ANZIAM J. 48(2006), 73–97

EXPLICIT EIGENVALUES AND INVERSES OF SEVERAL TOEPLITZ MATRICES

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(Received 16 February, 2005; revised 3 March, 2006)

Abstract

Based on the theory of difference equations, we derive necessary and sufficient conditions for the existence of eigenvalues and inverses of Toeplitz matrices with five different diagonals. In the course of derivations, we are also able to derive computational formulas for the eigenvalues, eigenvectors and inverses of these matrices. A number of explicit formulas are computed for illustration and verification.

2000 *Mathematics subject classification*: primary 15A09, 15A18; secondary 39A10.
Keywords and phrases: Toeplitz matrix, eigenvalue, eigenvector, inverse, difference equation, sequence.

1. Introduction

A Toeplitz matrix is a matrix with values constant along each (top-left to lower-right) diagonal. Several properties of these matrices are now known, including their eigenvalues, eigenvectors and inverses. In particular, in a recent paper [4] by Dow, Toeplitz matrices of the form

$$T_n = \begin{pmatrix} b & c & 0 & 0 & \cdots & 0 & \alpha \\ a & b & c & 0 & \cdots & 0 & 0 \\ 0 & a & b & c & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha & 0 & 0 & 0 & \cdots & a & b \end{pmatrix}_{n\times n},$$

where the corner elements are the same, are discussed and their explicit inverses are found. In many applications (such as boundary value problems for difference

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https://doi.org/10.1017/S1446181100003424 Published online by Cambridge University Press

equations), Toeplitz matrices of the form $A_n = T_n$ but with the α in the bottom left-hand side corner replaced by β, that is, $[a_{n1}] = \beta$, where $\alpha \neq \beta$, are also encountered. Therefore, it is of great interest to find out more about these matrices. In this paper, we derive the eigenvalues and their corresponding eigenvectors as well as the inverse of A_n when $ac \neq 0$ and at least one of the numbers α or β is not zero. When $\alpha = \beta = 0$, A_n reduces to the well-known tridiagonal matrix about which much is known. For general information about Toeplitz matrices, the references in [4] can be consulted.

For convenience, the set of integers, the set of nonnegative integers, the set of real numbers and the set of complex numbers are denoted by $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ respectively. The number $\sqrt{-1}$ is denoted by i. We will also set

$$\alpha\mathbb{Z} = \{m\alpha \mid m \in \mathbb{Z}\}, \quad \alpha \in \mathbb{C}.$$

In particular, $\pi\mathbb{Z}$ denotes the set $\{\dots, -2\pi, -\pi, 0, \pi, 2\pi, \dots\}$.

Toeplitz matrices are intimately related to boundary value problems involving difference equations. This relationship has been exploited in [2, 10] for finding eigenvalues or inverses of matrices arising from difference operators. We will again base our investigation here on the method of difference equations. For this reason, we recall some terminologies used in [1]. Let $l^{\mathbb{N}}$ be the set of complex sequences of the form $x = \{x_k\}_{k\in\mathbb{N}}$ endowed with the usual linear structure. A sequence of the form $\{\alpha, 0, 0, \dots\}$ is denoted by $\bar{\alpha}$ (or by α if no confusion is caused), and the sequence $\{0, 1, 0, 0, \dots\}$ is denoted by $\hbar$. Given two sequences $x = \{x_k\}$ and $y = \{y_k\}$ in $l^{\mathbb{N}}$, their convolution is denoted by $x * y$ (or xy if no confusion is caused) and is defined by

$$xy = \left\{\sum_{k=0}^{j} x_k y_{j-k}\right\}_{j\in\mathbb{N}}.$$

It is easily verified that $\hbar^2 = \hbar * \hbar = \{0, 0, 1, 0, 0, \dots\}$ and $\hbar^n = \{\hbar_j^n\}_{j\in\mathbb{N}}$, $n = 1, 2, \dots$, is given by $\hbar_j^n = 1$ if $n = j$ and $\hbar_j^n = 0$ otherwise. We will also set $\hbar^0 = \bar{1}$.

