Y \in \mathscr{B}.\ Y \cap \mathscr{B} = 0)$. Let $A \in \mathscr{B}$. $A \cap \mathscr{B} = 0$. From this and $\mathrm{Comp}(\mathscr{A})$, we can obtain $A \subseteq \mathscr{X}$. $A \in \mathscr{A}$. $\mathscr{O}_N(\mathscr{A})$ and hence, by 3.30, $A \in \mathscr{O}_N$ and hence, by *3.31, $A = 0 \vee \exists X(A = X'.\ X \in \mathscr{O}_N) \vee A \in \mathscr{L}im$. Let $A = 0$. By $0 \in \mathscr{X}$, $A \in \mathscr{X}$. Let $A = X'$. $X \in \mathscr{O}_N$. By $X \in A$ and $A \subseteq \mathscr{X}$, we obtain $X \in \mathscr{X}$. Hence, by $\forall X(X \in \mathscr{X} \supset X' \in \mathscr{X})$, $A = X' \in \mathscr{X}$. Let $A \in \mathscr{L}im$. By $A \subseteq \mathscr{X}$ and $\forall X(X \in \mathscr{L}im.\ X \subseteq \mathscr{X} :\supset.\ X \in \mathscr{X})$, $A \in \mathscr{X}$. On the other hand, by $A \in \mathscr{B}$, $A \notin \mathscr{X}$ which is a contradiction. Accordingly $\mathscr{B} = 0$ and hence $U \in \mathscr{X}$. Since $\mathscr{X}$ is arbitrary, we obtain $U \in \mathscr{O}$. Hence $\mathscr{A} \subseteq \mathscr{O}$. By $\mathscr{A} \neq \mathscr{O}$, $\mathscr{A} \subset \mathscr{O}$. By *3.24 and *3.27 and $\mathscr{O}(\mathscr{O})$, we obtain $\mathscr{O}_N(\mathscr{O})$. By the premise, $\mathrm{Comp}(\mathscr{A})$. That is, $\mathscr{O}_N(\mathscr{O})$. $\mathscr{A} \subset \mathscr{O}$. $\mathrm{Comp}(\mathscr{A})$. Hence, by *3.28, $\mathscr{A} \in \mathscr{O}$ and so $\mathscr{O}(\mathscr{A})$.

*3.33. $\mathscr{O}(\mathscr{A}) \equiv \mathscr{O}_I(\mathscr{A}) \equiv \mathscr{O}_N(\mathscr{A})$.   *3.34. $\mathscr{O} = \mathscr{O}_I = \mathscr{O}_N$.

Next we consider the class of ordinal numbers which are sets.

Def $\Omega \overset{D}{=} \{u \mid \forall \mathscr{X}[0 \in \mathscr{X}.\ \forall x(x \in \mathscr{X} \supset x' \in \mathscr{X}).$
$\forall x(x \in \mathscr{L}im.\ x \subseteq \mathscr{X} :\supset.\ x \in \mathscr{X}) :\supset.\ u \in \mathscr{X}]\}$.

Hereafter the convention is made that $\alpha, \beta, \gamma, \ldots,$ are variables whose range is Ω.

3.35. $0 \in \Omega$. 3.36. $\alpha' \in \Omega$. 3.37. $\mathfrak{a} \in \mathscr{L}im.\ \mathfrak{a} \subseteq \Omega \to \mathfrak{a} \in \Omega$.

3.38. $\Omega = \mathscr{O} \cap \mathscr{S}et$.

Proof. It can be easily proved that $\Omega \subseteq \mathscr{O} \cap \mathscr{S}et$. Put $\mathscr{A} \overset{D}{=} \{\mathfrak{a} \mid \mathfrak{a} \in \mathscr{S}et \supset \mathfrak{a} \in \Omega\}$. Assume $\mathfrak{a} \subseteq \mathscr{A}$ and $\mathfrak{a} \in \mathscr{S}et$. If $\mathfrak{a} = 0$, then, by 3.35, $\mathfrak{a} \in \Omega$. Assume $\mathfrak{a} = \mathfrak{b}'$. $\mathfrak{b} \in \mathfrak{a}$ and hence $\mathfrak{b} \in \mathscr{A}$. $\mathfrak{b} \in \mathscr{S}et$. So $\mathfrak{b} \in \Omega$ and hence, by 3.36, $\mathfrak{b}' \in \Omega$. Assume $\mathfrak{a} \in \mathscr{L}im$. By $\mathfrak{a} \subseteq \mathscr{A} \cap \mathscr{S}et$, $\mathfrak{a} \subseteq \Omega$. That is, $\mathfrak{a} \in \mathscr{S}et \cap \mathscr{L}im.\ \mathfrak{a} \subseteq \Omega$. Hence, by 3.37, $\mathfrak{a} \in \Omega$. Accordingly $\forall \mathfrak{a}(\mathfrak{a} \subseteq \mathscr{A} .\supset. \mathfrak{a} \in \mathscr{A})$ and hence $\mathscr{O} \subseteq \mathscr{A}$. From this, we obtain $\mathscr{O} \cap \mathscr{S}et \subseteq \Omega$.

3.39. $\Omega \in \mathscr{I}nduc$. 3.40. $\forall \alpha(\alpha \subseteq \mathscr{A} .\supset. \alpha \in \mathscr{A}) \to \Omega \subseteq \mathscr{A}$.

3.41. There holds one and only one of $\alpha = 0$, $\exists \beta(\alpha = \beta')$ and $\alpha \in \mathscr{L}im$.

3.42. $\Omega \in \mathscr{C}omp$. 3.43. $\Omega \in \mathscr{L}im$. 3.44. $\Omega \in \mathscr{P}roCls$. 3.45. $\Omega \in \mathscr{O}$.

3.46. $\forall u.\ u \in \Omega \equiv \forall X[0 \in X.\ \forall x(x \in X \supset x' \in X).$

$\forall x(x \in \mathscr{L}im.\ x \subseteq X : \supset .\ x \in X) : \supset .\ u \in X]$.

Finally we consider the set of natural numbers.

Def $\omega \overset{D}{=} \{u \mid \forall \mathscr{X}[0 \in X.\ \forall x(x \in \mathscr{X} \supset x' \in \mathscr{X}) : \supset .\ u \in \mathscr{X}]\}$.

We make the convention that k, l, m, n are variables whose range is ω.

3.47. $\omega \subseteq \Omega$. 3.48. $\omega \in \mathscr{S}et$. 3.49. $0 \in \omega$. 3.50. $n \in \omega \to n' \in \omega$.

3.51. $\neg \exists n(n \in 0)$. 3.52. $n' = m' \to n = m$.

3.53. $0 \in \mathscr{A}.\ \forall n(n \in \mathscr{A} \supset n' \in \mathscr{A}) \to \omega \subseteq \mathscr{A}$.

3.54. $\forall n(n \subseteq \mathscr{A} .\supset. n \in \mathscr{A}) \to \omega \subseteq \mathscr{A}$.

3.55. There holds one and only one between $n = 0$ and $\exists m(n = m')$.

3.56. $\omega \in \mathscr{C}omp$. 3.57. $\omega \in \mathscr{L}im$. 3.58. $\omega \in \Omega$.

3.59. $\forall u(u \in \omega \equiv \forall X[0 \in X.\ \forall x(x \in X \supset x' \in X) : \supset .\ u \in X])$.

3.60. $\forall u(u \in \omega \equiv \forall a[0 \in a.\ \forall x(x \in a \supset x' \in a) : \supset .\ u \in a])$.

3.61. $\forall u(u \in \omega \equiv \forall x(x \in u' .\supset. x = 0 \vee \exists n(x = n')))$.

3.62. $\langle n \mathfrak{a} \rangle \in \mathscr{C}ompar$. 3.63. $n = \mathfrak{a} \vee n \neq \mathfrak{a}$. 3.64. $\omega \in \mathscr{C}onnex$.

3.65. $\omega \in \mathscr{W}ord$. 3.66. $n \subset \mathfrak{a} \equiv n \in \mathfrak{a}$. 3.67. $\mathfrak{a} \subset n \equiv \mathfrak{a} \in n$.

§ 4. Addition and product and exponentiation of ordinal numbers.

Def $\mathscr{K}(\mathscr{G}) \overset{D}{=} \cup \{F \mid \exists \mathfrak{y}(F \text{ Fnc } \mathfrak{y}.\ \forall \mathfrak{a}[\mathfrak{a} \in \mathfrak{y} \supset F`\ \mathfrak{a} = \mathscr{G}`\ (\mathfrak{a} \upharpoonright F)])\}$,

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4.01. $\mathrm{Funct}(\mathscr{R}(\mathscr{G}))$. $\forall\mathfrak{a}[\mathfrak{a}\subseteq\mathscr{D}(\mathscr{R}(\mathscr{G})).\ \mathfrak{a}\upharpoonright\mathscr{R}(\mathscr{G})\in\mathscr{D}(\mathscr{G}):\supset.\ \mathscr{R}(\mathscr{G})‘\,\mathfrak{a}=\mathscr{G}‘\,(\mathfrak{a}\upharpoonright\mathscr{R}(\mathscr{G}))]$. $[\forall\mathfrak{a}(\mathfrak{a}\subseteq\mathscr{D}(\mathscr{R}(\mathscr{G})).\supset.\mathfrak{a}\upharpoonright\mathscr{R}(\mathscr{G}))$ $.\supset.\ \mathscr{D}(\mathscr{R}(\mathscr{G}))=\mathscr{O}]$.

Proof. Put $\mathscr{K}\overset{D}{=}\{F\mid\exists\mathfrak{y}(F\ \mathrm{Fnc}\ \mathfrak{y}.\ \forall\mathfrak{a}[\mathfrak{a}\in\mathfrak{y}.\supset.F‘\,\mathfrak{a}=\mathscr{G}‘\,(\mathfrak{a}\upharpoonright F)])\}$. It follows that $\mathscr{D}(\mathscr{R}(\mathscr{G}))=\cup\mathscr{D}o“\mathscr{K}\subseteq\mathscr{O}$. The proof falls into several lemmas.

4.011. $F,\ G\in\mathscr{K}.\ \mathfrak{a}\in\mathscr{D}o‘\,F\cap\mathscr{D}o‘\,G\rightarrow F‘\,\mathfrak{a}=G‘\,\mathfrak{a}$.

Proof. Put $\mathscr{A}\overset{D}{=}\{\mathfrak{a}\mid\mathfrak{a}\in\mathscr{D}o‘\,F\cap\mathscr{D}o‘\,G.\supset.F‘\,\mathfrak{a}=G‘\,\mathfrak{a}\}$. Assume $\mathfrak{a}\subseteq\mathscr{A}$ and $\mathfrak{a}\in\mathscr{D}o‘\,F\cap\mathscr{D}o‘\,G$. By the definition of $\mathscr{K}$, $F‘\,\mathfrak{a}=\mathscr{G}‘\,(\mathfrak{a}\upharpoonright F)$ and $G‘\,\mathfrak{a}=\mathscr{G}‘\,(\mathfrak{a}\upharpoonright G)$. Let $\langle AB\rangle\in\mathfrak{a}\upharpoonright F$. $A\in\mathfrak{a}.\ \langle AB\rangle\in F$. Hence $A\in\mathscr{A}\cap\mathscr{O}$. Since $\mathscr{D}o‘\,F,\ \mathscr{D}o‘\,G\in\mathscr{C}omp$, $A\in\mathscr{D}o‘\,F\cap\mathscr{D}o‘\,G$. Hence $F‘\,A=G‘\,A$. So $\langle AB\rangle\in G$ and hence we obtain $\mathfrak{a}\upharpoonright F\subseteq\mathfrak{a}\upharpoonright G$. Since the converse is similarly proved, $\mathfrak{a}\upharpoonright F=\mathfrak{a}\upharpoonright G$. Hence $F‘\,\mathfrak{a}=G‘\,\mathfrak{a}$. So $\mathfrak{a}\in\mathscr{A}$. Since $\mathfrak{a}$ is arbitrary, $\forall\mathfrak{a}(\mathfrak{a}\subseteq\mathscr{A}.\supset.\mathfrak{a}\in\mathscr{A})$. Hence $\mathscr{O}\subseteq\mathscr{A}$.

4.012. Funct $(\mathscr{R}(\mathscr{G}))$.

Proof. Assume $\langle AB\rangle,\ \langle AC\rangle\in\mathscr{R}(\mathscr{G})$. By definition, there are F and G such that $F, G\in\mathscr{K}$, $\langle AB\rangle\in F$ and $\langle AC\rangle\in G$. $A\in\mathscr{D}o‘\,F\cap\mathscr{D}o‘\,G$ and $\mathscr{D}o‘\,F\in\mathscr{O}$. Hence $A\in\mathscr{O}$. That is, $F,\ G\in\mathscr{K}.\ A\in\mathscr{O}\cap\mathscr{D}o‘\,F\cap\mathscr{D}o‘\,G$. Hence, by 4.011, $F‘\,A=G‘\,A$. Since $F,\ G\in\mathscr{F}unct$, $B=F‘\,A$ and $C=G‘\,A$. Hence $B=C$. Then $\mathrm{Un}(\mathscr{R}(\mathscr{G}))$ has been proved. By definition, $\mathrm{Rel}(\mathscr{R}(\mathscr{G}))$ is obvious. Accordingly $\mathrm{Funct}(\mathscr{R}(\mathscr{G}))$.

4.013. $F\in\mathscr{K}.\ \mathfrak{a}\subseteq\mathscr{D}o‘\,F\rightarrow\mathfrak{a}\upharpoonright F=\mathfrak{a}\upharpoonright\mathscr{R}(\mathscr{G})$.

Proof. Let $\langle AB\rangle\in\mathfrak{a}\upharpoonright F$. Then $A\in\mathfrak{a}.\ \langle AB\rangle\in F$. By $F\in\mathscr{K}$, $\langle AB\rangle\in\cup\mathscr{K}=\mathscr{R}(\mathscr{G})$. Hence $\langle AB\rangle\in\mathfrak{a}\upharpoonright\mathscr{R}(\mathscr{G})$. Conversely $\langle AB\rangle\in\mathfrak{a}\upharpoonright\mathscr{R}(\mathscr{G})$. Then $A\in\mathfrak{a}$ and there is a G such that $\langle AB\rangle\in G\in\mathscr{K}$. By $A\in\mathfrak{a}$ and $\mathfrak{a}\subseteq\mathscr{D}o‘\,F$, we obtain $A\in\mathscr{D}o‘\,F$. Hence $A\in\mathscr{D}o‘\,F\cap\mathscr{D}o‘\,G\cap\mathscr{O}$. Hence, by 4.011, $F‘\,A=G‘\,A$. Accordingly $\langle AB\rangle\in\mathfrak{a}\upharpoonright F$.

4.014. $\mathfrak{a}\in\mathscr{D}(\mathscr{R}(\mathscr{G}))\rightarrow\mathscr{R}(\mathscr{G})‘\,\mathfrak{a}=\mathscr{G}‘\,(\mathfrak{a}\upharpoonright\mathscr{R}(\mathscr{G}))$.

Proof. By the premise, there is an F such that $\mathfrak{a}\in\mathscr{D}o‘\,F.\ F\in\mathscr{K}.$ $\langle\mathfrak{a},\ F‘\,\mathfrak{a}\rangle\in F\in\mathscr{K}$ and hence $\langle\mathfrak{a},\ F‘\,\mathfrak{a}\rangle\in\mathscr{R}(\mathscr{G})$. So, by 4.012, $\mathscr{R}(\mathscr{G})‘\,\mathfrak{a}=F‘\,\mathfrak{a}$. By $F\in\mathscr{K}$, $F‘\,\mathfrak{a}=\mathscr{G}‘\,(\mathfrak{a}\upharpoonright F)$. By $\mathfrak{a}\in\mathscr{D}o‘\,F\in\mathscr{O}$, $\mathfrak{a}\subseteq\mathscr{D}o‘\,F$. Hence, by 4.013, $\mathfrak{a}\upharpoonright F=\mathfrak{a}\upharpoonright\mathscr{R}(\mathscr{G})$. Accordingly, $\mathscr{R}(\mathscr{G})‘\,\mathfrak{a}=\mathscr{G}‘\,(\mathfrak{a}\upharpoonright\mathscr{R}(\mathscr{G}))$.

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4.015. $\forall \mathfrak{a}[\mathfrak{a} \subseteq \mathscr{D}(\mathscr{R}(\mathscr{G})).\ \mathfrak{a} \upharpoonright \mathscr{R}(\mathscr{G}) \in \mathscr{D}(\mathscr{G}) : \supset.\ \mathfrak{a} \in \mathscr{D}(\mathscr{R}(\mathscr{G}))]$.

Proof. Assume $\mathfrak{a} \subseteq \mathscr{D}(\mathscr{R}(\mathscr{G})).\ \mathfrak{a} \upharpoonright \mathscr{R}(\mathscr{G}) \in \mathscr{D}(\mathscr{G})$. Hence $\mathscr{G}`\ (\mathfrak{a} \upharpoonright \mathscr{R}(\mathscr{G})) \in \mathscr{C}ls$. Put $G \overset{D}{=} (\mathfrak{a} \upharpoonright \mathscr{R}(\mathscr{G})) \cup \{\langle \mathfrak{a}, \mathscr{G}`\ (\mathfrak{a} \upharpoonright \mathscr{R}(\mathscr{G}))\rangle\}$. We prove that G Fnc $\mathfrak{a}'$. $\forall \mathfrak{b}(\mathfrak{b} \in \mathfrak{a}' \supset G`\ \mathfrak{b} = \mathscr{G}`\ (\mathfrak{b} \upharpoonright G))$. It is easy to prove G Fnc $\mathfrak{a}'$. Let $\mathfrak{b} \in \mathfrak{a}'$. Then $\mathfrak{b} \in \mathfrak{a} \vee \mathfrak{b} = \mathfrak{a}$. Assume $\mathfrak{b} \in \mathfrak{a}$. $\langle \mathfrak{b}, G`\ \mathfrak{b}\rangle \in G$ and hence $\langle \mathfrak{b}, G`\ \mathfrak{b}\rangle \in \mathfrak{a} \upharpoonright \mathscr{R}(\mathscr{G})$. So $G`\ \mathfrak{b} = \mathscr{R}(\mathscr{G})`\ \mathfrak{b}$. Since $\mathfrak{b} \in \mathscr{D}(\mathscr{R}(\mathscr{G}))$, we obtain, by 4.014, $\mathscr{R}(\mathscr{G})`\ \mathfrak{b} = \mathscr{G}`\ (\mathfrak{b} \upharpoonright \mathscr{R}(\mathscr{G}))$. By the definition of G, $\mathfrak{b} \upharpoonright \mathscr{R}(\mathscr{G}) = \mathfrak{b} \upharpoonright G$. Hence $G`\ \mathfrak{b} = \mathscr{G}`\ (\mathfrak{b} \upharpoonright G)$. Assume $\mathfrak{b} = \mathfrak{a}$. By the definition of G, $G`\ \mathfrak{b} = G`\ \mathfrak{a} = \mathscr{G}`\ (\mathfrak{a} \upharpoonright \mathscr{R}(\mathscr{G})) = G`\ (\mathfrak{b} \upharpoonright \mathscr{R}(\mathscr{G})) = \mathscr{G}`\ (\mathfrak{b} \upharpoonright G)$. Hence $G$ Fnc $\mathfrak{a}'.\ \forall \mathfrak{b}(\mathfrak{b} \in \mathfrak{a}' \supset G`\ \mathfrak{b} = \mathscr{G}`\ (\mathfrak{b} \upharpoonright G))$. So $G \in \mathscr{K}$. That is, $\mathfrak{a} \in \mathscr{D}o`\ G.\ G \in \mathscr{K}$. Hence $\mathfrak{a} \in \mathscr{D}(\mathscr{R}(\mathscr{G}))$. Then we obtain 4.015.

4.016. $\forall \mathfrak{a}[\mathfrak{a} \subseteq \mathscr{D}(\mathscr{R}(\mathscr{G})).\ \mathfrak{a} \upharpoonright \mathscr{R}(\mathscr{G}) \in \mathscr{D}(\mathscr{G}) : \supset.$
$\mathscr{R}(\mathscr{G})`\ \mathfrak{a} = \mathscr{G}`\ (\mathfrak{a} \upharpoonright \mathscr{R}(\mathscr{G}))]$.

Proof. By 4.014 and 4.015.

4.017. $\forall \mathfrak{a}(\mathfrak{a} \subseteq \mathscr{D}(\mathscr{R}(\mathscr{G})) .\supset. \mathfrak{a} \upharpoonright \mathscr{R}(\mathscr{G}) \in \mathscr{D}(\mathscr{G})) \rightarrow \mathscr{D}(\mathscr{R}(\mathscr{G})) = \mathscr{O}$.

Proof. Assume $\mathfrak{a} \subseteq \mathscr{D}(\mathscr{R}(\mathscr{G}))$. By the premise, $\mathfrak{a} \upharpoonright \mathscr{R}(\mathscr{G}) \in \mathscr{D}(\mathscr{G})$. Hence, by 4.015, $\mathfrak{a} \in \mathscr{D}(\mathscr{R}(\mathscr{G}))$. So it follows that $\forall \mathfrak{a}(\mathfrak{a} \subseteq \mathscr{D}(\mathscr{R}(\mathscr{G})) .\supset. \mathfrak{a} \in \mathscr{D}(\mathscr{R}(\mathscr{G})))$. Hence $\mathscr{O} \subseteq \mathscr{D}(\mathscr{R}(\mathscr{G}))$. On the other hand, $\mathscr{D}(\mathscr{R}(\mathscr{G})) \subseteq \mathscr{O}$ and hence $\mathscr{D}(\mathscr{R}(\mathscr{G})) = \mathscr{O}$. It completes the proof.

Def $\mathscr{G}_1(\mathscr{G}) \overset{D}{=} \cup \{F \mid \exists n(F \text{ Fnc } n.\ \forall m[m \in n \supset F`\ m = \mathscr{G}`\ (m \upharpoonright F)])\}$.

4.02. $\text{Funct}(\mathscr{R}_1(\mathscr{G})).\ \forall n[n \subseteq \mathscr{D}(\mathscr{R}_1(\mathscr{G})).\ n \upharpoonright \mathscr{R}_1(\mathscr{G}) \in \mathscr{D}(\mathscr{G})$
$: \supset.\ \mathscr{R}_1(\mathscr{G})`\ n = \mathscr{G}`\ (n \upharpoonright \mathscr{R}_1(\mathscr{G}))].\ [\forall n(n \subseteq \mathscr{D}(\mathscr{R}_1(\mathscr{G}))$
$.\supset. n \upharpoonright \mathscr{R}_1(\mathscr{G}) \in \mathscr{D}(\mathscr{G})) .\supset. \mathscr{D}(\mathscr{R}_1(\mathscr{G})) = \omega]$.

Def $\mathscr{S}c \overset{D}{=} \{\langle YZ\rangle \mid Z = Y'\}$.

Def $\mathscr{G}(\mathscr{P}, \mathscr{A}) \overset{D}{=} \{\langle YZ\rangle \mid (\mathscr{D}o`\ Y = 0.\ Z = \mathscr{A})$
$\vee (\mathscr{D}o`\ Y \in \mathscr{S}c``\ \mathscr{O}.\ Z = \mathscr{P}`\ Y`\ \mathscr{S}c^{-1}`\ \mathscr{D}o`\ Y)$
$\vee (\mathscr{D}o`\ Y \in \mathscr{L}im.\ Z = \cup \mathscr{W}`\ Y)$.

Def $\mathscr{I}t(\mathscr{P}, \mathscr{A}) \overset{D}{=} \mathscr{R}(\mathscr{G}(\mathscr{P}, \mathscr{A}))$.

4.03. $\mathscr{A} \in \mathscr{C}ls \rightarrow \mathscr{I}t(\mathscr{P}, \mathscr{A})`\ 0 = \mathscr{A}$.

Proof. By 4.01, $\forall \mathfrak{a}[\mathfrak{a} \subseteq \mathscr{D}(\mathscr{I}t(\mathscr{P}, \mathscr{A})).\ \mathfrak{a} \upharpoonright \mathscr{I}t(\mathscr{P}, \mathscr{A}) \in \mathscr{D}(\mathscr{G}(\mathscr{P}, \mathscr{A}))$ $: \supset.\ \mathscr{I}t(\mathscr{P}, \mathscr{A})`\ \mathfrak{a} = \mathscr{G}(\mathscr{P}, \mathscr{A})`\ (\mathfrak{a} \upharpoonright \mathscr{I}t(\mathscr{P}, \mathscr{A}))]$. Obviously, $0 \subseteq \mathscr{D}(\mathscr{I}t(\mathscr{P}, \mathscr{A}))$. Since $\mathscr{D}o`\ (0 \upharpoonright \mathscr{I}t(\mathscr{P}, \mathscr{A})) = 0$ and $\mathscr{A} \in \mathscr{C}ls$, $\langle 0 \upharpoonright \mathscr{I}t(\mathscr{P}, \mathscr{A}), \mathscr{A}\rangle$

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$\in \mathscr{G}(\mathscr{P}, \mathscr{A})$ and hence $0 \upharpoonright \mathscr{I}t(\mathscr{P}, \mathscr{A}) \in \mathscr{D}(\mathscr{G}(\mathscr{P}, \mathscr{A}))$. So, by the above, we obtain $\mathscr{I}t(\mathscr{P}, \mathscr{A})`0 = \mathscr{G}(\mathscr{P}, \mathscr{A})`(0 \upharpoonright \mathscr{I}t(\mathscr{P}, \mathscr{A}))$. Since it can be proved that Funct$(\mathscr{G}(\mathscr{P}, \mathscr{A}))$, $\mathscr{G}(\mathscr{P}, \mathscr{A})`(0 \upharpoonright \mathscr{I}t(\mathscr{P}, \mathscr{A})) = \mathscr{A}$. Accordingly $\mathscr{I}t(\mathscr{P}, \mathscr{A})`0 = \mathscr{A}$.

4.04. $\mathfrak{a}' \subseteq \mathscr{D}(\mathscr{I}t(\mathscr{P}, \mathscr{A}))$. $\mathscr{I}t(\mathscr{P}, \mathscr{A})`\mathfrak{a} \in \mathscr{D}(\mathscr{P}) \cup -\mathscr{D}(\mathscr{P})$
$\rightarrow \mathscr{I}t(\mathscr{P}, \mathscr{A})`\mathfrak{a}' = \mathscr{P}`\mathscr{I}t(\mathscr{P}, \mathscr{A})`\mathfrak{a}$.

Proof. By 4.01, Funct$(\mathscr{I}t(\mathscr{P}, \mathscr{A}))$. Also, by $\mathfrak{a}' \subseteq \mathscr{D}(\mathscr{I}t(\mathscr{P}, \mathscr{A}))$, $\mathfrak{a} \in \mathscr{D}(\mathfrak{a}' \upharpoonright \mathscr{I}t(\mathscr{P}, \mathscr{A}))$. Hence, by 2.29, $(\mathfrak{a}' \upharpoonright \mathscr{I}t(\mathscr{P}, \mathscr{A}))`\mathfrak{a} = \mathscr{I}t(\mathscr{P}, \mathscr{A})`\mathfrak{a} \in \mathscr{C}ls$. By $\mathscr{I}t(\mathscr{P}, \mathscr{A})`\mathfrak{a} \in \mathscr{D}(\mathscr{P}) \cup -\mathscr{D}(\mathscr{P})$ and 2.19, $\mathscr{P}`\mathscr{I}t(\mathscr{P}, \mathscr{A})`\mathfrak{a} \in \mathscr{C}ls$. Hence $\langle \mathfrak{a}' \upharpoonright \mathscr{I}t(\mathscr{P}, \mathscr{A}), \mathscr{P}`\mathscr{I}t(\mathscr{P}, \mathscr{A})`\mathfrak{a} \rangle \in \mathscr{G}(\mathscr{P}, \mathscr{A})$. So $\mathfrak{a}' \upharpoonright \mathscr{I}t(\mathscr{P}, \mathscr{A}) \in \mathscr{D}(\mathscr{G}(\mathscr{P}, \mathscr{A}))$. Together with $\mathfrak{a}' \subseteq \mathscr{D}(\mathscr{I}t(\mathscr{P}, \mathscr{A}))$, we obtain $\mathscr{I}t(\mathscr{P}, \mathscr{A})`\mathfrak{a}' = \mathscr{G}(\mathscr{P}, \mathscr{A})`(\mathfrak{a}' \upharpoonright \mathscr{I}t(\mathscr{P}, \mathscr{A})) = \mathscr{P}`\mathscr{I}t(\mathscr{P}, \mathscr{A})`\mathfrak{a}$, using that Funct$(\mathscr{G}(\mathscr{P}, \mathscr{A}))$.

4.05. $\mathfrak{a} \subseteq \mathscr{G}(\mathscr{I}t(\mathscr{P}, \mathscr{A}))$, $\mathfrak{a} \in \mathscr{L}im \rightarrow \mathscr{I}t(\mathscr{P}, \mathscr{A})`\mathfrak{a}$
$= \cup \mathscr{I}t(\mathscr{P}, \mathscr{A})``\mathfrak{a} \in \mathscr{C}ls$.

Proof. $\cup \mathscr{W}`(\mathfrak{a} \upharpoonright \mathscr{I}t(\mathscr{P}, \mathscr{A})) = \cup \mathscr{I}t(\mathscr{P}, \mathscr{A})``\mathfrak{a} \in \mathscr{C}ls$, by Funct$(\mathscr{I}t(\mathscr{P}, \mathscr{A}))$, 2.09 and 2.23. Hence $\langle \mathfrak{a} \upharpoonright \mathscr{I}t(\mathscr{P}, \mathscr{A}), \cup \mathscr{I}t(\mathscr{P}, \mathscr{A})``\mathfrak{a} \rangle \in \mathscr{G}(\mathscr{P}, \mathscr{A})$ and so $\mathfrak{a} \upharpoonright \mathscr{I}t(\mathscr{P}, \mathscr{A}) \in \mathscr{D}(\mathscr{G}(\mathscr{P}, \mathscr{A}))$. Together with $\mathfrak{a} \subseteq \mathscr{D}(\mathscr{I}t(\mathscr{P}, \mathscr{A}))$, we obtain $\mathscr{I}t(\mathscr{P}, \mathscr{A})`\mathfrak{a} = \mathscr{G}(\mathscr{P}, \mathscr{A})`(\mathfrak{a} \upharpoonright \mathscr{I}t(\mathscr{P}, \mathscr{A})) = \cup \mathscr{I}t(\mathscr{P}, \mathscr{A})``\mathfrak{a}$.

Def $\mathscr{F}$ Fnc $(\mathscr{A} \rightarrow \mathscr{B}) \overset{D}{\equiv} \mathscr{F}$ Fnc $\mathscr{A}$. $\mathscr{W}(\mathscr{G}) \subseteq \mathscr{B}$.

4.06. $\mathscr{A} \in \mathscr{B}$. $\forall \mathfrak{a}(\mathfrak{a}' \subseteq \mathscr{D}(\mathscr{I}t(\mathscr{P}, \mathscr{A})) . \supset . \mathscr{I}t(\mathscr{P}, \mathscr{A})`\mathfrak{a} \in \mathscr{B})$.
$\mathscr{B} \subseteq \mathscr{D}(\mathscr{P}) \cup -\mathscr{D}(\mathscr{P}) \rightarrow \mathscr{I}t(\mathscr{P}, \mathscr{A})$ Fnc $(\mathscr{O} \rightarrow \mathscr{B})$.
$\mathscr{I}t(\mathscr{P}, \mathscr{A})`0 = \mathscr{A}$. $\mathscr{I}t(\mathscr{P}, \mathscr{A})`\mathfrak{b}' = \mathscr{P}`\mathscr{I}t(\mathscr{P}, \mathscr{A})`\mathfrak{b}$.
$\mathscr{I}t(\mathscr{P}, \mathscr{A})`\mathfrak{l} = \cup \mathscr{I}t(\mathscr{P}, \mathscr{A})``\mathfrak{l}$.

Proof. By the premise and the proofs of 4.03-4.05, we obtain $\forall \mathfrak{a}(\mathfrak{a} \subseteq \mathscr{D}(\mathscr{I}t(\mathscr{P}, \mathscr{A})) . \supset . \mathfrak{a} \upharpoonright \mathscr{I}t(\mathscr{P}, \mathscr{A}) \in \mathscr{D}(\mathscr{P}, \mathscr{A})))$. Hence, by 4.01, $\mathscr{D}(\mathscr{I}t(\mathscr{P}, \mathscr{A})) = \mathscr{O}$. So, by the premise, $\forall \mathfrak{a}(\mathscr{I}t(\mathscr{P}, \mathscr{A})`\mathfrak{a} \in \mathscr{B})$ and hence $\mathscr{W}(\mathscr{I}t(\mathscr{P}, \mathscr{A})) \subseteq \mathscr{B}$. By 4.01, Funct$(\mathscr{I}t(\mathscr{P}, \mathscr{A}))$. From these, we obtain $\mathscr{I}t(\mathscr{P}, \mathscr{A})$ Fnc $(\mathscr{O} \rightarrow \mathscr{B})$. The premises of 4.03-4.05 are easily proved and hence the other assertions follow from 4.03-4.05.

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Def $\mathscr{A}+\mathscr{B} \overset{D}{=} \mathscr{I}t(\mathscr{S}c, \mathscr{A})`\mathscr{B}$. Def $1 \overset{D}{=} 0'$ Def $2 \overset{D}{=} 1'$

4.07. $A+\mathfrak{b} \in \mathscr{C}ls$.

Proof. $\mathscr{D}(\mathscr{S}c)=\mathscr{C}ls$. Hence, if we substitute $\mathscr{C}ls$ for $\mathscr{B}$, $\mathscr{S}c$ for $\mathscr{P}$ and A for $\mathscr{A}$ in 4.06, we obtain $\mathscr{I}t(\mathscr{S}c, A)$ Fnc $(\mathscr{O} \rightarrow \mathscr{C}ls)$. Hence, by definition, $A+\mathfrak{b} \in \mathscr{C}ls$.

4.08. $\mathscr{I}t(\mathscr{S}c, \mathfrak{a})$ Fnc $\mathscr{O}$. $\mathfrak{a}+0=\mathfrak{a}$. $\mathfrak{a}+\mathfrak{b}'=(\mathfrak{a}+\mathfrak{b})'$.
$\mathfrak{a}+\mathfrak{l}= \bigcup\{\mathfrak{a}+\mathfrak{y} \mid \mathfrak{y} \in \mathfrak{l}\}_{\mathfrak{y}}$.

4.09. $\mathfrak{a}+\mathfrak{b}=\mathfrak{a} \cup \{\mathfrak{a}+\mathfrak{y} \mid \mathfrak{y} \in \mathfrak{b}\}$. 4.10. $0+\mathfrak{a}=\mathfrak{a}$. 4.11. $\mathfrak{a}+1=\mathfrak{a}'$.

4.12. $\mathfrak{a}+\mathfrak{b}=0. \equiv : \mathfrak{a}=0. \mathfrak{b}=0$. 4.13. $\mathfrak{c} \in \mathfrak{b} \rightarrow \mathfrak{a}+\mathfrak{c} \in \mathfrak{a}+\mathfrak{b}$.

4.14. $0 \in \mathfrak{b} \rightarrow \mathfrak{a} \in \mathfrak{a}+\mathfrak{b}$. 4.15. $\mathfrak{a}+\mathfrak{b} \in \mathscr{C}omp$. 4.16. $\mathfrak{a}+\mathfrak{l} \in \mathscr{L}im$.

4.17. $\mathfrak{a}+\mathfrak{b} \in \mathscr{O}$. 4.18. $(\mathfrak{a}+\mathfrak{b})+\mathfrak{c}=\mathfrak{a}+(\mathfrak{b}+\mathfrak{c})$.

Def $\mathfrak{a}+\mathfrak{b}+\mathfrak{c} \overset{D}{=} (\mathfrak{a}+\mathfrak{b})+\mathfrak{c}$.

4.19. $\mathfrak{a}+\mathfrak{b}=\mathfrak{a}+\mathfrak{c}.\equiv.\mathfrak{b}=\mathfrak{c}$. 4.20. $\mathfrak{a}+\mathfrak{b} \in \mathfrak{a}+\mathfrak{c} \equiv \mathfrak{b} \in \mathfrak{c}$.

4.21. $\mathfrak{a}+\mathfrak{c} \in \mathfrak{b}+\mathfrak{c} \rightarrow \mathfrak{a} \in \mathfrak{b}$.

4.22. $\mathfrak{a} \subseteq \mathfrak{b}$. $\mathfrak{a}+\mathfrak{c}$, $\mathfrak{b}+\mathfrak{c} \in \mathfrak{z} \in \mathscr{C}onnex \rightarrow \mathfrak{a}+\mathfrak{c} \mathrel{\underline{\subseteq}} \mathfrak{b}+\mathfrak{c}$.

*4.23. $\mathfrak{a} \subseteq \mathfrak{b} \rightarrow \mathfrak{a}+\mathfrak{c} \mathrel{\underline{\subseteq}} \mathfrak{b}+\mathfrak{c}$. 4.24. $\mathfrak{a}$, $\mathfrak{b} \in \mathscr{C}onnex \rightarrow \mathfrak{a}+\mathfrak{b} \in \mathscr{C}onnex$.

Def $\mathscr{I}ndecomp \overset{D}{=} \{\mathfrak{a} \mid \neg \exists \mathfrak{b}\mathfrak{c}(\mathfrak{a}=\mathfrak{b}+\mathfrak{c}.\ \mathfrak{b}, \mathfrak{c} \in \mathfrak{a})\}$.

4.25. $\forall \mathfrak{y}(\mathfrak{y} \in \mathfrak{a} \supset \mathfrak{y}+\mathfrak{a}=\mathfrak{a}) \rightarrow \mathfrak{a} \in \mathscr{I}ndecomp$.

*4.26. $\mathfrak{a} \in \mathscr{I}ndecomp \equiv \forall \mathfrak{y}(\mathfrak{y} \in \mathfrak{a} \supset \mathfrak{y}+\mathfrak{a}=\mathfrak{a})$.

Def $\mathfrak{a} \leq \mathfrak{b} \overset{D}{=} \exists \mathfrak{y}(\mathfrak{a}+\mathfrak{y}=\mathfrak{b})$. Def $\mathfrak{a}<\mathfrak{b} \overset{D}{=} \exists \mathfrak{y}(\mathfrak{a}+\mathfrak{y}=\mathfrak{b}.\ \mathfrak{y} \neq 0)$.

4.27. $0 \leq \mathfrak{a}$. 4.28. $\mathfrak{a} \leq \mathfrak{b}$. $\mathfrak{b} \leq \mathfrak{c} \rightarrow \mathfrak{a} \leq \mathfrak{c}$. 4.29. $\mathfrak{a}<\mathfrak{b}$. $\mathfrak{b} \leq \mathfrak{c} \rightarrow \mathfrak{a}<\mathfrak{c}$.

4.30. $\mathfrak{a} \leq \mathfrak{b}$. $\mathfrak{b}<\mathfrak{c} \rightarrow \mathfrak{a}<\mathfrak{c}$. 4.31. $\mathfrak{a}<\mathfrak{b}$. $\mathfrak{b}<\mathfrak{c} \rightarrow \mathfrak{a}<\mathfrak{c}$.