In the following discussions, we need, among other things, the well-known properties of the complex functions e^z, $\sin z$ and $\cos z$ from the theory of complex analysis. In particular, let $z = x + iy \in \mathbb{C}$ where $x, y \in \mathbb{R}$. Then (i) $\sin z = 0$ if and only if $y = 0$ and $x = k\pi$ for some $k \in \mathbb{Z}$, (ii) $\cos z = \pm 1$ if and only if $y = 0$ and $x = k\pi$ for some $k \in \mathbb{Z}$, and (iii) if $z \neq k\pi$ for any $k \in \mathbb{Z}$, then $\sin z \neq 0$, $\cos z \neq \pm 1$, $\sin(z/2) \neq 0$ and $\cos(z/2) \neq 0$.

2. Necessary conditions for the eigenvalues

Consider the eigenvalue problem $A_n u = \lambda u$, where a, b, c and α, β are numbers in the complex plane $\mathbb{C}$. We will assume that $ac \neq 0$. When $ac = 0$, the corresponding analysis is quite different and is treated elsewhere. To avoid trivial conditions, we will also assume $n \geq 3$ in the following discussions.

Let λ be an eigenvalue (which may be complex) and $(u_1, \dots, u_n)^\dagger$ a corresponding eigenvector of A_n. We may view the numbers $u_1, u_2, \dots, u_n$ respectively as the first, second, ..., and the n-th term of an infinite (complex) sequence $u = \{u_i\}_{i=0}^\infty$. Since $A_n u = \lambda u$ can be written as

$$\begin{aligned} u_0 &= 0, \\ au_0 + bu_1 + cu_2 &= \lambda u_1 - \alpha u_n, \\ au_1 + bu_2 + cu_3 &= \lambda u_2, \\ &\cdots\cdots\cdots \\ au_{n-2} + bu_{n-1} + cu_n &= \lambda u_{n-1}, \\ au_{n-1} + bu_n + cu_{n+1} &= \lambda u_n - \beta u_1, \\ u_{n+1} &= 0, \end{aligned} \tag{2.1}$$

we see that the sequence $u = \{u_k\}_{k=0}^\infty$ satisfies $u_0 = 0$, $u_{n+1} = 0$ and

$$au_{k-1} + bu_k + cu_{k+1} = \lambda u_k + f_k, \quad k = 1, 2, \dots, \tag{2.2}$$

where $f_1 = -\alpha u_n$ and $f_n = -\beta u_1$, while $f_k = 0$ for $k \neq 1, n$. Note that u_n and u_1 cannot be 0 simultaneously, for otherwise from (2.1), $u_2 = 0$ and inductively $u_3 = u_4 = \cdots = u_n = 0$, which is contrary to the definition of an eigenvector.

Let $f = \{f_k\}_{k=0}^\infty$ be defined above. Then (2.2) can be expressed as

$$c\{u_{k+2}\}_{k=0}^\infty + b\{u_{k+1}\}_{k=0}^\infty + a\{u_k\}_{k=0}^\infty = \lambda\{u_{k+1}\}_{k=0}^\infty + \{f_{k+1}\}_{k=0}^\infty.$$

By taking the convolution of the above equation with $\hbar^2 = \hbar\hbar$, and noting that

$$\hbar\{u_{n+1}\} = \hbar\{u_1, u_2, \dots\} = \{0, u_1, u_2, \dots\} = u - \overline{u}_0$$

and

$$\hbar^2\{u_{n+2}\} = \hbar^2\{u_2, u_3, \dots\} = \{0, 0, u_2, u_3, \dots\} = u - \overline{u}_0 - u_1\hbar,$$

we have $c(u - \overline{u}_0 - u_1\hbar) + (b - \lambda)\hbar(u - \overline{u}_0) + a\hbar^2 u = \hbar(f - \overline{f}_0)$. Solving for u, and substituting $f_0 = u_0 = 0$, we obtain

$$\left(a\hbar^2 + (b - \lambda)\hbar + \bar{c}\right)u = (c\overline{u}_1 + f)\hbar.$$

Since $c \neq 0$, we can divide the above equation by $a\hbar^2 + (b - \lambda)\hbar + \bar{c}$ to obtain [1, Theorem 24]

$$u = \frac{(c\overline{u}_1 + f)\hbar}{a\hbar^2 + (b - \lambda)\hbar + \bar{c}}. \tag{2.3}$$

Let

$$\gamma_{\pm} = \frac{-(b-\lambda) \pm \sqrt{\omega}}{2a}$$

be the two roots of $az^2 + (b-\lambda)z + c = 0$, where $\omega = (b-\lambda)^2 - 4ac$, which may or may not be zero.

Based on the value of ω, there are two cases to be considered.

Case I. Suppose $\omega \neq 0$ so that γ_+ and γ_- are distinct. Since $\gamma_+\gamma_- = c/a \neq 0$, we may write $\gamma_{\pm} = e^{\pm i\theta}/\rho$ for some θ in the strip $\{z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z < 2\pi\}$, where $\rho = \sqrt{a/c}$ and

$$\cos\theta = (\lambda - b)/2\rho c. \tag{2.4}$$

Since $\sin\theta = \sqrt{\omega}/(2i\rho c) \neq 0$, we must also have $\theta \notin \pi\mathbb{Z}$.

By the method of partial fractions, we can then write (2.3) in the form

$$\begin{aligned} u &= \frac{1}{\sqrt{\omega}}\left(\frac{1}{\gamma_- - \hbar} - \frac{1}{\gamma_+ - \hbar}\right)(c\bar{u}_1 + f)\hbar \\ &= \frac{1}{\sqrt{\omega}}\left\{\gamma_-^{-(j+1)} - \gamma_+^{-(j+1)}\right\}_{j\in\mathbb{N}}(c\bar{u}_1 + f)\hbar \\ &= \frac{2i}{\sqrt{\omega}}\{\rho^j \sin j\theta\} * \{cu_1, -\alpha u_n, 0, \ldots, -\beta u_1, 0, \ldots\}. \end{aligned}$$

By evaluating the convolution product, we obtain the j-th term of u,

$$\begin{aligned} u_j = \frac{2i}{\sqrt{\omega}}\big(&cu_1\rho^j \sin j\theta - \alpha u_n \rho^{j-1}\sin(j-1)\theta \\ &- H(j-n)\beta u_1 \rho^{j-n}\sin(j-n)\theta\big) \end{aligned} \tag{2.5}$$

for $j \geq 1$, where $H(x)$ is the unit step function defined by $H(x) = 1$ if $x \geq 0$ and $H(x) = 0$ if $x < 0$.

In particular,

$$\frac{\sqrt{\omega}}{2i}u_n = cu_1\rho^n \sin n\theta - \alpha u_n \rho^{n-1}\sin(n-1)\theta \tag{2.6}$$

and

$$\frac{\sqrt{\omega}}{2i}u_{n+1} = cu_1\rho^{n+1}\sin(n+1)\theta - \alpha u_n\rho^n \sin n\theta - \beta u_1 \rho \sin\theta. \tag{2.7}$$

By (2.6), we have

$$c\rho^n(\sin n\theta)u_1 - \left(\frac{\sqrt{\omega}}{2i} + \alpha\rho^{n-1}\sin(n-1)\theta\right)u_n = 0, \tag{2.8}$$

and by (2.7), and the condition $u_{n+1} = 0$, we have

$$\left(c\rho^{n+1}\sin(n+1)\theta - \beta\rho\sin\theta\right)u_1 - \alpha\rho^n(\sin n\theta)u_n = 0. \tag{2.9}$$

Since u_1 and u_n cannot be both zero, we must necessarily have

$$\begin{vmatrix} c\rho^n\sin n\theta & -\left(\frac{\sqrt{\omega}}{2i} + \alpha\rho^{n-1}\sin(n-1)\theta\right) \\ c\rho^{n+1}\sin(n+1)\theta - \beta\rho\sin\theta & -\alpha\rho^n\sin n\theta \end{vmatrix} = 0,$$

which leads to the necessary condition

$$\alpha c\rho^{2n}\sin\theta - \rho^n\left(ac\sin(n+1)\theta - \alpha\beta\sin(n-1)\theta\right) + a\beta\sin\theta = 0. \tag{2.10}$$