4.32. $\mathfrak{a}+\mathfrak{b} \leq \mathfrak{a}+\mathfrak{c}.\equiv.\mathfrak{b} \leq \mathfrak{c}$. 4.33. $\mathfrak{a}+\mathfrak{b}<\mathfrak{a}+\mathfrak{c}.\equiv.\mathfrak{b}<\mathfrak{c}$.

4.34. $\mathfrak{a} \leq \mathfrak{b}.\equiv.\mathfrak{a}<\mathfrak{b} \vee \mathfrak{a}=\mathfrak{b}$. 4.35. $\neg(\mathfrak{a}<\mathfrak{b}.\ \mathfrak{b} \leq \mathfrak{a})$. 4.36. $\mathfrak{a}<\mathfrak{b}'.\equiv.\mathfrak{a} \leq \mathfrak{b}$.

4.37. $\mathfrak{a}<\mathfrak{b} \rightarrow \mathfrak{a} \in \mathfrak{b}$. *4.38. $\mathfrak{a}<\mathfrak{b} \equiv \mathfrak{a} \in \mathfrak{b}$.

Proof. By 4.37, it suffices to prove that $\mathfrak{a} \in \mathfrak{b} \rightarrow \mathfrak{a}<\mathfrak{b}$. Put $\mathscr{A} \overset{D}{=} \{\mathfrak{b} \mid \mathfrak{a} \in \mathfrak{b} \supset \mathfrak{a}<\mathfrak{b}\}$. Assume $\mathfrak{b} \subseteq \mathscr{A}$ and $\mathfrak{a} \in \mathfrak{b}$. Then $\mathfrak{b} \neq 0$. Assume that $\mathfrak{b}=\mathfrak{c}'$ for a $\mathfrak{c}$. So $\mathfrak{a} \in \mathfrak{c}'$. If $\mathfrak{a} \in \mathfrak{c}$, then, by $\mathfrak{c} \in \mathscr{A}$, $\mathfrak{a}<\mathfrak{c}$ and hence $\mathfrak{a}<\mathfrak{b}$. If $\mathfrak{a}=\mathfrak{c}$, then $\mathfrak{a}<\mathfrak{a}'=\mathfrak{b}$. Assume $\mathfrak{b} \in \mathscr{L}im$. Put $\mathscr{B} \overset{D}{=} \{\mathfrak{y} \mid \mathfrak{a}+\mathfrak{y} \in \mathfrak{b}\}$. We prove $\mathscr{B} \in \mathscr{O}$. Let $\mathfrak{c} \in \mathscr{B}$. Then $\mathfrak{a}+\mathfrak{c} \in \mathfrak{b}$. By $\mathfrak{b} \in \mathscr{L}im$ and 3.16, $\mathfrak{a}+\mathfrak{c}' \in \mathfrak{b}$ and hence $\mathfrak{c}' \in \mathscr{B}$. So $\mathfrak{c} \in \bigcup \mathscr{B}$. Conversely, $\mathfrak{c} \in \bigcup \mathscr{B}$. Let $\mathfrak{c} \in \mathfrak{d} \in \mathscr{B}$. By 4.13 and the definition, $\mathfrak{a}+\mathfrak{c} \in \mathfrak{a}+\mathfrak{d}$

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$\in \mathfrak{b}$. Hence $\mathfrak{a}+\mathfrak{c}\in\mathfrak{b}$ and so $\mathfrak{c}\in\mathscr{B}$. Accordingly $\mathscr{B}=\cup\mathscr{B}$. By $\mathfrak{a}\in\mathfrak{b}$, $0\in\mathscr{B}$ and hence $\exists U(U\in\mathscr{B})$. Hence $\mathrm{Lim}(\mathscr{B})$. Let $\mathfrak{c}\in\mathscr{B}$. Then $\mathfrak{a}+\mathfrak{c}\in\mathfrak{b}$. By *4.23, $\mathfrak{c}\leqq\mathfrak{a}+\mathfrak{c}$ and so $\mathfrak{c}\in\mathfrak{b}$. Hence $\mathscr{B}\subseteq\mathfrak{b}$. Since $\mathfrak{b}\in\mathscr{C}ls$, by 2.14, $\mathscr{B}\in\mathscr{C}ls$. Together with $\mathrm{Lim}(\mathscr{B})$, we obtain $\mathscr{B}\in\mathscr{L}im$. Evidently $\mathscr{B}\subseteq\mathscr{O}$ and hence, by 3.03, $\mathscr{B}\in\mathscr{O}$. Next we prove $\mathfrak{a}+\mathscr{B}=\mathfrak{b}$. Let $\mathfrak{c}\in\mathfrak{a}+\mathscr{B}$. Since $\mathscr{B}\in\mathscr{L}im\cap\mathscr{O}$, there is a $\mathfrak{y}$ such that $\mathfrak{c}\in\mathfrak{a}+\mathfrak{y}$. $\mathfrak{y}\in\mathscr{B}$. Then $\mathfrak{a}+\mathfrak{y}\in\mathfrak{b}$ and hence $\mathfrak{c}\in\mathfrak{b}$. So $\mathfrak{a}+\mathscr{B}\subseteq\mathfrak{b}$. Conversely $\mathfrak{c}\in\mathfrak{b}$. By Connex($\mathscr{O}$), $\langle\mathfrak{a}\mathfrak{c}\rangle\in\mathscr{C}ompar$. Let $\mathfrak{a}\in\mathfrak{c}$. By $\mathfrak{c}\in\mathscr{A}$, we obtain $\mathfrak{a}<\mathfrak{c}$ and hence there is a $\mathfrak{y}$ such that $\mathfrak{a}+\mathfrak{y}=\mathfrak{c}$. Hence $\mathfrak{a}+\mathfrak{y}\in\mathfrak{b}$ and so $\mathfrak{y}\in\mathscr{B}$. Hence, by 4.13, $\mathfrak{c}=\mathfrak{a}+\mathfrak{y}\in\mathfrak{a}+\mathscr{B}$. Let $\mathfrak{c}\leqq\mathfrak{a}$. Then $\mathfrak{c}\in\mathfrak{a}+\mathscr{B}$. Hence $\mathfrak{b}\subseteq\mathfrak{a}+\mathscr{B}$ and so $\mathfrak{a}+\mathscr{B}=\mathfrak{b}$. By definition, $\mathfrak{a}<\mathfrak{b}$ and so $\mathfrak{b}\in\mathscr{A}$. Accordingly $\forall\mathfrak{b}(\mathfrak{b}\subseteq\mathscr{A}.\supset.\mathfrak{b}\in\mathscr{A})$ and so $\mathscr{O}\subseteq\mathscr{A}$.

4.39. $\mathfrak{a}'\leq\mathfrak{b}\to\mathfrak{a}<\mathfrak{b}$. *4.40. $\mathfrak{a}'\leq\mathfrak{b}\equiv\mathfrak{a}<\mathfrak{b}$. 4.41. $\boldsymbol{n}<\boldsymbol{m}\equiv\boldsymbol{n}\in\boldsymbol{m}\equiv\boldsymbol{n}\subset\boldsymbol{m}$.

4.42. $\boldsymbol{n}+\boldsymbol{m}=\boldsymbol{m}+\boldsymbol{n}$. 4.43. $\boldsymbol{n}+\boldsymbol{m}\in\omega$. 4.44. $\omega\in\mathscr{I}ndecomp$.

Def $\mathscr{P}_4(\mathscr{A})\overset{D}{=}\{\langle XY\rangle\mid Y=X+\mathscr{A}\}$.

Def $\mathscr{A}\mathscr{B}\overset{D}{=}\mathscr{A}\cdot\mathscr{B}\overset{D}{=}\mathscr{I}t(\mathscr{P}_4(\mathscr{A}),0)`\mathscr{B}$.

4.45. $\mathscr{I}t(\mathscr{P}_4(\mathfrak{a}),0)$ Fnc $(\mathscr{O}\to\mathscr{C}ls)$. $\mathfrak{a}0=0$. $\mathfrak{a}\mathfrak{b}'=\mathfrak{a}\mathfrak{b}+\mathfrak{a}$. $\mathfrak{a}\mathfrak{l}=\cup\{\mathfrak{a}\mathfrak{y}\mid\mathfrak{y}\in\mathfrak{l}\}$.

Proof. By 4.07, $\mathscr{D}(\mathscr{P}_4(\mathfrak{a}))=\mathscr{C}ls$. Hence if we substitute 0 for $\mathscr{A}$ and $\mathscr{C}ls$ for $\mathscr{B}$ and $\mathscr{P}_4(\mathfrak{a})$ for $\mathscr{P}$ in 4.06, then the premise is satisfied. Hence the conclusion of 4.06 gives the theorem.

4.46. $0\mathfrak{a}=0$.

The following three theorems are proved simultaneously.

4.47. $0\in\mathfrak{c}$. $\mathfrak{a}\in\mathfrak{b}\to\mathfrak{c}\mathfrak{a}\in\mathfrak{c}\mathfrak{b}$. 4.48. $0\in\mathfrak{a}$. $\mathfrak{b}\in\mathscr{L}im\to\mathfrak{a}\mathfrak{b}\in\mathscr{L}im$.

4.49. $\mathfrak{a}\mathfrak{b}\in\mathscr{O}$. 4.50. $\mathfrak{a}(\mathfrak{b}+\mathfrak{c})=\mathfrak{a}\mathfrak{b}+\mathfrak{a}\mathfrak{c}$. 4.51. $(\mathfrak{a}\mathfrak{b})\mathfrak{c}=\mathfrak{a}(\mathfrak{b}\mathfrak{c})$.

4.52. $\mathfrak{a}\mathfrak{b}=0.\equiv.\mathfrak{a}=0\vee\mathfrak{b}=0$. 4.53. $\mathfrak{c}\mathfrak{a}\in\mathfrak{c}\mathfrak{b}$. $\langle\mathfrak{a}\mathfrak{b}\rangle\in\mathscr{C}ompar\to\mathfrak{a}\in\mathfrak{b}$.

*4.54. $\mathfrak{c}\mathfrak{a}\in\mathfrak{b}\mathfrak{c}\to\mathfrak{a}\in\mathfrak{b}$. 4.55. $\mathfrak{a}\leqq\mathfrak{b}$. $\mathfrak{a}\mathfrak{c}$, $\mathfrak{b}\mathfrak{c}\in\mathfrak{z}\in\mathscr{C}onnex\to\mathfrak{a}\mathfrak{c}\leqq\mathfrak{b}\mathfrak{c}$.

*4.56. $\mathfrak{a}\leqq\mathfrak{b}\to\mathfrak{a}\mathfrak{c}\leqq\mathfrak{b}\mathfrak{c}$. 4.57. $\mathfrak{a}$, $\mathfrak{b}\in\mathscr{C}onnex\to\mathfrak{a}\mathfrak{b}\in\mathscr{C}onnex$. 4.58. $\boldsymbol{n}\boldsymbol{m}\in\omega$.

4.59. $\exists!\,\mathfrak{b}\boldsymbol{n}((\mathfrak{a}=\mathfrak{b}+2\boldsymbol{n}\vee\mathfrak{a}=\mathfrak{b}+2\boldsymbol{n}+1).\ \mathfrak{b}\in 1\cup\mathscr{L}\mathrm{im})$.

Def $h(\mathscr{A})\overset{D}{=}\{U\mid(\mathscr{A}=0.\ U\in 0)\vee(0\in\mathscr{A}.\ U\in 1)$.

4.60. $[\mathfrak{a}=0\supset h(\mathfrak{a})=0]$. $[0\in\mathfrak{a}\supset h(\mathfrak{a})=1]$. 4.61. $h(\mathfrak{a})\in\mathscr{O}$.

Def $\mathscr{P}_5(\mathscr{A})\overset{D}{=}\{\langle XY\rangle\mid Y=X\cdot\mathscr{A}\}$.

Def $\mathscr{A}^{\mathscr{B}}\overset{D}{=}\mathscr{E}xp(\mathscr{A},\mathscr{B})\overset{D}{=}\mathscr{I}t(\mathscr{P}_5(\mathscr{A}),h(\mathscr{A}))`\mathscr{B}$.

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4.62. $[\mathfrak{a}=0 \supset \mathfrak{a}^0=0]$. $[0 \in \mathfrak{a} \supset \mathfrak{a}^0=1]$.

The following six theorems are proved simultaneously.

4.63. $\mathscr{I}t(\mathscr{P}_5(\mathfrak{a}),\ h(\mathfrak{a}))$ Fnc $(\mathscr{O} \to \mathscr{O})$. $\mathfrak{a}^{\mathfrak{b}'}=\mathfrak{a}^{\mathfrak{b}}\mathfrak{a}$. $\mathfrak{a}^{\mathfrak{l}}=\bigcup\{\mathfrak{a}^{\mathfrak{y}} \mid \mathfrak{y} \in \mathfrak{l}\}$.

4.46. $0^{\mathfrak{b}}=0$. 4.65. $1^{\mathfrak{b}}=1$. 4.66. $1 \in \mathfrak{a}.\ \mathfrak{c} \in \mathfrak{b} \to \mathfrak{a}^{\mathfrak{c}} \in \mathfrak{a}^{\mathfrak{b}}$.

4.67. $1 \in \mathfrak{a}.\ \mathfrak{b} \in \mathscr{L}im \to \mathfrak{a}^{\mathfrak{b}} \in \mathscr{L}im$. 4.68. $\mathfrak{a}^{\mathfrak{b}} \in \mathscr{O}$. 4.69. $\mathfrak{a}^{\mathfrak{b}+\mathfrak{c}}=\mathfrak{a}^{\mathfrak{b}} \cdot \mathfrak{a}^{\mathfrak{c}}$.

4.70. $(\mathfrak{a}^{\mathfrak{b}})^{\mathfrak{c}}=\mathfrak{a}^{\mathfrak{b}\mathfrak{c}}$. 4.71. $\mathfrak{a} \in \mathfrak{b}.\ \mathfrak{a}^{\mathfrak{c}},\ \mathfrak{b}^{\mathfrak{c}} \in \mathfrak{z} \in \mathscr{C}onnex \to \mathfrak{a}^{\mathfrak{c}} \leqq \mathfrak{b}^{\mathfrak{c}}$.

*4.72. $\mathfrak{a} \in \mathfrak{b} \to \mathfrak{a}^{\mathfrak{c}} \subseteq \mathfrak{b}^{\mathfrak{c}}$. 4.73. $\mathfrak{a},\ \mathfrak{b} \in \mathscr{C}onnex \to \mathfrak{a}^{\mathfrak{b}} \in \mathscr{C}onnex$.

4.74. $\mathfrak{a},\ \mathfrak{b} \in \mathscr{W}ord \to \mathfrak{a}+\mathfrak{b},\ \mathfrak{a}\mathfrak{b},\ \mathfrak{a}^{\mathfrak{b}} \in \mathscr{W}ord$. 4.75. $n^m \in \omega$.

4.76. $1 \in n \to n^{\omega}=\omega$. 4.77. $n+m,\ nm,\ n^m \in \mathscr{W}ord$.

4.78. $\alpha+\beta,\ \alpha\beta,\ \alpha^{\beta} \in \Omega$.

§ 5. Recursive construction of ordinal numbers.

Def $\mathscr{G}_1(\mathscr{P},\ \mathscr{A}) \overset{D}{=} \{\langle YZ\rangle \mid (\mathscr{D}o`\ Y=0.\ Z=\mathscr{A})$
$\vee(\mathscr{D}o`\ Y \in \mathscr{S}c``\omega.\ Z=\mathscr{P}`\ Y`\ \mathscr{S}c^{-1}`\ \mathscr{D}o`\ Y)\}$.

Def $\mathrm{It}(\mathscr{P},\ \mathscr{A}) \overset{D}{=} \mathscr{R}_1(\mathscr{G}_1(\mathscr{P},\ \mathscr{A}))$.

The following theorems may be proved in an analogous fashion to 4.03, 4.04 and 4.06.

5.01. $\mathscr{A} \in \mathscr{C}ls \to \mathrm{It}(\mathscr{P},\ \mathscr{A})`\ 0=\mathscr{A}$.

5.02. $n' \subseteq \mathscr{D}(\mathrm{It}(\mathscr{P},\ \mathscr{A}))$. $\mathrm{It}(\mathscr{P},\ \mathscr{A})`\ n \in \mathscr{D}(\mathscr{P}) \cup -\mathscr{D}(\mathscr{P})$
$\to \mathrm{It}(\mathscr{P},\ \mathscr{A})`\ n'=\mathscr{P}`\ \mathrm{It}(\mathscr{P},\ \mathscr{A})`\ n$.

5.03. $\mathscr{A} \in \mathscr{B}$. $\forall m(m' \subseteq \mathscr{D}(\mathrm{It}(\mathscr{P},\ \mathscr{A})) .\supset. \mathrm{It}(\mathscr{P},\ \mathscr{A})`\ m \in \mathscr{B})$.
$\mathscr{B} \subseteq \mathscr{D}(\mathscr{P}) \cup -\mathscr{D}(\mathscr{P}) \to \mathrm{It}(\mathscr{P},\ \mathscr{A})$ Fnc $(\omega \to \mathscr{B})$.
$\mathrm{It}(\mathscr{P},\ \mathscr{A})`\ 0=\mathscr{A}$. $\mathrm{It}(\mathscr{P},\ \mathscr{A})`\ n'=\mathscr{P}`\ \mathrm{It}(\mathscr{P},\ \mathscr{A})`\ n$.

Def $n-m=\mathrm{It}(\mathscr{S}c^{-1},\ n)`\ m$.

5.04. $\mathrm{It}(\mathscr{S}c^{-1},\ n)$ Fnc $(\omega \to \omega)$. $n-0=n$. $n-m'=\mathscr{S}c^{-1}`\ (n-m)$.

Proof. Put $a \overset{D}{=} \{m \mid m' \subseteq \mathscr{D}(\mathrm{It}(\mathscr{S}c^{-1},\ n)) .\supset. n-m \in \omega\}$. Assume $m \subseteq a$ and $m' \subseteq \mathscr{D}(\mathrm{It}(\mathscr{S}c^{-1},\ n))$. If $m=0$, then, by 5.01, $n-0=n \in \omega$. Assume $m=k'$. Since $k \in a$ and $k' \subseteq \mathscr{D}(\mathrm{It}(\mathscr{S}c^{-1},\ n))$, we obtain $n-k \in \omega$. On the other hand, $\omega \subseteq \mathscr{D}(\mathscr{S}c^{-1}) \cup -\mathscr{D}(\mathscr{S}c^{-1})$. Hence $n-k \in \mathscr{D}(\mathscr{S}c^{-1}) \cup -\mathscr{D}(\mathscr{S}c^{-1})$. Sc, by 5.02, $n-k'=\mathscr{S}c^{-1}`\ (n-k)$ and hence $n-m \in \omega$. Accordingly $m \in a$ and hence $\omega \subseteq a$. So if we substitute n for $\mathscr{A}$, ω for $\mathscr{B}$ and

$\mathscr{S}c^{-1}$ for $\mathscr{P}$ in 5.03, then the premise holds and the conclusion gives the theorem.

5.05. $\mathscr{S}c^{-1}{}^{\text{‘}}\ n = n - 1$. 5.06. $(n-m)-k = n-(m+k)$.

5.07. $(n-m)-k = (n-k)-m$. 5.08. $m \leq n \rightarrow (n-m)+m = n$.

5.09. $1 \leq m \rightarrow (n-1)-(m-1) = n-m$. 5.10. $(n+m)-m = n$.

5.11. $n \leq m \rightarrow n-m = 0$. 5.12. $m \leq n \rightarrow n-(n-m) = m$.

Def $\mathscr{P}_6 \overset{D}{=} \{\langle YZ \rangle | Z = Y \times \Omega\}$.

5.13. $\text{It}(\mathscr{P}_6,\ A)\ \text{Fnc}\ (\omega \rightarrow \mathscr{C}ls)$. $\text{It}(\mathscr{P}_6,\ A)^{\text{‘}}\ 0 = A$.
$\text{It}(\mathscr{P}_6,\ A)^{\text{‘}}\ n' = \text{It}(\mathscr{P}_6,\ A)^{\text{‘}}\ n \times \Omega$.

5.14. $A \subseteq B \rightarrow \text{It}(\mathscr{P}_6,\ A)^{\text{‘}}\ n \subseteq \text{It}(\mathscr{P}_6,\ B)^{\text{‘}}\ n$.

Def $T = \{\langle 00 \rangle\} \cup \{\langle n',\ \text{It}(\mathscr{P}_6,\ \Omega)^{\text{‘}}\ n \rangle | n \in \omega\} \cup \{\langle \omega,\ \cup \text{It}(\mathscr{P}_6,\ \Omega)\text{“}\ \omega \rangle\}$.

5.15. $T\ \text{Fnc}\ (\omega' \rightarrow \mathscr{C}ls)$. $T^{\text{‘}}\ 0 = 0$. $T^{\text{‘}}\ 1 = \Omega$. $T^{\text{‘}}\ n'' = T^{\text{‘}}\ n' \times \Omega$.
$T^{\text{‘}}\ \omega = \cup T\text{“}\ \omega$.

The elements of $T^{\text{‘}}\ n$ are n-tuples whose members are elements of Ω. $T^{\text{‘}}\ \omega$ is the class of all n-tuples where n is an arbitrary natural number.

5.16. $T \in \mathscr{C}ls$. 5.17. $A \subseteq T^{\text{‘}}\ n \rightarrow \text{It}(\mathscr{P}_6,\ A)^{\text{‘}}\ m \subseteq T^{\text{‘}}\ (n+m)$.

5.18. $T^{\text{‘}}\ n \subseteq T^{\text{‘}}\ \omega \subseteq \mathscr{S}et$. 5.19. $a \in T^{\text{‘}}\ n \cap T^{\text{‘}}\ m \rightarrow n = m$.

Proof. Put $b \overset{D}{=} \{n | \forall am(a \in T^{\text{‘}}\ n \cap T^{\text{‘}}\ m . \supset . n = m)\}$. Assume $n \subseteq b$ and $a \in T^{\text{‘}}\ n \cap T^{\text{‘}}\ m$. By $T^{\text{‘}}\ 0 = 0$, we may assume that $1 \leq n$. $1 \leq m$. Let $n = 1$. If $1 < m$, then there are c and α such that $a = \langle c\alpha \rangle$. $c \in T^{\text{‘}}\ (m-1)$. By $a \in T^{\text{‘}}\ 1 = \Omega$, $\langle c\alpha \rangle \in \Omega$. Since then $c \in \{c\} \in \langle c\alpha \rangle \in \mathscr{C}omp$, $c \in \langle c\alpha \rangle$ and hence $c = \{c\}$ or $c = \{c\alpha\}$ which is impossible. Hence $m = 1$ and so $n = m$. Let $1 < n$. If $m = 1$, then we have a contradiction in a similar way. Then assume $1 < n$. $1 < m$. By $a \in T^{\text{‘}}\ n$, there is c and α such that $a = \langle c\alpha \rangle$. $c \in T^{\text{‘}}\ (n-1)$. By $a \in T^{\text{‘}}\ m$, $c \in T^{\text{‘}}\ (m-1)$. Since $n-1 \in b$ and $c \in T^{\text{‘}}\ (n-1) \cap T^{\text{‘}}\ (m-1)$, we obtain $n-1 = m-1$. Hence, by 5.08, $n - m$. So, $n \in b$ and hence $\omega \subseteq b$.

5.20. $\mathscr{P}_1{}^{\text{‘}}\ \alpha = \alpha$. $\mathscr{P}_2{}^{\text{‘}}\ \alpha = \alpha$. 5.21. $T^{\text{‘}}\ \omega \subseteq \mathscr{D}(\mathscr{P}_1)$. $T^{\text{‘}}\ \omega \subseteq \mathscr{D}(\mathscr{P}_2)$.

5.22. $a \in T^{\text{‘}}\ \omega \rightarrow \text{It}(\mathscr{P}_1,\ a)\ \text{Fnc}\ (\omega \rightarrow T^{\text{‘}}\ \omega)$. $\text{It}(\mathscr{P}_1,\ a)^{\text{‘}}\ 0$
$= a$. $\text{It}(\mathscr{P}_1,\ a)^{\text{‘}}\ n' = \mathscr{P}_1{}^{\text{‘}}\ \text{It}(\mathscr{P}_1,\ a)^{\text{‘}}\ n$.

Proof. Put $b \overset{D}{=} \{m | m' \subseteq \mathscr{D}(\text{It}(\mathscr{P}_1,\ a)) . \supset . \text{It}(\mathscr{P}_1,\ a)^{\text{‘}}\ m \in T^{\text{‘}}\ \omega\}$. Assume

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$m \subseteq b$ and $m' \subseteq \mathscr{D}(\mathrm{It}(\mathscr{P}_1, a))$. If $m=0$, then, by 5.01 and $a \in T`\omega$, $\mathrm{It}(\mathscr{P}_1, a)`0 \in T`\omega$. Assume $m = n'$. By $n \in b$ and $n' \subseteq \mathscr{D}(\mathrm{It}(\mathscr{P}_1, a))$, $\mathrm{It}(\mathscr{P}_1, a)`n \in T`\omega$. By 5.21, $\mathrm{It}(\mathscr{P}_1, a)`n \in \mathscr{D}(\mathscr{P}_1) \cup -\mathscr{D}(\mathscr{P}_1)$ and hence, by 5.02, $\mathrm{It}(\mathscr{P}_1, a)`n = \mathscr{P}_1`\mathrm{It}(\mathscr{P}_1, a)`n$. By $\mathrm{It}(\mathscr{P}_1, a)`n \in T`\omega$ and 5.15, there is $k$ such that $\mathrm{It}(\mathscr{P}_1, a)`n \in T`k$. If $k=1$, then, by 5.20, $\mathscr{P}_1`\mathrm{It}(\mathscr{P}_1, a)`n \in T`k$. Let $1<k$. Then there is c and α such that $\mathrm{It}(\mathscr{P}_1, a)`n = \langle c\alpha\rangle . c \in T`(k-1)$. $\mathscr{P}_1`\mathrm{It}(\mathscr{P}_1, a)`n = c$ and hence $\mathscr{P}_1`\mathrm{It}(\mathscr{P}_1, a)`n \in T`\omega$. So $n \in b$. Accordingly $\omega \subseteq b$. Hence if we substitute $a$ for $\mathscr{A}$, $T`\omega$ for $\mathscr{B}$ and $\mathscr{P}_1$ for $\mathscr{P}$ in 5.03, then we can obtain the theorem.

5.23. $a \in T`\omega \to \mathrm{It}(\mathscr{P}_1, \mathrm{It}(\mathscr{P}_1, a)`n)`m = \mathrm{It}(\mathscr{P}_1, a)`(n+m)$.

5.24. $a \in T`n.\ m<n \to \mathrm{It}(\mathscr{P}_1, a)`m \in T`(n-m)$.

Def $M_k = \{\langle YZ\rangle \mid \exists n[Y \in T`n.\ (1 \le k \le n .\supset. Z = \mathscr{P}_2`\mathrm{It}(\mathscr{P}_1, Y)`(n-k)).$
$(\neg(1 \le k \le n) .\supset. Z=0)]\}$.

For a and k such that $a \in T`n$ and $1 \le k \le n$, $M_k`a$ gives the k-th member of a.

5.25. M_k Fnc $(T`\omega \to \Omega)$.

Proof. By definition, Funct (M_k) is easily proved, using 5.19 and $(1 \le k \le n) \vee \neg(1 \le k \le n)$. Evidently $\mathscr{D}(M_k) \subseteq T`\omega$. We prove that $T`\omega \subseteq \mathscr{D}(M_k)$ and $\mathscr{W}(M_k) \subseteq \Omega$. Let $Y \in T`\omega$. Then there is $n$ such that $Y \in T`n$. $(1 \le k \le n) \vee \neg(1 \le k \le n)$ holds. Let $1 \le k \le n$. $n-k<n$ and hence, by 5.24 and 5.12, $\mathrm{It}(\mathscr{P}_1, Y)`(n-k) \in T`(n-(n-k)) = T`k$ and so $\mathscr{P}_2`\mathrm{It}(\mathscr{P}_1, Y)`(n-k) \in \Omega$. Accordingly $\langle Y, \mathscr{P}_2`\mathrm{It}(\mathscr{P}_1, Y)`(n-k)\rangle \in M_k$. If $\neg(1 \le k \le n)$, then $\langle Y, 0\rangle \in M_k$. Hence $Y \in \mathscr{D}(M_k)$. So $T`\omega \subseteq \mathscr{D}(M_k)$. From the above, we also see that $\mathscr{W}(M_k) \subseteq \Omega$. Accordingly M_k Fnc $(T`\omega \to \Omega)$.

5.26. $M_k \in \mathscr{C}ls$.

5.27. $a \in T`n.\ 1 \le k \le n.\ m \le n-k \to M_k`a = M_k`\mathrm{It}(\mathscr{P}_1, a)`m$.

Proof. By definition and the premise, $M_k`a = \mathscr{P}_2`\mathrm{It}(\mathscr{P}_1, a)`(n-k)$ and $\mathrm{It}(\mathscr{P}_1, a)`m \in T`(n-m)$. By the premise, we can obtain $k \le n-m$. Hence, $M_k`\mathrm{It}(\mathscr{P}_1, a)`m = \mathscr{P}_2`\mathrm{It}(\mathscr{P}_1, \mathrm{It}(\mathscr{P}_1, a)`m)`((n-m)-k) = \mathscr{P}_2`\mathrm{It}(\mathscr{P}_1, a)`(n-k) = M_k`a$.

5.28. $a, b \in T`n.\ \forall k(1 \le k \le n .\supset. M_k`a = M_k`b) \to a=b$.

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5.29. $a \in T`\ n \rightarrow \forall k(1 \leq k \leq n \supset M_k`\ a = 0) \vee \exists k(1 \leq k \leq n.\ \exists u(u \in M_k`\ a).$
$\forall m(k < m \leq n \supset M_m`\ a = 0)).$

Def $\mathscr{L}(a,\ n) \overset{D}{=} \{\langle kZ\rangle \mid k \in n' \cap -2.\ Z = \mathrm{It}(\mathscr{P}_6,\ \{\mathrm{It}(\mathscr{P}_1,\ a)`\ (n-k)'\}$
$\times M_k`\ a)`\ (n-k)\}.$

Def $L \overset{D}{=} \{\langle aZ\rangle \mid \exists n(a \in T`\ n.\ Z = \mathrm{It}(\mathscr{P}_6,\ M_1`\ a)`\ (n-1)$
$\cup (\cup \mathscr{L}(a,\ n)``\ (n' \cap -2))\}.$

The following two theorems are proved simultaneously.

5.30. L Fnc $(T`\ \omega \rightarrow \mathscr{P}ow(T`\ \omega))$.   5.31. $a \in T`\ n \rightarrow L`\ a \subseteq T`\ n.$

Proof of 5.30 and 5.31. It may be easily proved that Funct(L) and $\mathscr{D}(L) \subseteq T`\ \omega$. We prove that $T`\ \omega \subseteq \mathscr{D}(L)$ and $\mathscr{W}(L) \subseteq \mathscr{P}ow(T`\ \omega)$. Let $a \in T`\ \omega$. Then there is n such that $a \in T`\ n$. We may assume $0 < n$. $M_1`\ a \in \Omega$ and hence $M_1`\ a \subseteq \Omega$. By 5.14, $\mathrm{It}(\mathscr{P}_6,\ M_1`\ a)`\ (n-1) \subseteq \mathrm{It}(\mathscr{P}_6,\ \Omega)`\ (n-1) = T`\ n$. Next we show that $\cup \mathscr{L}(a, n)``\ (n' \cap -2) \subseteq T`\ n$. Let $A \in B \in \mathscr{L}(a, n)``\ (n' \cap -2)$ for arbitrary classes A and B. By definition, there is k such that $\langle kB\rangle \in (n' \cap -2) \upharpoonright \mathscr{L}(a,n)$. Then $B = \mathrm{It}(\mathscr{P}_6,\ \{\mathrm{It}(\mathscr{P}_1,\ a)`\ (n-k)'\} \times M_k`\ a)`\ (n-k)$. By $k \in n' \cap -2$, $(n-k)' < n$. Hence, by 5.24 and $a \in T`\ n$, $\mathrm{It}(\mathscr{P}_1,\ a)`\ (n-k)' \in T`\ (n-(n-k)') = T`\ (k-1)$. Hence $\{\mathrm{It}(\mathscr{P}_1,\ a)`\ (n-k)'\} \times M_k`\ a \subseteq T`\ (k-1) \times \Omega = T`\ k$. So, by 5.17, $B \subseteq T`\ (k+(n-k)) = T`\ n$. By $A \in B$, $A \in T`\ n$. Accordingly we can obtain $\cup \mathscr{L}(a,\ n)``\ (n' \cap -2) \subseteq T`\ n$. Put $E \overset{D}{=} \mathrm{It}(\mathscr{P}_6,\ M_1`\ a)`\ (n-1) \cup (\cup \mathscr{L}(a,\ n)``\ (n' \cap -2))$. Hence $E \subseteq T`\ n$. By $T`\ n \in \mathscr{C}ls$ and 2.14, $E \in \mathscr{C}ls$. So $\langle aE\rangle \in L$ and hence $a \in \mathscr{D}(L)$ and $L`\ a \subseteq T`\ n$. Thence, it follows that $T`\ \omega \subseteq \mathscr{D}(L)$ and $\mathscr{W}(L) \subseteq \mathscr{P}ow(T`\ \omega)$. Hence L Fnc $(T`\ \omega \rightarrow \mathscr{P}ow(T`\ \omega))$. From the above, 5.31 also holds.

5.32. $L \in \mathscr{C}ls$.   5.33. $a \in T`\ 1 \rightarrow L`\ a = a.$

5.34. $a \in T`\ n.\ 2 \leq n \rightarrow L`\ a = (L`\ \mathscr{P}_1`\ a \times \Omega) \cup (\{\mathscr{P}_1`\ a\} \times \mathscr{P}_2`\ a).$

Proof. $L`\ a = \mathrm{It}(\mathscr{P}_6,\ M_1`\ a)`\ (n-1) \cup (\cup \mathscr{L}(a, n)``\ (n' \cap -2))$. By $2 \leq n$ and 5.27, $M_1`\ a = M_1`\ \mathscr{P}_1`\ a$ and hence $\mathrm{It}(\mathscr{P}_6,\ M_1`\ a)`\ (n-1) = \mathrm{It}(\mathscr{P}_6,\ M_1`\ \mathscr{P}_1`\ a)`(n-2) \times \Omega$. We can prove that $\cup \mathscr{L}(a, n)``(n \cap -2) = (\cup \mathscr{L}(\mathscr{P}_1`a,\ n-1)``\ (n \cap -2)) \times \Omega$ and $\mathscr{L}(a,\ n)`\ n = \{\mathscr{P}_1`\ a\} \times \mathscr{P}_2`\ a$. Since $\mathscr{P}_1`\ a \in T`\ (n-1)$, we obtain $L`\ a = ([\mathrm{It}(\mathscr{P}_6,\ M_1`\ \mathscr{P}_1`\ a)`\ (n-2) \cup (\cup \mathscr{L}(\mathscr{P}_1`\ a,\ n-1)``(n \cap -2))] \times \Omega) \cup (\{\mathscr{P}_1`\ a\} \times \mathscr{P}_2`\ a) = (L`\ \mathscr{P}_1`\ a \times \Omega) \cup (\{\mathscr{P}_1`\ a\} \times \mathscr{P}_2`\ a)$.

5.35. $a \in T`\ \omega \rightarrow a \notin L`\ a.$

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5.36. $a \in T' \omega \to L' \langle a\beta' \rangle = L' \langle a\beta \rangle \cup \{\langle a\beta \rangle\}$.

5.37. $b \in T' n.\ a \in L' b \to L' a \subseteq L' b$.

Proof. Put $c \overset{D}{=} \{n | \forall ab(b \in T' n.\ a \in L' b : \supset .\ L' a \subseteq L' b)\}$. Assume $n \subseteq c$ and $b \in T' n.\ a \in L' b$. If $n = 0$ or $n = 1$, then we have $L' a \subseteq L' b$. Let $1 < n$. By 5.34, $L' b = (L' \mathscr{P}_1' b \times \Omega) \cup (\{\mathscr{P}_1' b\} \times \mathscr{P}_2' b)$. Let $a \in L' \mathscr{P}_1' b \times \Omega$. Then there is d such that $a = \langle d\alpha \rangle$. $d \in L' \mathscr{P}_1' b$. Since $\mathscr{P}_1' b \in T' (n-1)$, $L' d \subseteq L' \mathscr{P}_1' b$ and hence $L' d \times \Omega \subseteq L' \mathscr{P}_1' b \times \Omega$. Also $\{d\} \times \alpha \subseteq L' \mathscr{P}_1' b \times \Omega$ and hence $L' a = (L' d \times \Omega) \cup (\{d\} \times \alpha) \subseteq L' b$. Let $a \in \{\mathscr{P}_1' b\} \times \mathscr{P}_2' b$. Then there is β such that $a = \langle \mathscr{P}_1' b, \beta \rangle$. $\beta \in \mathscr{P}_2' b$. $L' a = (L' \mathscr{P}_1' b \times \Omega) \cup (\{\mathscr{P}_1' b\} \times \beta)$ and $\{\mathscr{P}_1' b\} \times \beta \subseteq \{\mathscr{P}_1' b\} \times \mathscr{P}_2' b$. Hence $L' a \subseteq L' b$.

5.38. $b \in T' \omega.\ a \in L' b \to L' a \subseteq L' b$.

5.39. $a \in T' n \to \exists k(1 \leq k \leq n.\ \exists u(u \in M_k' a)) .\equiv. \exists u(u \in L' a)$.

Proof. Put $b \overset{D}{=} \{n | \forall a(a \in T' n.\ \exists k(1 \leq k \leq n.\ \exists u(u \in M_k' a)) : \supset .\ \exists u(u \in L' a))\}$. Assume $n \subseteq b$. If $n = 0$ or $n = 1$, then $n \in b$. Assume $a \in T' n.\ 1 \leq k \leq n.\ \exists u(u \in M_k' a)$. Then $k < n \vee k = n$. Let $k < n$. By 5.27, $M_k' \mathscr{P}_1' a = M_k' a$. Hence $\mathscr{P}_1' a \in T' (n-1)$. $\exists k(1 \leq k \leq n-1.\ \exists u(u \in M_k' \mathscr{P}_1' a))$. By $n - 1 \in b$, we obtain $\exists u(u \in L' \mathscr{P}_1' a)$. It is easy to prove $\exists u(u \in L' a)$. Let $k = n$. $M_k' a = M_n' a = \mathscr{P}_2' a$ and hence $\exists u(u \in \mathscr{P}_2' a)$. Hence. by 5.34, $\exists u(u \in L' a)$. So $n \in b$ and hence $\omega \subseteq b$. Next we put $c \overset{D}{=} \{n | \forall a(a \in T' n.\ \exists u(u \in L' a) : \supset .\ \exists k(1 \leq k \leq n.\ \exists u(u \in M_k' a)))\}$. Assume $n \subseteq c$. If $n = 0$ or $n = 1$, then $n \in b$. Assume $1 < n$ and $a \in T' n.\ \exists u(u \in L' a)$. Let $d \in L' a$. Assume $d \in L' \mathscr{P}_1' a \times \Omega$. Then $\exists u(u \in L' \mathscr{P}_1' a)$. Since $\mathscr{P}_1' a \in c$, $\exists k(1 \leq k \leq n-1.\ \exists u(u \in M_k' \mathscr{P}_1' a))$. Then, by 5.27, $\exists k(1 \leq k \leq n-1.\ \exists u(u \in M_k' a))$. Assume $d \in \{\mathscr{P}_1' a\} \times \mathscr{P}_2' a$. $\exists u(u \in \mathscr{P}_2' a)$, $\mathscr{P}_2' a = M_n' a$ and hence $\exists u(u \in M_n' a)$. Accordingly we have $\exists k(1 \leq k \leq n.\ \exists u(u \in M_k' a))$. So $n \in b$ and we obtain $\omega \subseteq b$.