Once we have found a θ that satisfies (2.10), we obtain by (2.4)

$$\lambda = b + 2\rho c\cos\theta, \quad \theta \neq m\pi,\ m \in \mathbb{Z}. \tag{2.11}$$

Case II. Suppose $\omega = 0$ so that $\gamma_+ = \gamma_-$. In this case, $(\lambda - b)^2 = 4ac$, and (2.3) can be written as

$$\begin{aligned} u &= \frac{(c\overline{u}_1 + f)\hbar}{c\left(\overline{1} - 2((\lambda-b)/2c)\hbar + ((\lambda-b)/2c)^2\hbar^2\right)} = \frac{1}{\widetilde{\rho}c}\frac{\widetilde{\rho}\hbar}{\left(\overline{1} - \widetilde{\rho}\hbar\right)^2}(c\overline{u}_1 + f) \\ &= \frac{1}{\widetilde{\rho}c}\left\{j\widetilde{\rho}^j\right\}_{j\in\mathbf{N}} * \{cu_1, -\alpha u_n, 0, \ldots, -\beta u_1, 0, \ldots\}, \end{aligned}$$

where

$$\widetilde{\rho} = (\lambda - b)/2c = \pm\sqrt{a/c} = \pm\rho. \tag{2.12}$$

The j-th term of u now becomes

$$u_j = \frac{1}{\widetilde{\rho}c}\left(cu_1 j\widetilde{\rho}^j - \alpha u_n(j-1)\widetilde{\rho}^{j-1} - H(j-n)\beta u_1(j-n)\widetilde{\rho}^{j-n}\right), \quad j \geq 1. \tag{2.13}$$

In particular,

$$u_n = \left(cu_1 n\widetilde{\rho}^n - \alpha u_n(n-1)\widetilde{\rho}^{n-1}\right)/\widetilde{\rho}c \tag{2.14}$$

and

$$u_{n+1} = 0 = \left(cu_1(n+1)\widetilde{\rho}^{n+1} - \alpha u_n n\widetilde{\rho}^n - \beta u_1\widetilde{\rho}\right)/\widetilde{\rho}c. \tag{2.15}$$

This leads to the necessary condition

$$\alpha c\widetilde{\rho}^{2n} - \widetilde{\rho}^n\left(ac(n+1) - \alpha\beta(n-1)\right) + a\beta = 0. \tag{2.16}$$

Once we have found the $\tilde{\rho}$ that satisfies (2.16), then we obtain by (2.12)

$$\lambda = b + 2\tilde{\rho}c. \tag{2.17}$$

We remark that since (2.17) may be written as

$$\lambda = b \pm 2\rho c = b + 2\rho c \cos\theta, \quad \theta = m\pi, \ m \in \mathbb{Z},$$

we may combine (2.17) and (2.11) and assert that an eigenvalue λ of A_n is necessarily of the form $\lambda = b + 2\rho c \cos\theta$.

According to the above discussions, when λ is an eigenvalue of A_n, it is then necessary that either (2.10) or (2.16) holds.

THEOREM 2.1. *Let λ be an eigenvalue of the matrix A_n and $u = (u_1, \ldots, u_n)^\dagger$ its corresponding eigenvector. If* (2.10) *is satisfied for some*

$$\theta \in \{z \in \mathbb{C} \mid 0 \le \operatorname{Re} z < 2\pi\} \backslash \pi\mathbb{Z},$$

then (2.5) *and* (2.11) *hold.*

THEOREM 2.2. *Let λ be an eigenvalue of the matrix A_n and $u = (u_1, \ldots, u_n)^\dagger$ its corresponding eigenvector. If* (2.16) *is satisfied for $\tilde{\rho} = \sqrt{a/c}$ or $\tilde{\rho} = -\sqrt{a/c}$, then* (2.13) *and* (2.17) *hold.*

Recall that the first and last components u_1 and u_n cannot be zero simultaneously. There are some other interesting properties for the eigenvector u if $\omega \neq 0$.