5.40. $a \in T' n \to \forall k(1 \leq k \leq n .\supset. M_k' a = 0) .\equiv. L' a = 0$.

5.41. $a \in T' \omega \to L' a = 0 \vee \exists u(u \in L' a)$.

Proof. By 5.29, 5.39 and 5.40.

5.42. $a, b \in T' n.\ L' a = L' b \to a = b$.

Proof. Put $c \overset{D}{=} \{n | \forall ab(a, b \in T' n.\ L' a = L' b : \supset .\ a = b)\}$. Assume $n \subseteq c$. If $n = 0$ or $n = 1$, then $n \in c$. Assume $1 < n$ and $a, b \in T' n.\ L' a = L' b$. Let

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$x \in L` \mathscr{P}_1` a$. $\langle x, \mathscr{P}_2` b\rangle \in L` \mathscr{P}_1` a \times \Omega \subseteq L` a$ and hence, by the assumption, $\langle x, \mathscr{P}_2` b\rangle \in L` b$. On the other hand, $\langle x, \mathscr{P}_2` b\rangle \notin \{\mathscr{P}_1` b\} \times \mathscr{P}_2` b$ and hence $\langle x, \mathscr{P}_2` b\rangle \in L` \mathscr{P}_1` b \times \Omega$. So $x \in L` \mathscr{P}_1` b$ and hence $L` \mathscr{P}_1` a \subseteq L` \mathscr{P}_1` b$. Similar for $L` \mathscr{P}_1` b \subseteq L` \mathscr{P}_1` a$ and hence $L` \mathscr{P}_1` a = L` \mathscr{P}_1` b$. Since $\mathscr{P}_1` a$, $\mathscr{P}_1` b \in T` (n-1)$ and $n-1 \in c$, we obtain $\mathscr{P}_1` a = \mathscr{P}_1` b$. Let $y \in \mathscr{P}_2` a$. $\langle \mathscr{P}_1` a, y\rangle \in \{\mathscr{P}_1` a\} \times \mathscr{P}_2` a \subseteq L` a$ and hence $\langle \mathscr{P}_1` a, y\rangle \in L` b$. Since $\langle \mathscr{P}_1` a, y\rangle \notin L` \mathscr{P}_1` b \times \Omega$, it follows that $y \in \mathscr{P}_2` b$. Hence $\mathscr{P}_2` a \subseteq \mathscr{P}_2` b$. Likewise for $\mathscr{P}_2` b \subseteq \mathscr{P}_2` a$. Hence $a = \langle \mathscr{P}_1` a, \mathscr{P}_2` a\rangle = \langle \mathscr{P}_1` b, \mathscr{P}_2` b\rangle = b$.

5.43. $a \in T` \omega$. $\exists u(u \in L` a) \rightarrow L^{-1}` a = a$.

Proof. By $a \in T` \omega$ and 5.30, $\langle a, L` a\rangle \in L$. Assume $\langle x, L` a\rangle$, $\langle y, L` a\rangle \in L$. $x, y \in \mathscr{D}o` L = T` \omega$. Then there are n and m such that $x \in T` n$ and $y \in T` m$. By 5.31, $L` x \subseteq T` n$ and $L` y \subseteq T` m$. By $\exists u(u \in L` a)$, there is $b$ such that $b \in L` a$. By the assumption, $L` x = L` a$ and $L` y = L` a$. Hence $b \in T` n \cap T` m$ and so, by 5.19, $n = m$. Hence $x, y \in T` n$ and together with $L` x = L` y$, we obtain $x = y$, by 5.42. Hence $\exists ! x(\langle x, L` a\rangle \in L)$ and $\exists ! x(\langle L` a, x\rangle \in L^{-1})$. Since $\langle L` a, a\rangle \in L^{-1}$, we have $L^{-1}` L` a = a$.

5.44. $a, b \in T` \omega$. $\exists u(u \in L` a)$. $L` a = L` b \rightarrow a = b$.

Proof. $a = L^{-1}` L` a = L^{-1}` L` b = b$, by 5.43.

5.45. $\forall u(u \in T` n.\ L` u \subseteq \mathscr{A} : \supset.\ u \in \mathscr{A}) \rightarrow T` n \subseteq \mathscr{A}$.

Proof. Put $a \overset{D}{=} \{n \mid \forall \mathscr{A} [\forall u(u \in T` n.\ L` u \subseteq \mathscr{A} : \supset.\ u \in \mathscr{A}) .\supset.\ T` n \subseteq \mathscr{A}]\}$. Assume $n \subseteq a$. If $n = 0$ or $n = 1$, then $n \in a$. Assume $1 < n$ and $\forall u(u \in T` n.\ L` u \subseteq \mathscr{A} : \supset.\ u \in \mathscr{A})$. Put $A(b) \overset{D}{=} \{\alpha \mid L` \langle b\alpha\rangle \subseteq \mathscr{A}\}$ and $B \overset{D}{=} \{x \mid x \in T` (n-1).\ \Omega \subseteq A(x)\}$. Let $\alpha \subseteq A(b)$. $L` b \subseteq B$. $b \in T` (n-1)$ and let $c \in L` \langle b\alpha\rangle$. By 5.34, $c \in L` b \times \Omega$ or $c \in \{b\} \times \alpha$. Assume $c \in L` b \times \Omega$. Then $\mathscr{P}_1` c \in L` b$ and hence $\mathscr{P}_1` c \in B$. So, $L` \langle \mathscr{P}_1` c, (\mathscr{P}_2` c)'\rangle \subseteq \mathscr{A}$. By 5.36, $c \in L` \langle \mathscr{P}_1` c, (\mathscr{P}_2` c)'\rangle$ and hence $c \in \mathscr{A}$. Assume $c \in \{b\} \times \alpha$. Then evidently $\alpha \neq 0$. Assume $\alpha = \beta'$. Since $\beta \in A(b)$, $L` \langle b\beta\rangle \subseteq \mathscr{A}$. $\mathscr{P}_2` c \in \alpha$ and hence $\mathscr{P}_2` c \in \beta \vee \mathscr{P}_2` c = \beta$. If $\mathscr{P}_2` c \in \beta$, then $c \in L` \langle b\beta\rangle$ and hence $c \in \mathscr{A}$. If $\mathscr{P}_2` c = \beta$, then, by $\forall u(u \in T` n.\ L` u \subseteq \mathscr{A} : \supset.\ u \in \mathscr{A})$ and $\langle b\beta\rangle \in T` n$, we obtain $c = \langle b\beta\rangle \in \mathscr{A}$. Assume $\alpha \in \mathscr{L}im$. Then $\mathscr{P}_2` c \in (\mathscr{P}_2` c)' \in \alpha$. $(\mathscr{P}_2` c)' \in A(b)$ and hence $L` \langle b, (\mathscr{P}_2` c)'\rangle \subseteq \mathscr{A}$. So $c = \langle b, \mathscr{P}_2` c\rangle \in \mathscr{A}$. Since $c$ is arbitrary, $L` \langle b\alpha\rangle \subseteq \mathscr{A}$ and hence $\alpha \in A(b)$. So $\forall \alpha(\alpha \subseteq A(b) .\supset. \alpha \in A(b))$

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and hence $\Omega \subseteq A(b)$. So $b \in B$. We obtain $\forall b(b \in T'(n-1).\ L'b \subseteq B :\supset.\ b \in B)$. Together with $n-1 \in a$, we have $T'(n-1) \subseteq B$. Let $u \in T'n$. $\mathscr{P}_1'u \in T'(n-1)$ and hence $\mathscr{P}_1'u \in B$. By definition, $L'u = L'\langle \mathscr{P}_1'u, \mathscr{P}_2'u\rangle \subseteq \mathscr{A}$. By $\forall u(u \in T'n.\ L'u \subseteq \mathscr{A} :\supset.\ u \in \mathscr{A})$, $u \in \mathscr{A}$ and so it follows that $T'n \subseteq \mathscr{A}$. Accordingly $\forall u(u \in T'n.\ L'u \subseteq \mathscr{A} :\supset.\ u \in \mathscr{A}).\supset.T'n \subseteq \mathscr{A}$, It implies $n \in a$ and hence we obtain $\omega \subseteq a$.

5.46. $\forall u(u \in T'\omega.\ L'u \subseteq \mathscr{A} :\supset.\ u \in \mathscr{A}) \to T'\omega \subseteq \mathscr{A}$.

Def $\mathscr{R}_2(\mathscr{G}) \overset{D}{=} \cup\{F \mid \exists u(F \text{ Fnc } L'u.\ u \in T'\omega.\ \forall x[x \in L'u.\supset.F'x = \mathscr{G}'(L'x \upharpoonright F)])\}$.

5.47. $\text{Funct}(\mathscr{R}_2(\mathscr{G}))$. $\forall u[u \in T'\omega.\ L'u \subseteq \mathscr{D}o(\mathscr{R}_2(\mathscr{G})).\ L'u \upharpoonright \mathscr{R}_2(\mathscr{G}) \in \mathscr{D}(\mathscr{G}) :\supset.\ \mathscr{R}_2(\mathscr{G})'u = \mathscr{G}'(L'u \upharpoonright \mathscr{R}_2(\mathscr{G}))]$. $[\forall u(u \in T'\omega.\ L'u \subseteq \mathscr{D}(\mathscr{R}_2(\mathscr{G})) :\supset.\ L'u \upharpoonright \mathscr{R}_2(\mathscr{G}) \in \mathscr{D}(\mathscr{G})).\supset.\mathscr{D}(\mathscr{R}_2(\mathscr{G})) = T'\omega]$.

Proof. Analogous to the proof of 4.01. Put $\mathscr{K} \overset{D}{=} \{F \mid \exists u(F \text{ Fnc } L'u.\ u \in T'\omega.\forall x[x \in L'u.\supset.F'x = \mathscr{G}'(L'x \upharpoonright F)])\}$. It can be proved that $\mathscr{D}(\mathscr{R}_2(\mathscr{G})) = \cup\mathscr{D}o``\mathscr{K} \subseteq T'\omega$.

5.471. $F, G \in \mathscr{K}.\ a \in \mathscr{D}o'F \cap \mathscr{D}o'G \to F'a = G'a$.

5.472. $\text{Funct}(\mathscr{R}_2(\mathscr{G}))$.

5.473. $F \in \mathscr{K}.\ a \in \mathscr{D}o'F \to L'a \upharpoonright F = L'a \upharpoonright \mathscr{R}_2(\mathscr{G})$.

5.474. $a \in \mathscr{D}(\mathscr{R}_2(\mathscr{G})) \to \mathscr{R}_2(\mathscr{G})'a = \mathscr{G}'(L'a \upharpoonright \mathscr{R}_2(\mathscr{G}))$.

5.475. $\forall u[u \in T'\omega.\ L'u \subseteq \mathscr{D}(\mathscr{R}_2(\mathscr{G})).\ L'u \upharpoonright \mathscr{R}_2(\mathscr{G}) \in \mathscr{D}(\mathscr{G}) :\supset.\ u \in \mathscr{D}(\mathscr{R}_2(\mathscr{G}))]$.

5.476. $\forall u[u \in T'\omega.\ L'u \subseteq \mathscr{D}(\mathscr{R}_2(\mathscr{G})).\ L'u \upharpoonright \mathscr{R}_2(\mathscr{G}) \in \mathscr{D}(\mathscr{G}) :\supset.\ \mathscr{R}_2(\mathscr{G})'u = \mathscr{G}'(L'u \upharpoonright \mathscr{R}_2(\mathscr{G}))]$.

5.477. $\forall u(u \in T'\omega.\ L'u \subseteq \mathscr{D}(\mathscr{R}_2(\mathscr{G})) :\supset.\ L'u \upharpoonright \mathscr{R}_2(\mathscr{G}) \in \mathscr{D}(\mathscr{G})) \to \mathscr{D}(\mathscr{R}_2(\mathscr{G})) = T'\omega$.

Def $\mathscr{G}_2(A) \overset{D}{=} \{\langle YZ\rangle \mid (\mathscr{D}o'Y = 0.\ Z = 0) \vee (\mathscr{D}o'Y = 1.\ Z = A'1) \vee (\mathscr{D}o'Y \in \mathscr{S}c``\mathscr{S}c``\omega.\ Z = \langle Y'\mathscr{S}c^{-1}{}'\mathscr{D}o'Y, A'\mathscr{D}o'Y\rangle)\}$.

5.48. $\omega \subseteq \mathscr{D}o'A.\ \mathscr{W}'A \subseteq \Omega \to \mathscr{R}_1(\mathscr{G}_2(A))$ Fnc $(\omega \to T'\omega)$.

5.49. $\omega \subseteq \mathscr{D}o'A.\ \mathscr{W}'A \subseteq \Omega \to \mathscr{R}_1(\mathscr{G}_2(A))'n \in T'n'$.

5.50. $\omega \subseteq \mathscr{D}o'A.\ \mathscr{W}'A \subseteq \Omega \to \mathscr{R}_1(\mathscr{G}_2(A))'0 = 0.\ \mathscr{R}_1(\mathscr{G}_2(A))'1 = A'1.\ \mathscr{R}_1(\mathscr{G}_2(A))'n'' = \langle \mathscr{R}_1(\mathscr{G}_2(A))'n', A'n''\rangle$.

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Def $S(\alpha, k, m, a) \overset{D}{=} \{\langle n, M_n` a\rangle \mid n < k \vee m < n\} \cup \{\langle n\alpha\rangle \mid k \leq n \leq m\}$.

Def $S(\alpha, k, a) = S(\alpha, k, k, a)$.

Def $Sb(\alpha, k, m) = \{\langle xZ\rangle \mid \exists n(x \in T` n.\ Z = \mathscr{R}_1(\mathscr{G}_2(S(\alpha, k, m, x)))` n)\}$.

Def $Sb(\alpha, k) = Sb(\alpha, k, k)$.

5.51. $Sb(\alpha, k, m)$ Fnc $(T` \omega \to T` \omega)$. 5.52. $Sb(\alpha, k, m) \in \mathscr{C}ls$.

5.53. $a \in T` n \to Sb(\alpha, k, m)` a \in T` n$.

5.54. $a, b \in T` \omega.\ \forall l(l \leq n. \supset. S(\alpha, k, m, a)` l = S(\alpha, k, m, b)` l)$

$$\mathscr{R}_1(\mathscr{G}_2(S(\alpha, k, m, a)))` n = \mathscr{R}_1(\mathscr{G}_2(S(\alpha, k, m, b)))` n.$$

5.55. $a \in T` n.\ n = 1 \to Sb(\alpha, k, m)` a = S(\alpha, k, m, a)` 1$.

5.56. $a \in T` n.\ 1 < n \to Sb(\alpha, k, m)` a$

$$= \langle Sb(\alpha, k, m)` \mathscr{P}_1` a, S(\alpha, k, m, a)` n\rangle.$$

Proof. Put $b \overset{D}{=} \{n \mid \forall a(a \in T` n.\ 1 < n : \supset.\ Sb(\alpha, k, m)` a = \langle Sb(\alpha, k, m)` \mathscr{P}_1` a, S(\alpha, k, m, a)` n\rangle)\}$. Assume $n \subseteq b$ and $a \in T` n.\ 1 < n$. By definition, $Sb(\alpha, k, m)` a = \mathscr{R}_1(\mathscr{G}_2(S(\alpha, k, m, a)))` n = \langle \mathscr{R}_1(\mathscr{G}_2(S(\alpha, k, m, a)))` (n-1), S(\alpha, k, m, a)` n\rangle$ and $Sb(\alpha, k, m)\mathscr{P}_1` a = \mathscr{R}_1(\mathscr{G}_2(S(\alpha, k, m, \mathscr{P}_1` a)))` (n-1)$. Since $\forall l(l \leq n-1. \supset. S(\alpha, k, m, a)` l = S(\alpha, k, m, \mathscr{P}_1` a)` l)$, by 5.54, $\mathscr{R}_1(\mathscr{G}_2(S(\alpha, k, m, a)))` (n-1) = \mathscr{R}_1(\mathscr{G}_2(S(\alpha, k, m, \mathscr{P}_1` a)))` (n-1)$ and hence $Sb(\alpha, k, m)` a = \langle Sb(\alpha, k, m)` \mathscr{P}_1` a, S(\alpha, k, m, a)` n\rangle$.

5.57. $a \in T` \omega.\ n < k \vee m < n \to M_n` Sb(\alpha, k, m)` a = M_n` a$.

Proof. Assume $n < k \vee m < n$. Put $b \overset{D}{=} \{l \mid \forall a(a \in T` l. \supset. M_n` Sb(\alpha, k, m)` a = M_n` a)\}$. Assume $l \subseteq b$ and $a \in T` l$. Evidently $l \neq 0$. By 5.53, $Sb(\alpha, k, m)` a \in T` l$. If $\neg(1 \leq n \leq l)$, then, by definition, $M_n` Sb(\alpha, k, m)` a = 0$. $M_n` a = 0$ and hence $M_n` Sb(\alpha, k, m)` a = M_n` a$. So we may assume $1 \leq n \leq l$. Let $l = 1$. Then $n = 1$. By 5.55, $Sb(\alpha, k, m)` a = S(\alpha, k, m, a)` 1$ and hence, by $1 < k \vee m < 1$, $S(\alpha, k, m, a)` 1 = M_1` a = a$. So we have $M_n` Sb(\alpha, k, m)` a = M_n` a$. Assume $1 < l$. Let $1 \leq n < l$. Then, by 5.27 and 5.56, $M_n` Sb(\alpha, k, m)` a = M_n` \mathscr{P}_1` Sb(\alpha, k, m)` a = M_n` Sb(\alpha, k, m)` \mathscr{P}_1` a$. Since $l-1 \in b$ and $\mathscr{P}_1` a \in T` (l-1)$, $M_n` Sb(\alpha, k, m)` \mathscr{P}_1` a = M_n` \mathscr{P}_1` a = M_n` a$ and hence we obtain $M_n` Sb(\alpha, k, m)` a = M_n` a$. Let $n = l$. Then $M_n` Sb(\alpha, k, m)` a = \mathscr{P}_2` Sb(\alpha, k, m)` a = S(\alpha, k, m, a)` l$. By $n < k \vee m < n$, $l < k \vee m < l$ and hence $S(\alpha, k, m, a)` l = M_n` a$. So we obtain $M_n` Sb(\alpha, k, m)` a = M_n` a$. Accordingly $l \in b$ and we obtain $\omega \subseteq b$.

5.58. $a \in T` n. n < k \rightarrow Sb(\alpha, k, m)` a = a$.

Proof. Let $1 \leq l \leq n$. $l < k$ and hence, by 5.57, $M_l` Sb(\alpha, k, m)` a = M_l` a$. Since $a, Sb(\alpha, k, m)` a \in T` n$, by 5.28, $Sb(\alpha, k, m)` a = a$.

5.59. $a \in T` n. k \leq l \leq m. 1 \leq l \leq n \rightarrow M_l` Sb(\alpha, k, m)` a = \alpha$.

Proof. Assume $k \leq l \leq m$. Put $b \overset{D}{=} \{n | \forall a(a \in T` n. 1 \leq l \leq n : \supset. M_l` Sb(\alpha, k, m)` a = \alpha)\}$. Assume $n \subseteq b$ and $a \in T` n. 1 \leq l \leq n$. Let $n = 1$. Then $l = 1$. By 5.55, $Sb(\alpha, k, m)` a = S(\alpha, k, m, a)` 1$ and hence, by $k \leq l \leq m$, $S(\alpha, k, m, a)` 1 = \alpha$. Hence $M_l` Sb(\alpha, k, m)` a = M_l` \alpha = \alpha$. Let $1 < n$. Assume $1 \leq l < n$. Since $n - 1 \in b$, $M_l` Sb(\alpha, k, m)` \mathscr{P}_1` a = \alpha$. By 5.27, $M_l` Sb(\alpha, k, m)` a = M_l` \mathscr{P}_1` Sb(\alpha, k, m)` a = M_l` Sb(\alpha, k, m)` \mathscr{P}_1` a = \alpha$. Assume $l = n$. By $k \leq l \leq m$, $k \leq n \leq m$ and hence $M_l` Sb(\alpha, k, m)` a = \mathscr{P}_2` Sb(\alpha, k, m)` a = S(\alpha, k, m, a)` n = \alpha$. Hence $n \in b$ and we obtain $\omega \subseteq b$.

5.60. $a \in T` n. n < m \rightarrow Sb(\alpha, k, m)` a = Sb(\alpha, k, n)` a$.

Proof. Let $1 \leq l \leq n$. It holds that $l < k \vee k \leq l$. If $l < k$, then, by 5.57, $M_l` Sb(\alpha, k, m)` a = M_l` a$ and $M_l` Sb(\alpha, k, n)` a = M_l` a$ and hence $M_l` Sb(\alpha, k, m)` a = M_l` Sb(\alpha, k, m)` a$. If $k \leq l \leq n$, then, by 5.59, $M_l` Sb(\alpha, k, m)` a = \alpha = M_l` Sb(\alpha, k, n)` a$. Hence, by 5.28, $Sb(\alpha, k, m)` a = Sb(\alpha, k, n)` a$.

5.61. $a \in T` \omega. m < k \rightarrow Sb(\alpha, k, m)` a = a$.

5.62. $a \in T` n. 1 \leq k \leq n. k' \leq m. \alpha \in M_k` a$

$$\rightarrow Sb(\beta, k', m)` Sb(\alpha, k)` a \in \bar{L}` a$$

Proof. Assume $k' \leq m$. Put $b \overset{D}{=} \{n | \forall a(a \in T` n. 1 \leq k \leq n. \alpha \in M_k` a : \supset. Sb(\beta, k', m)` Sb(\alpha, k)` a \in \bar{L}` a)\}$. Assume $n \subseteq b$ and $a \in T` n. 1 \leq k \leq n. \alpha \in M_k` a$. Let $n = 1$. Then $k = 1$. Hence we can prove that $Sb(\beta, k', m)` Sb(\alpha, k)` a = Sb(\beta, 2, m)` \alpha = \alpha \in a = \bar{L}` a$. Let $1 < n$. By $k \leq n$, $k < n$ or $k = n$. Assume $k < n$. Since $n - 1 \in b$ and $1 \leq k \leq n - 1$, we obtain $Sb(\beta, k', m)` Sb(\alpha, k)` \mathscr{P}_1` a \in \bar{L}` \mathscr{P}_1` a$. Further $m < n \vee n \leq m$. Assume $m < n$. By 5.56, $Sb(\beta, k', m)` Sb(\alpha, k)` a = Sb(\beta, k', m)` \langle Sb(\alpha, k)` \mathscr{P}_1` a, S(\alpha, k, a)` n \rangle = Sb(\beta, k', m)` \langle Sb(\alpha, k)` \mathscr{P}_1` a, M_n` a \rangle = \langle Sb(\beta', k, m)` Sb(\alpha, k)` \mathscr{P}_1` a, M_n` a \rangle$. Assume $n \leq m$. Then $k' \leq n \leq m$. Hence, $Sb(\beta', k, m)` Sb(\alpha, k)` a = \langle Sb(\beta', k, m)` Sb(\alpha, k)` \mathscr{P}_1` a, \beta \rangle$. In both cases, $Sb(\beta, k', m)` Sb(\alpha, k)` a \in \bar{L}` \mathscr{P}_1` a \times \Omega \subseteq L` a$ and hence $Sb(\beta, k', m)` Sb(\alpha, k)` a \in \bar{L}` a$. Assume $k = n$. By 5.59, $Sb(\beta, k', m)` Sb(\alpha, k)` a = Sb(\beta, k', m)` \langle \mathscr{P}_1` a, \alpha \rangle = \langle \mathscr{P}_1` a, \alpha \rangle \in \{\mathscr{P}_1` a\} \times M_k` a$

$=\{\mathscr{P}_1\text{‘ } a\}\times\mathscr{P}_2\text{‘ } a\subseteq L\text{‘ } a$ and hence $Sb(\beta, k', m)\text{‘ } Sb(\alpha, k)\text{‘ } a\in L\text{‘ } a$. Hence $n\in b$ and we obtain $\omega\subseteq b$.

5.63. $a\in T\text{‘ } n.\ 1\leq k\leq n.\ \alpha\in M_k\text{‘ } a\rightarrow Sb(\alpha, k)\text{‘ } a\in L\text{‘ } a.$

5.64. $a\in T\text{‘ } n\rightarrow\forall k(0<k<n.\supset.M_k\text{‘ } a=0)$
$\vee\exists k(0<k<n-1.\ M_k\text{‘ } a\in\mathscr{S}c\text{“ }\Omega.\ \forall m(k<m<n.\supset.M_m\text{‘ } a=0))$
$\vee(1<n.\ M_{n-1}\text{‘ } a\in\mathscr{S}c\text{“ }\Omega)$
$\vee\exists k(0<k<n.\ M_k\text{‘ } a\in\mathscr{L}im.\ \forall m(k<m<n.\supset.M_m\text{‘ } a=0).$

Proof. We may assume $1\leq n$. Let $n=1$. Then $\forall k(0<k<n\supset M_k\text{‘ } a=0)$ and hence the theorem follows. Let $1<n$. Since $\mathscr{P}_1\text{‘ } a\in T\text{‘ } (n-1)$ and $M_k\text{‘ } \mathscr{P}_1\text{‘ } a=M_k\text{‘ } a$ for k such that $k<n$, we obtain, by 5.29, $\forall k(0<k<n \supset M_k\text{‘ } a=0)\vee\exists k(0<k<n.\ \exists u(u\in M_k\text{‘ } a).\ \forall m(k<m<n\supset M_m\text{‘ } a=0))$. By 3.05, $\exists u(u\in M_k\text{‘ } a).\equiv.M_k\text{‘ } a\in\mathscr{S}c\text{“ }\Omega\vee M_k\text{‘ } a\in\mathscr{L}\mathrm{im}$. Then we can prove the theorem.

Def $\mathscr{G}_3\overset{D}{=}\{\langle YZ\rangle|\exists n[(\exists u(\mathscr{D}o\text{‘ } Y=L\text{‘ } u.\ u\in T\text{‘ } n.\ \forall k(0<k<n\supset M_k\text{‘ } u$
$=0)).\ Z=\mathscr{E}xp(M_n\text{‘ } L^{-1}\text{‘ } \mathscr{D}o\text{‘ } Y, M_n\text{‘ } L^{-1}\text{‘ } \mathscr{D}o\text{‘ } Y))$
$\vee\exists k(0<k<n-1.\ \exists u(\mathscr{D}o\text{‘ } Y=L\text{‘ } u.\ u\in T\text{‘ } u.\ M_k\text{‘ } u\in\mathscr{S}c\text{“ }\Omega.$
$\forall m(k<m<n\supset M_m\text{‘ } u=0)).\ Z=Y\text{‘ } Sb(M_n\text{‘ } L^{-1}\text{‘ } \mathscr{D}o\text{‘ } Y, k', n-1)\text{‘ }$
$Sb(\mathscr{S}c^{-1}\text{‘ } M_k\text{‘ } L^{-1}\text{‘ } \mathscr{D}o\text{‘ } Y, k)\text{‘ } L^{-1}\text{‘ } \mathscr{D}o\text{‘ } Y)$
$\vee(\exists u(\mathscr{D}o\text{‘ } Y=L\text{‘ } u.\ u\in T\text{‘ } n.\ 1<n.\ M_{n-1}\text{‘ } u\in\mathscr{S}c\text{“ }\Omega).$
$Z=Y\text{‘ } Sb(Y\text{‘ } Sb(\mathscr{S}c^{-1}\text{‘ } M_{n-1}\text{‘ } L^{-1}\text{‘ } \mathscr{D}o\text{‘ } Y, n-1)\text{‘ } L^{-1}\text{‘ } \mathscr{D}o\text{‘ } Y, n)\text{‘ }$
$Sb(\mathscr{S}c^{-1}\text{‘ } M_{n-1}\text{‘ } L^{-1}\text{‘ } \mathscr{D}o\text{‘ } Y,n-1)\text{‘ } L^{-1}\text{‘ } \mathscr{D}o\text{‘ } Y)$
$\vee\exists k(0<k<n.\ \exists u(\mathscr{D}o\text{‘ } Y=L\text{‘ } u.\ u\in T\text{‘ } n.\ M_k\text{‘ } u\in\mathscr{L}im.\ \forall m(k<m<n$
$\supset M_m\text{‘ } u=0)).\ Z=\cup\{Y\text{‘ } Sb(\theta, k)\text{‘ } L^{-1}\text{‘ } \mathscr{D}o\text{‘ } Y|\theta$
$\in M_k\text{‘ } L^{-1}\text{‘ } \mathscr{D}o\text{‘ } Y\})]\}.$

Def $F_1\overset{D}{=}\mathscr{R}_2(\mathscr{G}_3)$.

The following theorems are proved simultaneously.

5.65. F_1 Fnc $(T\text{‘ } \omega\rightarrow\Omega)$.

5.66. $a\in T\text{‘ } n.\ \forall k(0<k<n\supset M_k\text{‘ } a=0)\rightarrow F_1\text{‘ } a=\mathscr{E}xp(M_n\text{‘ } a, M_n\text{‘ } a).$
$a\in T\text{‘ } n.\ 0<k<n-1.\ M_k\text{‘ } a\in\mathscr{S}c\text{“ }\Omega.\ \forall m(k<m<n\supset M_m\text{‘ } a=0)$
$\rightarrow F_1\text{‘ } a=F_1\text{‘ } Sb(M_n\text{‘ } a, k', n-1)\text{‘ } Sb(\mathscr{S}c^{-1}\text{‘ } M_k\text{‘ } a, k)\text{‘ } a.$
$a\in T\text{‘ } n.\ 1<n.\ M_{n-1}\text{‘ } a\in\mathscr{S}c\text{“ }\Omega$
$\rightarrow F_1\text{‘ } a=F_1\text{‘ } Sb(F_1\text{‘ } Sb(\mathscr{S}c^{-1}\text{‘ } M_{n-1}\text{‘ } a, n-1)\text{‘ } a, n)\text{‘ }$
$Sb(\mathscr{S}c^{-1}\text{‘ } M_{n-1}\text{‘ } a, n-1)\text{‘ } a.$

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$a \in T^{\text{`}}\ n.\ 0<k<n.\ M_k{}^{\text{`}}\ a \in \mathscr{L}im.\ \forall m(k<m<n \supset M_m{}^{\text{`}}\ a=0)$
$\rightarrow F_1{}^{\text{`}}\ a = \cup\{F_1{}^{\text{`}}\ Sb(\theta,\ k)^{\text{`}}\ a \mid \theta \in M_k{}^{\text{`}}\ a\}.$

5.67. $a \in T^{\text{`}}\ n.\ M_n{}^{\text{`}}\ a=0 \rightarrow F_1{}^{\text{`}}\ a=0.$

5.68. $a \in T^{\text{`}}\ n.\ M_n{}^{\text{`}}\ a=1 \rightarrow F_1{}^{\text{`}}\ a=1.$

5.69. $a \in T^{\text{`}}\ n.\ 1 \in M_n{}^{\text{`}}\ a \rightarrow 1 \in F_1{}^{\text{`}}\ a.$

5.70. $a \in T^{\text{`}}\ n \rightarrow F_1{}^{\text{`}}\ a \in \Omega.$

5.71. $a \in T^{\text{`}}\ n.\ 0<k<n.\ \alpha \in M_k{}^{\text{`}}\ a.\ \forall m(k<m<n \supset M_m{}^{\text{`}}\ a=0).\ 1 \in M_n{}^{\text{`}}\ a$
$\rightarrow F_1{}^{\text{`}}\ Sb(\alpha,\ k)^{\text{`}}\ a \in F_1{}^{\text{`}}\ a.$

5.72. $a \in T^{\text{`}}\ n.\ 0<k<n.\ M_k{}^{\text{`}}\ a \in \mathscr{L}im.\ \forall m(k<m<n \supset M_m{}^{\text{`}}\ a$
$=0).\ 1 \in M_n{}^{\text{`}}\ a \rightarrow F_1{}^{\text{`}}\ a \in \mathscr{L}im.$

Before entering the proof, we give illustrations of these theorems informally. If we write elements of $T^{\text{`}}\ n$ in such a form as $\langle\alpha_1 \cdots \alpha_n\rangle$ where $\alpha_1, \ldots, \alpha_n \in \Omega$, then the four formulas of 5.66 wiil mean the following respectively.

$F_1{}^{\text{`}}\ \langle 0 \cdots 0\ \alpha_n\rangle = \alpha_n^{\alpha_n}$ where $0<n$.

$F_1{}^{\text{`}}\ \langle\alpha_1 \cdots \alpha_{k-1}\alpha_k' 0 \cdots 0\alpha_n\rangle = F_1{}^{\text{`}}\ \langle\alpha_1 \cdots \alpha_{k-1}\alpha_k\alpha_n \cdots \alpha_n\rangle$
where $0<k<n-1$.

$F_1{}^{\text{`}}\ \langle\alpha_1 \cdots \alpha_{n-2}\alpha_{n-1}'\alpha_n\rangle = F_1{}^{\text{`}}\ \langle\alpha_1 \cdots \alpha_{n-1}F_1{}^{\text{`}}\ \langle\alpha_1 \cdots \alpha_n\rangle\rangle$ where $1<n$.

$F_1{}^{\text{`}}\ \langle\alpha_1 \cdots \alpha_{k-1}\alpha_k 0 \cdots 0\alpha_n\rangle = \cup\{F_1{}^{\text{`}}\ \langle\alpha_1 \cdots \alpha_{k-1}\theta 0 \cdots 0\alpha_n\rangle \mid \theta \in \alpha_k\}$
where $0<k<n$ and $\alpha_k \in \mathscr{L}im$.

5.67-5.70 means respectively: $F_1{}^{\text{`}}\ \langle\alpha_1 \cdots \alpha_{n-1}0\rangle=0$. $F_1{}^{\text{`}}\ \langle\alpha_1 \cdots \alpha_{n-1}1\rangle =1$. $1 \in \alpha_n \rightarrow 1 \in F_1{}^{\text{`}}\ \langle\alpha_1 \cdots \alpha_n\rangle$. 5.71 means: If $1 \in \alpha_n$ and $\alpha \in \alpha_k$, then $F_1{}^{\text{`}}\ \langle\alpha_1 \cdots \alpha_{k-1}\alpha 0 \cdots 0\alpha_n\rangle \in F_1{}^{\text{`}}\ \langle\alpha_1 \cdots \alpha_k 0 \cdots 0\alpha_n\rangle$ where $0<k<n$. The corresponding proposition when $k=n$ is Theorem 5.76 whose assertion is considerably different from that of 5.71. In general, the last member α_n of $\langle\alpha_1 \cdots \alpha_n\rangle$ plays a distinct role from $\alpha_1, \ldots, \alpha_{n-1}$ when we consider properties of $F_1{}^{\text{`}}\ \langle\alpha_1 \cdots \alpha_n\rangle$. 5.72 may be written thus: If $1 \in \alpha_n$ and $\alpha_k \in \mathscr{L}im$, then $F_1{}^{\text{`}}\ \langle\alpha_1 \cdots \alpha_k 0 \cdots 0\alpha_n\rangle \in \mathscr{L}\mathrm{im}$ where $0<k<n$. Theorem 5.73 is the corresponding proposition when $k=n$.

The class of $F_1{}^{\text{`}}\ \langle\alpha_1 \cdots \alpha_n\rangle$ is not ordered lexicografically, since e.g. $F_1{}^{\text{`}}\ \langle\omega 1\rangle = 1 \in F_1{}^{\text{`}}\ \langle 02\rangle$.

Proof of 5.65-5.72. Put $\mathscr{A} \overset{D}{=} \{a \mid (M_n{}^{\text{`}}\ a=0 \supset F_1{}^{\text{`}}\ a=0).\ (M_n{}^{\text{`}}\ a=1 \supset F_1{}^{\text{`}}\ a =1)\}$. $\mathscr{B} \overset{D}{=} \{a \mid 1 \in M_n{}^{\text{`}}\ a \supset [1 \in F_1{}^{\text{`}}\ a \in \Omega.\ \forall k[0<k<n.\ \forall m(k<m<n \supset M_m{}^{\text{`}}\ a =0):\supset.\quad (\forall\alpha(\alpha \in M_k{}^{\text{`}}\ a \supset F_1{}^{\text{`}}\ Sb(\alpha,\ k)^{\text{`}}\ a \in F_1{}^{\text{`}}\ a).\quad (M_k{}^{\text{`}}\ a \in \mathscr{L}im \supset F_1{}^{\text{`}}\ a$

$\in \mathscr{S}im))]]\}$. $\mathscr{C} \overset{D}{=} \{a|L' \ a \upharpoonright F_1 \in \mathscr{D}(\mathscr{G}_3).\ a \in \mathscr{A} \cap \mathscr{B}\}$. $\mathscr{D} \overset{D}{=} \{a|L' \ a \subseteq \mathscr{D}(F_1)$ $.\supset.a \in \mathscr{C}\}$. Assume $a \in T' \ n$. $L' \ a \subseteq \mathscr{D}$ and $L' \ a \subseteq \mathscr{D}(F_1)$. By 5.64, we divide the proof of $a \in \mathscr{C}$ in four cases.

i) Assume $\forall k(0<k<n \supset M_k' \ a=0)$. By 5.41, $M_n' \ a=0 \vee \exists x(x \in M_n' \ a)$. Let $M_n' \ a=0$. Then $M_n' \ L^{-1'} \ L' \ a=M_n' \ L^{-1'} \ 0=0=M_n' \ a$. Let $\exists x(x \in M_n' \ a)$. By 5.43, $M_n' \ L^{-1'} \ L' \ a=M_n' \ a$. Hence, by 4.78, $\mathscr{E}xp(M_n' \ L^{-1'} \ L' \ a, M_n' \ L^{-1'} \ L' \ a)$ $=\mathscr{E}xp(M_n' \ a, M_n' \ a) \in \Omega$. So we can prove $\langle L' \ a \upharpoonright F_1, \mathscr{E}xp(M_n' \ a, M_n' \ a)\rangle$ $\in \mathscr{G}_3$. Hence $L' \ a \upharpoonright F_1 \in \mathscr{D}(\mathscr{G}_3)$. So, by 5.47, $F_1' \ a=\mathscr{G}_3' \ (L' \ a \upharpoonright F_1)$. Since $\mathrm{Funct}(\mathscr{G}_3)$, $\mathscr{G}_3' \ (L' \ a \upharpoonright F_1)=\mathscr{E}xp(M_n' \ a, M_n' \ a)$. Accordingly, $F_1' \ a=\mathscr{E}xp(M_n'$ $a, M_n' \ a)$. By 4.64 and 4.65, $a \in \mathscr{A}$. By 4.66 and 4.78, we obtain $1 \in M_n' \ a$ $\supset 1 \in F_1' \ a \in \Omega$, whence we have $a \in \mathscr{B}$. Accordingly $a \in \mathscr{C}$.

ii) Assume $0<k<n-1$. $M_k' \ a \in \mathscr{S}c'' \ \Omega$. $\forall m(k<m<n \supset M_m' \ a=0)$. Put $b \overset{D}{=} Sb(M_n' \ a, k'', n-1)' \ Sb(\mathscr{S}c^{-1'} \ M_k' \ a, k)' \ a$. By the assumption, $\mathscr{S}c^{-1'} \ M_k' \ a$ $\in M_k' \ a$ and hence, by 5.62, $b \in L' \ a$. By $L' \ a \subseteq \mathscr{D}(F_1)$, $b \in \mathscr{D}(F_1)$. So, by 2.19, $F_1' \ b \in \mathscr{C}ls$. Using that $L^{-1'} \ \mathscr{D}o' \ (L' \ a \upharpoonright F_1)=a$ and $Sb(\mathscr{S}c^{-1'} \ M_k' \ L^{-1'} \ L' \ a,$ $k)=Sb(\mathscr{S}c^{-1'} \ M_k' \ a, k)$, we can obtain $\langle L' \ a \upharpoonright F_1, F_1' \ b\rangle \in \mathscr{G}_3$. From this and 5.47, we obtain that $L' \ a \upharpoonright F_1 \in \mathscr{D}(\mathscr{G}_3)$ and $F_1' \ a=F_1' \ b$. By $b \in L' \ a$ and 5.37, $L' \ b \subseteq L' \ a$ and hence $L' \ b \subseteq \mathscr{D}(F_1)$. On the other hand, by $b \in L' \ a$, $b \in \mathscr{D}$ and hence $b \in \mathscr{A} \cap \mathscr{B}$. By the definition, $M_n' \ b=M_n' \ a$. So, together with $F_1' \ a$ $=F_1' \ b$, we obtain, by $b \in \mathscr{A}$, $a \in \mathscr{A}$. Let $1 \in M_n' \ a$. By $b \in \mathscr{B}$ and $M_n' \ b$ $=M_n' \ a$, we have $1 \in F_1' \ b \in \Omega$ and hence $1 \in F_1' \ a \in \Omega$. Assume $0<l<n$. $\forall m(l<m<n \supset M_m' \ a=0)$ and $\alpha \in M_l' \ a$. Then $l=k$. By $M_k' \ a \in \mathscr{S}c'' \ \Omega$. there is ζ such that $M_k' \ a=\zeta'$. Then $\alpha \leqq \zeta$. By 5.63, $Sb(\zeta, k)' \ a \in L' \ a$ and hence $Sb(\zeta, k)' \ a \in \mathscr{D}$ and, a fortiori, $Sb(\zeta, k)' \ a \in \mathscr{B}$. Hence, by $1 \in M_n' \ Sb(\zeta, k)' \ a$. $\forall m(k<m<n \supset M_m' \ Sb(\zeta, k)' \ a=0)$, we can obtain $F_1' \ Sb(\alpha, k)' \ a \leqq F_1' \ Sb(\zeta,$ $k)' \ a$. Put $c \overset{D}{=} \{m|k<m<n \supset F_1' \ Sb(\zeta, k)' \ a \in F_1' \ Sb(M_n' \ a, k', m)' \ Sb(\zeta, k)' \ a$. Assume $m \subseteq c$ and $k<m<n$. By $0<k$, $0<m$ and hence $\mathscr{S}c^{-1'} \ m \in m$. So, by $m \subseteq c$, $\mathscr{S}c^{-1'} \ m \in c$ and $k \leq \mathscr{S}c^{-1'} \ m<m$. Hence $F_1' \ Sb(\zeta, k)' \ a \leqq F_1' \ Sb(M_n' \ a,$ $k', \mathscr{S}c^{-1'} \ m)' \ Sb(\zeta, k)' \ a$. By $\zeta \in M_k' \ a$ and 5.62, $Sb(M_n' \ a, k', m)' \ Sb(\zeta, k)' \ a$ $\in L' \ a$ and hence $Sb(M_n' \ a, k', m)' \ Sb(\zeta, k)' \ a \in B$. Hence we obtain $F_1' \ Sb(0,$ $m)' \ Sb(M_n' \ a, k', m)' \ Sb(\zeta, k)' \ a \in F_1' \ Sb(M_n' \ a, k', m)' \ Sb(\zeta, k)' \ a$, using that $1 \in M_n' \ Sb(M_n' \ a, k', m)' \ Sb(\zeta, k)' \ a$ and $\forall y(m<y<n \supset M_y' \ Sb(M_n' \ a, k',$ $m)' \ Sb(\zeta, k)' \ a=0)$ and $0 \in M_m' \ Sb(M_n' \ a, k', m)' \ Sb(\zeta, k)' \ a$. Since $F' \ Sb(0,$ $m)' \ Sb(M_n' \ a, k', m)' \ Sb(\zeta, k)' \ a=F_1' \ Sb(M_n' \ a, k'', \mathscr{S}c^{-1'} \ m)' \ Sb(\zeta, k)' \ a$

$=F_1`\, Sb(\zeta, k)`\, a$, we obtain $F_1`\, Sb(\zeta, k)`\, a \in F_1`\, Sb(M_n`\, a, k', m)`\, Sb(\zeta, k)`\, a$. So $m \in c$ and hence $\omega \subseteq c$. So, $F_1`\, Sb(\zeta, k)`\, a \in F_1`\, Sb(M_n`\, a, k', n-1)`\, Sb(\zeta, k)`\, a = F_1`\, a$. Since $\alpha$ is arbitrary, we have $\forall\alpha(\alpha \in M_k`\, a \supset F_1`\, Sb(\zeta, k)`\, a \in F_1`\, a)$. By $M_k`\, a \in \mathscr{S}c``\, \Omega$, $M_k`\, a \notin \mathscr{L}im$. Accordingly $a \in \mathscr{B}$ and hence $a \in \mathscr{C}$.

iii) Assume $1 < n$. $M_{n-1}`\, a \in \mathscr{S}c``\, \Omega$. Put $d \overset{D}{=} Sb(\mathscr{S}c^{-1}``\, M_{n-1}`\, a, n-1)`\, a$ and $e \overset{D}{=} Sb(F_1`\, d, n)`\, d$. By the assumption $\mathscr{S}c^{-1}``\, M_{n-1}`\, a \in M_{n-1}`\, a$ and hence, by 5.63 $d \in L`\, a$. So $d \in \mathscr{D}$ and by $L`\, d \subseteq \mathscr{D}(F_1)$, $d \in \mathscr{A} \cap \mathscr{B}$. Thence we can prove that $F_1`\, d \in \Omega$. Hence by 5.62 $e \in L`\, a$. By $L`\, a \subseteq \mathscr{D}(F_1)$, $e \in \mathscr{D}(F_1)$ and hence by 2.19 $F_1`\, e \in \mathscr{C}ls$. We can see that $\langle L`\, a \upharpoonright F_1, F_1`\, e\rangle \in \mathscr{G}_3$ and hence $L`\, a \upharpoonright F_1 \in \mathscr{D}(\mathscr{G}_3)$ and $F_1`\, a = F_1`\, e$. We prove $a \in \mathscr{A} \cap \mathscr{B}$. Let $M_n`\, a = 0$. By the definition, $M_n`\, d = M_n`\, a$. Hence, by $d \in \mathscr{A}$, $F_1`\, d = 0$ and so $M_n`\, e = M_n`\, Sb(F_1`\, d, n)`\, d = 0$. Hence by $e \in \mathscr{A}$, $F_1`\, a = F_1`\, e = 0$. So $M_n`\, a = 0 \supset F_1`\, a = 0$. Likewise for $M_n`\, a = 1 \supset F_1`\, a = 1$ and we obtain $a \in \mathscr{A}$. Assume $1 \in M_n`\, a$. By $e \in L`\, a$, $e \in \mathscr{D}$ and, together with $L`\, e \subseteq L`\, a \subseteq \mathscr{D}(F_1)$, $e \in \mathscr{A} \cap \mathscr{B}$. Hence, by $1 \in F_1`\, d = M_n`\, e$, $1 \in F_1`\, e \in \Omega$ and hence $1 \in F_1`\, a \in \Omega$. Assume $0 < l < n$. $\forall m(l < m < n \supset M_m`\, a = 0)$. Let $\alpha \in M_l`\, a$. Then $l = n-1$. By $d \in \mathscr{B}$ and $1 \in M_n`\, a = M_n`\, d$, we obtain $\forall\alpha(\alpha \in M_l`\, d \supset F_1`\, Sb(\alpha, l)`\, d \in F_1`\, d)$. By $\alpha \in M_l`\, a$ and $M_{n-1}`\, a \in \mathscr{S}c``\, \Omega$, we have $\alpha \leqq \mathscr{S}c^{-1}``\, M_l`\, a = M_l`\, d$. Also $Sb(\alpha, l)`\, d = Sb(\alpha, l)`\, a$ and hence $F_1`\, Sb(\alpha, l)`\, a \leqq F_1`\, d$. Put $f \overset{D}{=} \{m \mid 0 \leq m < n \supset F_1`\, d \in F_1`\, Sb(0, m', l)`\, e\}$. Assume $m \subseteq f$ and $0 \leq m < n$. Let $m = 0$. Since $\forall y(0 < y < n \supset M_y`\, Sb(0, 1, l)`\, e = 0)$, we obtain, by the case i), $F_1`\, Sb(0, 1, l)`\, e = \mathscr{E}xp(F_1`\, d, F_1`\, d)$. By $1 \in F_1`\, d \in \Omega$, we can prove, by 4.66, $F_1`\, d \in \mathscr{E}xp(F_1`\, d, F_1`\, d)$. Hence $F_1`\, d \in F_1`\, Sb(0, 1, l)`\, e$. Let $0 < m$. Then $0 \leq \mathscr{S}c^{-1}``\, m < n$ and $\mathscr{S}c^{-1}``\, m \in f$. So, $F_1`\, d \in F_1`\, Sb(0, m, l)`\, e$. If $M_m`\, Sb(0, m', l)`\, e = 0$, then $F_1`\, Sb(0, m, l)`\, e = F_1`\, Sb(0, m', l)`\, e$ and hence $F_1`\, d \in F_1`\, Sb(0, m', l)`\, e$. Hence assume $0 \in M_m`\, Sb(0, m', l)`\, e$. Since $Sb(0, m', l)`\, e \in L`\, a$, $Sb(0, m', l)`\, e \in \mathscr{B}$. Using that $1 \in M_n`\, Sb(0, m', l)`\, e$ and $\forall y(m < y < n \supset M_y`\, Sb(0, m', l)`\, e = 0)$, we obtain $F_1`\, Sb(0, m)`\, Sb(0, m', l)`\, e \in F_1`\, Sb(0, m', l)`\, e$. Namely $F_1`\, Sb(0, m, l)`\, e \in F_1`\, Sb(0, m', l)`\, e$ and hence $F_1`\, d \in F_1`\, Sb(0, m', l)`\, e$. So $m \in f$ and we have $\omega \subseteq f$. Hence $F_1`\, d \in F_1`\, Sb(0, n, l)`\, e = F_1`\, e = F_1`\, a$. Combining $F_1`\, Sb(\alpha, l)`\, a \leqq F_1`\, d$, we obtain $F_1`\, Sb(\alpha, l)`\, a \in F_1`\, a$. Accordingly $\forall\alpha(\alpha \in M_l`\, a \supset F_1`\, Sb(Sb(\alpha, l)`\, a \in F_1`\, a)$. By $M_l`\, a \in \mathscr{S}c``\, \Omega$, $M_l`\, a \in \mathscr{L}im$. Since l is arbitrary, we have $a \in \mathscr{B}$ and hence $a \in \mathscr{C}$.

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iv) Assume $0<k<n$. M_k‘ $a \in \mathscr{L}im$. $\forall m(k<m<n \supset M_m$‘ $a=a)$. Put $\mathscr{F} \stackrel{D}{=} \{\langle XY\rangle | X \in M_k$‘ a. $Y \in F_1$‘ $Sb(X, k)$‘ $a\}$ and $A \stackrel{D}{=} \cup\{F_1$‘ $Sb(\theta, k)$‘ $a | \theta \in M_k$‘ $a\}$. Evidently $\mathrm{Un}(\mathscr{F})$. Since $\mathscr{F}$ “ M_k‘ $a = \{F_1$‘ $Sb(\theta, k)$‘ $a | \theta \in M_k$‘ $a\}$, we can prove $A \in \mathscr{C}ls$, by 2.09 and 2.23. Hence $\langle L$‘ $a \upharpoonright F_1, A\rangle \in \mathscr{G}_3$ and we obtain that L‘ $a \upharpoonright F_1 \in \mathscr{D}(\mathscr{G}_3)$ and F_1‘ $a = A$. Next we prove $a \in \mathscr{A} \cap \mathscr{B}$. Assume M_n‘ $a=0$. Let $\theta \in M_k$‘ a. By 5.63, $Sb(\theta, k)$‘ $a \in L$‘ a and hence $Sb(\theta, k)$‘ $a \in \mathscr{A} \cap \mathscr{B}$. So, by M_n‘ $Sb(\theta, k)$‘ $a = M_n$‘ $a = 0$, F_1‘ $Sb(\theta, k)$‘ $a=0$ and hence $A = \cup\{0\} = 0$. Hence M_n‘ $a = 0 \supset F_1$‘ $a=0$. Likewise for M_n‘ $a=1 \supset F_1$‘ $a=1$ and hence $a \in \mathscr{A}$. Assume $1 \in M_n$‘ a. Let $\alpha \in M_k$‘ a. By M_k‘ $a \in \mathscr{L}im$, $\alpha \in \alpha' \in M_k$‘ a. Hence $Sb(\alpha', k)$‘ $a \in L$‘ a and so $Sb(\alpha', k)$‘ $a \in \mathscr{B}$. Since $1 \in M_n$‘ $Sb(\alpha', k)$‘ a and $\forall m(k<m<n \supset M_m$‘ $Sb(\alpha', k)$‘ $a=0)$, we obtain F_1‘ $Sb(\alpha, k)$‘ $Sb(\alpha', k)$‘ $a \in F_1$‘ $Sb(\alpha', k)$‘ a and hence F_1‘ $Sb(\alpha, k)$‘ $a \in F_1$‘ $Sb(\alpha', k)$‘ a. So, by the definition of A, F_1‘ $Sb(\alpha, k)$‘ $a \in A$. Hence we have $\forall \alpha(\alpha \in M_k$‘ $a \supset F_1$‘ $Sb(\alpha, k)$‘ $a \in F_1$‘ $a)$. Let $B \in A$. Then there is a θ such that $B \in F_1$‘ $Sb(\theta, k)$‘ a. $\theta \in M_k$‘ a. Hence, by F_1‘ $Sb(\theta, k)$‘ $a \in F$‘ a, $B \in \cup A$. Conversely let $B \in C \in A$. Then there is a θ such that $C \in F_1$‘ $Sb(\theta, k)$‘ a. $\theta \in M_k$‘ a. We can prove that $Sb(\theta, k)$‘ $a \in \mathscr{B}$ and hence F_1‘ $Sb(\theta, k)$‘ $a \in \Omega$. So we have $B \in F_1$‘ $Sb(\theta, k)$‘ a and hence, by the definition of A, $B \in A$. Accordingly $A = \cup A$. $0 \in M_k$‘ a and hence F_1‘ $Sb(0, k)$‘ $a \in F$‘ a. That is, $\exists U(U \in F_1$‘ $a)$ and hence F_1‘ $a \in \mathscr{L}im$. Since F_1‘ $a \subseteq \Omega$ is easily obtained, by 3.03, F_1‘ $a \in \Omega$. $Sb(0, k)$‘ $a \in \mathscr{B}$ and hence $1 \in Sb(0, k)$‘ a. So $1 \in F_1$‘ a. Accordingly $1 \in F_1$‘ $a \in \Omega$. Assume $0<l<n$. $\forall m(l<m<n \supset M_m$‘ $l=0)$. Let $\alpha \in M_l$‘ a. Then $l=k$. Hence, by what we have just proved, $\forall \alpha(\alpha \in M$‘ $a \supset F_1$‘ $Sb(\alpha, l)$‘ $a \in F_1$‘ $a)$ and M_l‘ $a \in \mathscr{L}im \supset F_1$‘ $a \in \mathscr{L}im$. So $a \in \mathscr{B}$ and hence $a \in \mathscr{C}$.

Accordingly we have proved $a \in \mathscr{C}$ for all cases and hence L‘ $a \subseteq \mathscr{G}(F_1) .\supset. a \in \mathscr{C}$. So $a \in \mathscr{D}$. Since α is arbitrary, $\forall a(a \in T$‘ $n. L$‘ $a \subseteq \mathscr{D} :\supset. a \in \mathscr{D})$ and hence, by 5.45, T‘ $n \subseteq \mathscr{D}$. So $\forall u(u \in T$‘ n. L‘ $u \subseteq \mathscr{D}(F_1) :\supset. L$‘ $u \upharpoonright F_1 \in (\mathscr{G}_3))$. Since we can obtain this formula for arbitrary n, it follows that $\forall u(u \in T$‘ ω. L‘ $u \subseteq \mathscr{D}(F_1) :\supset. L$‘ $u \upharpoonright F_1 \in \mathscr{D}(\mathscr{G}_3))$ and hence, by 5.47, $\mathscr{D}(F_1) = T$‘ ω. By 5.47, $\mathrm{Funct}(F_1)$ and hence F_1 Fnc T‘ ω. It is not difficult to prove that $\mathscr{W}(F_1) \subseteq \Omega$ and other assertions.

5.73. $a \in T$‘ n. M_n‘ $a \in \mathscr{L}im \to F_1$‘ $a \in \mathscr{L}im$.

Proof. Put $\mathscr{A} \stackrel{D}{=} \{a | M_n$‘ $a \in \mathscr{L}im \supset F_1$‘ $a \in \mathscr{L}im\}$. Assume $a \in T$‘ n. L‘ a

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$\subseteq \mathscr{A}$ and $M_n`\ a \in \mathscr{L}im$. By 5.64, we divide the proof in four cases. Assume $\forall k(0<k<n \supset M_k`\ a=0)$. By $M_n`\ a \in \mathscr{L}im$, $1 \in M_n`\ a$ and hence, by 5.66 and 4.67, $F_1`\ a = \mathscr{E}xp(M_n`\ a, M_n`\ a) \in \mathscr{L}im$. Assume $0<k<n-1$. $M_k`\ a \in \mathscr{S}c``\ \Omega$. $\forall m(k<m<n \supset M_m`\ a=0)$. Put $b \overset{D}{=} Sb(M_n`\ a, k', n-1)`\ Sb(\mathscr{S}c^{-1}`\ M_k`\ a, k)`\ a$. $F_1`\ a = F_1`\ b$. $M_{n-1}`\ b = M_n`\ a$. Hence, by the premise and 5.27, $F_1`\ b \in \mathscr{L}im$ and so $F_1`\ a \in \mathscr{L}im$. Assume $1<n$. $M_{n-1}`\ a \in \mathscr{S}c``\ \Omega$. Put $c \overset{D}{=} Sb(\mathscr{S}c^{-1}`\ M_{n-1}`\ a, n-1)`\ a$ and $d \overset{D}{=} Sb(F_1`\ c, n)`\ c$. By 5.63, $c \in L`\ a$ and hence $c \in \mathscr{A}$. Since $M_n`\ c = M_n`\ a \in \mathscr{L}im$, we obtain $F_1`\ c \in \mathscr{L}im$. By 5.62, $d \in L`\ a$ and hence $d \in \mathscr{A}$. $M_n`\ d = F_1`\ c \in \mathscr{L}im$ and hence $F_1`\ d \in \mathscr{L}im$. By 5.66, $F_1`\ a = F_1`\ d$ and hence $F_1`\ a \in \mathscr{L}im$. Assume $0<k<n$. $M_k`\ a \in \mathscr{L}im$. $\forall m(k<m<n \supset M_m`\ a=0)$. By $M_n`\ a \in \mathscr{L}im$, $1 \in M_n`\ a$ and hence, by 5.72, $F_1`\ a \in \mathscr{L}im$. So $a \in \mathscr{A}$ and we obtain $\forall a(a \in T`\ n.\ L`\ a \subseteq \mathscr{A} :\supset.\ a \in \mathscr{A})$. Accordingly $T`\ n \subseteq \mathscr{A}$.

5.74. $a \in T`\ n.\ \forall k(1 \leq k \leq n \supset M_k`\ a \in \mathscr{C}onnex) \rightarrow F_1`\ a \in \mathscr{C}onnex$.

Proof. Put $\mathscr{A} \overset{D}{=} \{a | \forall k(1 \leq k \leq n \supset M_k`\ a \in \mathscr{C}onnex) \supset F_1`\ a \in \mathscr{C}onnex\}$. Assume $a \in T`\ n$. $L`\ a \subseteq \mathscr{A}$ and $\forall k(1 \leqq k \leq n \supset M_k`\ a \in \mathscr{C}onnex)$. By 5.64, we divide the proof in four cases. Assume $\forall k(0<k<n \supset M_k`\ a=0)$. By 5.66, $F_1`\ a = \mathscr{E}xp(M_n`\ a, M_n`\ a)$. By the assumption, $M_n`\ a \in \mathscr{C}onnex$ and hence, by 4.73, $F_1`\ a \ni \mathscr{C}onnex$. Assume $0<k<n-1$. $M_k`\ a \in \mathscr{S}c``\ \Omega$. $\forall m(k<n \supset M_m`\ a=0)$. Put $b \overset{D}{=} Sb(M_n`\ a, k', n-1)`\ Sb(\mathscr{S}c^{-1}`\ M_k`\ a, k)`\ a$. $b \in L`\ a$ and hence $b \in \mathscr{A}$. By $M_n`\ a$, $\mathscr{S}c^{-1}`\ M_h`\ a \in \mathscr{C}onnex$, we obtain $\forall k(1 \leq k \leq n \supset M_k`\ b \in \mathscr{C}onnex)$ and hence $F`\ b \in \mathscr{C}onnex$. So, by 5.66, $F`\ a = F`\ b \in \mathscr{C}onnex$. Thirdly, assume $1<n$. $M_{n-1}`\ a \in \mathscr{S}c``\ \Omega$. Put $c \overset{D}{=} Sb(\mathscr{S}c^{-1}`\ M_{n-1}`\ a, n-1)`\ a$ and $d \overset{D}{=} Sb(F_1`\ c, n)`\ c$. $c \in L`\ a$ and hence $c \in \mathscr{A}$. Since $M_{n-1}`\ a \in \mathscr{C}onnex$, then $\mathscr{S}c^{-1}`\ M_{n-1}`\ a \in \mathscr{C}onnex$. So $\forall k(1 \leq k \leq n \supset M_k`\ c \in \mathscr{C}onnex)$ and hence, by $c \in \mathscr{A}$, $F_1`\ c \in \mathscr{C}onnex$. Hence $\forall k(1 \leq k \leq n \supset M_k`\ d \in \mathscr{C}onnex)$. $d \in L`\ a$ and hence $d \in \mathscr{A}$. So, together with 5.66, $F_1`\ a = F_1`\ d \in \mathscr{C}onnex$. Finally, assume $0<k<n$. $M_k`\ a \in \mathscr{L}im$. $\forall m(k<n<n \supset M_m`\ a=0)$. Let $\alpha, \beta \in F_1`\ a$. Then there are μ and ν such that $\alpha \in F_1`\ Sb(\mu, k)`\ a$. $\beta \in F_1`\ Sb(\nu, k)`\ a$. μ, ν $M_k`\ a$. By the assumption, $M_k`\ a \in \mathscr{C}onnex$ and hence $\langle \mu\nu \rangle \in \mathscr{C}ompar$. Let $\mu \in \nu$. If $M_n`\ Sb(\nu, k)`\ a=0$ or $M_n`\ Sb(\nu, k)`\ a=1$, then, by 5.67 or 5.68 respectively, $F_1`\ Sb(\mu, k)`\ a = F_1`\ Sb(\nu, k)`\ a$. Assume $1 \in M_n`\ Sb(\nu, k)`\ a$. Then $\mu \in M_k`\ Sb(\nu, k)`\ a$ and hence, by 5.71, $F_1`\ Sb(\mu, k)`\ Sb(\nu, k)`\ a \in F_1`\ Sb(\nu, k)`\ a$. Hence $F_1`\ Sb(\mu, k)`\ a \in F_1`\ Sb(\nu, k)`\ a$. On the other hand, by 5.70, $F_1`\ Sb(\nu,$

$k)^{\text{‘}}\ a \in \Omega$ and hence $F_1{}^{\text{‘}}\ Sb(\nu, k)^{\text{‘}}\ a \in \mathscr{C}omp$. So α, $\beta \in F_1{}^{\text{‘}}\ Sb(\nu, k)^{\text{‘}}\ a$. $Sb(\nu, k)^{\text{‘}}\ a \in L^{\text{‘}}\ a$ and hence $Sb(\nu, k)^{\text{‘}}\ a$. Since $\forall k(1 \leq k \leq n \supset M_k{}^{\text{‘}}\ Sb(\nu, k)^{\text{‘}}\ a \in \mathscr{C}onnex)$, $F_1{}^{\text{‘}}\ Sb(\nu, k)^{\text{‘}}\ a \in \mathscr{C}onnex$. Accordingly $\langle\alpha\,\beta\rangle \in \mathscr{C}ompar$. Likewise for $\mu = \nu$ and for $\nu \in \mu$. Hence $F_1{}^{\text{‘}}\ a \in \mathscr{C}onnex$. So we have $a \in \mathscr{A}$ and hence $T^{\text{‘}}\ n \subseteq \mathscr{A}$.

5.75. $a \in T^{\text{‘}}\ n.\ \forall k(1 \leq k \leq n \supset M_k{}^{\text{‘}}\ a \in \mathscr{W}ord) \rightarrow F_1{}^{\text{‘}}\ a \in \mathscr{W}ord.$

Proof. By the premise and definition, $\forall k(1 \leq k \leq n \supset M_k{}^{\text{‘}}\ a \in \mathscr{C}onnex)$ and hence, by 5.74, $F_1{}^{\text{‘}}\ a \in \mathscr{C}onnex$. By 5.70, $F_1{}^{\text{‘}}\ a \in \Omega$ and hence, by 3.08, $F_1{}^{\text{‘}}\ a \in \mathscr{I}nduc$. Hence, by definition, $F_1{}^{\text{‘}}\ a \in \mathscr{W}ord$.

5.76. $a \in T^{\text{‘}}\ n.\ \alpha \leqq M_n{}^{\text{‘}}\ a.\ F_1{}^{\text{‘}}\ Sb(\alpha, n)^{\text{‘}}\ a,\ F_1{}^{\text{‘}}\ a \in \delta \in \mathscr{C}onnex$

$\rightarrow F_1{}^{\text{‘}}\ Sb(\alpha, n)^{\text{‘}}\ a \leqq F_1{}^{\text{‘}}\ a.$

Proof. Put $\mathscr{A} \overset{D}{=} \{a \mid \forall\alpha(\alpha \leqq M_n{}^{\text{‘}}\ a.\ F_1{}^{\text{‘}}\ Sb(\alpha, n)^{\text{‘}}\ a,\ F_1{}^{\text{‘}}\ a \in \delta \in \mathscr{C}onnex : \supset.\ F_1{}^{\text{‘}}\ Sb(\alpha, n)^{\text{‘}}\ a \leqq F_1{}^{\text{‘}}\ a)\}$. Assume $a \in T^{\text{‘}}\ n$. $L^{\text{‘}}\ a \subseteq \mathscr{A}$ and $\alpha \leqq M_n{}^{\text{‘}}\ a.\ F_1{}^{\text{‘}}\ Sb(\alpha, n)^{\text{‘}}\ a,\ F_1{}^{\text{‘}}\ a \in \delta \in \mathscr{C}onnex$. If $\alpha = M_n{}^{\text{‘}}\ a$, then $F_1{}^{\text{‘}}\ Sb(\alpha, n)^{\text{‘}}\ a = F_1{}^{\text{‘}}\ a$ and hence we may assume $\alpha \in M_n{}^{\text{‘}}\ a$. If $\alpha = 0$ or $\alpha = 1$, then, by 5.67–5.69, $F_1{}^{\text{‘}}\ Sb(\alpha, n)^{\text{‘}}\ a \in F_1{}^{\text{‘}}\ a$. So assume $1 \in \alpha$. By 5.64, we divide the proof in four cases.

i) Assume $\forall k(0 < k < n \supset M_k{}^{\text{‘}}\ a = 0)$. By 5.66, $F_1{}^{\text{‘}}\ Sb(\alpha, n) = \mathscr{E}xp(\alpha, \alpha)$ and $F_1{}^{\text{‘}}\ a = \mathscr{E}xp(M_n{}^{\text{‘}}\ a, M_n{}^{\text{‘}}\ a)$. By 4.66, $\mathscr{E}xp(M_n{}^{\text{‘}}\ a, \alpha) \in \mathscr{E}xp(M_n{}^{\text{‘}}\ a, M_n{}^{\text{‘}}\ a)$. Hence, by the fiirst assumption, $\mathscr{E}xp(\alpha, \alpha)$, $\mathscr{E}xp(M_n{}^{\text{‘}}\ a, \alpha) \in \delta \in \mathscr{C}onnex$. So, by 4.71 and $\alpha \in M_n{}^{\text{‘}}\ a$, $\mathscr{E}xp(\alpha, \alpha) \leqq \mathscr{E}xp(M_n{}^{\text{‘}}\ a, \alpha)$ and hence $F_1{}^{\text{‘}}\ Sb(\alpha, n) \in F_1{}^{\text{‘}}\ a$.

ii) Assume $0 < k < n-1.\ M_k{}^{\text{‘}}\ a \in \mathscr{S}c^{\text{‘‘}}\ \Omega.\ \forall m(k < m < n \supset M_m{}^{\text{‘}}\ a = 0)$. Put $b \overset{D}{=} Sb(M_n{}^{\text{‘}}\ a, k', n-1)^{\text{‘}}\ Sb(\mathscr{S}c^{-1\text{‘}}\ M_k{}^{\text{‘}}\ a, k)^{\text{‘}}\ a$. By 5.66, $F_1{}^{\text{‘}}\ a = F_1{}^{\text{‘}}\ b$ and $F_1{}^{\text{‘}}\ Sb(\alpha, n)^{\text{‘}}\ a = F_1{}^{\text{‘}}\ Sb(\alpha, k', n)^{\text{‘}}\ b$. We prove $F_1{}^{\text{‘}}\ Sb(\alpha, k', n-1)^{\text{‘}}\ b \in F_1{}^{\text{‘}}\ b$. Put $A \overset{D}{=} \{m \mid k < m < n \supset F_1{}^{\text{‘}}\ Sb(\alpha, k', m)^{\text{‘}}\ b \in F_1{}^{\text{‘}}\ b\}$. Assume $m \subseteq A$ and $k < m < n$. By $\alpha \in M^{\text{‘}}\ b$, $\alpha' \leqq M_n{}^{\text{‘}}\ a = M_n{}^{\text{‘}}\ b$. So, by 5.66 and 5.71, $F_1{}^{\text{‘}}\ Sb(\alpha, k', m)^{\text{‘}}\ b = F_1{}^{\text{‘}}\ Sb(\alpha, k', \mathscr{S}c^{-1\text{‘}}\ m)^{\text{‘}}\ Sb(\alpha', m)^{\text{‘}}\ Sb(0, m', n-1)^{\text{‘}}\ b \leqq F_1{}^{\text{‘}}\ Sb(\alpha, k', \mathscr{S}c^{-1\text{‘}}\ m)^{\text{‘}}\ Sb(M_n{}^{\text{‘}}\ b, m)^{\text{‘}}\ Sb(0, m', n-1)^{\text{‘}}\ b$. We can prove the following: $F_1{}^{\text{‘}}\ Sb(\alpha, k', \mathscr{S}c^{-1\text{‘}}\ m)^{\text{‘}}\ Sb(M_n{}^{\text{‘}}\ b, m)^{\text{‘}}\ Sb(0, m', n-1)^{\text{‘}}\ b \in F_1{}^{\text{‘}}\ Sb(\alpha, k', \mathscr{S}c^{-1\text{‘}}\ m)^{\text{‘}}\ Sb(M_n{}^{\text{‘}}\ b, m)^{\text{‘}}\ Sb(M_n{}^{\text{‘}}\ b, m', n-1)^{\text{‘}}\ b = F_1{}^{\text{‘}}\ Sb(\alpha, k', \mathscr{S}c^{-1\text{‘}}\ m)^{\text{‘}}\ b$. Hence $F_1{}^{\text{‘}}\ Sb(\alpha, k', m)^{\text{‘}}\ b \in F_1{}^{\text{‘}}\ Sb(\alpha, k', \mathscr{S}c^{-1\text{‘}}\ m)^{\text{‘}}\ b$. By the assumption, $k \leq \mathscr{S}c^{-1\text{‘}}\ m$. If $k < \mathscr{S}c^{-1\text{‘}}\ m$, then, by $\mathscr{S}c^{-1\text{‘}}\ m \in A$, $F_1{}^{\text{‘}}\ Sb(\alpha, k', \mathscr{S}c^{-1\text{‘}}\ m)^{\text{‘}}\ b \in F_1{}^{\text{‘}}\ b$. If $k = \mathscr{S}c^{-1\text{‘}}\ m$, then $F_1{}^{\text{‘}}\ Sb(\alpha, k', \mathscr{S}c^{-1\text{‘}}\ m)^{\text{‘}}\ b = F_1{}^{\text{‘}}\ b$. Hence $F_1{}^{\text{‘}}\ Sb(\alpha, k', m)^{\text{‘}}\ b \in F_1{}^{\text{‘}}\ b$. That is, m

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$\in A$ and hence $\omega \subseteq A$. So we have F_1‘$Sb(\alpha, k', n-1)$‘$b \in F_1$‘q. Since $Sb(\alpha, k', n-1)$‘$b \in L$‘a, $Sb(\alpha, k', n-1)$‘$b \in \mathscr{A}$. Since F_1‘$Sb(\alpha, k', n)$‘$b = F_1$‘$Sb(\alpha, n)$‘$Sb(\alpha, k', n-1)$‘b and F_1‘$Sb(\alpha, k', n)$‘b, F_1‘$Sb(\alpha, k', n-1)$‘$b \in \delta \in \mathscr{C}onnex$ and $\alpha \in M_n$‘$Sb(\alpha, k', n-1)$‘b, we obtain F_1‘$Sb(\alpha, n)$‘$Sb(\alpha, k', n-1)$‘$b \leqq F_1$‘$Sb(\alpha, k', n-1)$‘b. Hence, by F_1‘$Sb(\alpha, n)$‘$a = F_1$‘$Sb(\alpha, k', n)$‘b, we have F_1‘$Sb(\alpha, n)$‘$a \in F$‘a.

iii) Assume $1 < n$. M_{n-1}‘$a \in \mathscr{S}c$“Ω. Put $c \overset{D}{=} Sb(\mathscr{S}c^{-1}$‘$M_{n-1}$‘$a, n-1)$‘$a$ and $d \overset{D}{=} Sb(F$‘$c, n)$‘c. By 5.66 and 5.71, F_1‘$a = F_1$‘d. F‘$c \in F$‘a. F_1‘$Sb(\alpha, n)$‘$c \in F_1$‘$Sb(\alpha, n)$‘a. Hence, by the first assumption, F_1‘$Sb(\alpha, n)$‘c, F_1‘$c \in \delta \in \mathscr{C}onnex$. So, by $c \in \mathscr{A}$ and $\alpha \in M_n$‘c, we obtain F_1‘$Sb(\alpha, n)$‘$c \leqq F_1$‘c. F_1‘$c = M_n$‘d and hence F_1‘$Sb(\alpha, n)$‘$c \leqq M_n$‘d. We can prove that F_1‘$Sb(\alpha, n)$‘$a = F_1$‘$Sb(F_1$‘$Sb(\mathscr{S}c^{-1}$‘M_{n-1}‘$a, n-1)$‘$Sb(\alpha, n)$‘$a, n)$‘$Sb(\mathscr{S}c^{-1}$‘M_{n-1}‘$a, n-1)$‘$Sb(\alpha, n)$‘$a = F_1$‘$Sb(F_1$‘$Sb(\alpha, n)$‘$c, n)$‘$Sb(\mathscr{S}c^{-}$‘M_{n-1}‘$a, n-1)$‘$a = F_1$‘$Sb(F_1$‘$Sb(\alpha, n)$‘$c, n)$‘$Sb(F_1$‘$c, n)$‘$c = F_1$‘$Sb(F_1$‘$Sb(\alpha, n)$‘$c, n)$‘d. Hence, by the first assumption, F_1‘$Sb(F$‘$Sb(\alpha, n)$‘$c, n)$‘d, F_1‘$d \in \delta \in \mathscr{C}onnex$. Hence by $d \in \mathscr{A}$ and F_1‘$Sb(\alpha, n)$‘$c \leqq M_n$‘d, we obtain F_1‘$Sb(F_1$‘$Sb(\alpha, n)$‘$c, n)$‘$d \leqq F_1$‘d. Hence F_1‘$Sb(\alpha, n)$‘$a \leqq F_1$‘a.

iv) Assume $0 < k < n$. M_k‘$a \in \mathscr{L}im$. $\forall m(k < m < n \supset M_m$‘$a = 0)$. Let $\xi \in F_1$‘$Sb(\alpha, n)$‘a. By 5.66 there is a θ such that $\xi \in F_1$‘$Sb(\theta, k)$‘$Sb(\alpha, n)$‘a. $\theta \in M_k$‘a. By 5.72 F_1‘$Sb(\theta, k)$‘$Sb(\alpha, n)$‘$a \in F_1$‘$Sb(\alpha, n)$‘a and F_1‘$Sb(\theta, k)$‘$a \in F_1$‘a. Hence F_1‘$Sb(\alpha, n) Sb(\theta, k)$‘$a$, F_1‘$Sb(\theta, k)$‘$a \in \delta \in \mathscr{C}onnex$. Since $\alpha \in M_n$‘$Sb(\theta, k)$‘a and $Sb(\theta, k)$‘$a \in \mathscr{A}$, we obtain F_1‘$Sb(\alpha, n)$‘$Sb(\theta, k)$‘$a \leqq F_1$‘$Sb(\theta, k)$‘a. Combining these, it follows that $\xi \in F_1$‘a and hence F_1‘$Sb(\alpha, n)$‘$a \subseteq F_1$‘a. $\langle F_1$‘$Sb(\alpha, n)$‘a, F_1‘$a\rangle \in \mathscr{C}ompar$ and hence F_1‘$Sb(\alpha, n)$‘$a \leqq F_1$‘a.

Accordingly, for all cases, F_1‘$Sb(\alpha, n)$‘$a \leqq F_1$‘a and hence $a \in \mathscr{A}$. So, $\forall a(a \in T$‘n. L‘$a \subseteq \mathscr{A} :\supset. a \in \mathscr{A})$ and hence by 5.45, T‘$n \subseteq \mathscr{A}$.

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