COROLLARY 2.3. *Let λ be an eigenvalue of the matrix A_n and $u = (u_1, \ldots, u_n)^\dagger$ its corresponding eigenvector such that $\omega = (b-\lambda)^2 - 4ac \neq 0$. Let θ be the number found in Theorem* 2.1.

(i) *If $u_1 = 0$, then $\alpha \neq 0$.*
(ii) *If $u_n = 0$ or $u_1 = 0$ then $\sin n\theta = 0$.*
(iii) *If $\sin n\theta = 0$, then either $u_n = 0$ or $\alpha = \pm a\rho^{-n}$, and either $u_1 = 0$ or $\beta = \pm c\rho^n$.*
(iv) *If $\beta \neq \pm c\rho^n$, then $u_n \neq 0$.*
(v) *If $\alpha \neq \pm a\rho^{-n}$, then $u_1 \neq 0$.*
(vi) *If $\beta \neq \pm c\rho^n$ and $u_1 \neq 0$, then $\sin n\theta \neq 0$.*
(vii) *If $\alpha \neq \pm a\rho^{-n}$ and $u_n \neq 0$, then $\sin n\theta \neq 0$.*
(viii) *If $\beta \neq \pm c\rho^n$ and $\alpha \neq \pm a\rho^{-n}$, then $u_1 \neq 0$, $u_n \neq 0$ and $\sin n\theta \neq 0$.*

PROOF. If $u_1 = 0$, then by (2.8) $\sqrt{\omega}/(2i) + \alpha\rho^{n-1}\sin(n-1)\theta = 0$. Since $\omega \neq 0$, we must have $\alpha \neq 0$.

If $u_n = 0$, then by (2.8) $c\rho^n u_1 \sin n\theta = 0$. Since $c\rho^n u_1 \neq 0$, we must have $\sin n\theta = 0$. Similarly, if $u_1 = 0$, then by (2.9) $\alpha\rho^n u_n \sin n\theta = 0$. Since $\alpha\rho^n u_n \neq 0$, we must have $\sin n\theta = 0$.

If $\sin n\theta = 0$, then by (2.8), either $u_n = 0$ or $\rho c \sin\theta + \alpha\rho^{n-1}\sin(n-1)\theta = 0$. Since $\sin(n-1)\theta = \sin n\theta\cos\theta - \cos n\theta \sin\theta = \mp\sin\theta$ when $\sin n\theta = 0$; the latter implies $\rho c = \pm\alpha\rho^{n-1}$ or $\alpha = \pm a\rho^{-n}$. Similarly, if $\sin n\theta = 0$, then by (2.9), either $u_1 = 0$ or $c\rho^{n+1}\sin(n+1)\theta - \beta\rho\sin\theta = 0$. The latter implies $\beta = \pm c\rho^n$.

Suppose $\beta \neq \pm c\rho^n$. If $u_n = 0$, then $\sin n\theta = 0$. Since $u_1 \neq 0$, (iii) implies $\beta = \pm c\rho^n$, which is a contradiction.

Suppose $\alpha \neq \pm a\rho^{-n}$. If $u_1 = 0$, then $\sin n\theta = 0$. Since $u_n \neq 0$, (iii) implies $\alpha = \pm a\rho^{-n}$, which is a contradiction.

Suppose $\beta \neq \pm c\rho^n$ and $u_1 \neq 0$. If $\sin n\theta = 0$, then by (iii), either $u_1 = 0$ or $\beta = \pm c\rho^n$. This is a contradiction.

Suppose $\alpha \neq \pm a\rho^{-n}$ and $u_n \neq 0$. If $\sin n\theta = 0$, then by (iii), either $u_n = 0$ or $\alpha = \pm a\rho^{-n}$. This is a contradiction.

The last assertion (viii) follows from (iv), (v) and (vi), (vii). $\square$

3. Additional conditions for eigenvectors

Given an eigenvalue λ of the matrix A_n and its corresponding eigenvector $u = (u_1, \ldots, u_n)^\dagger$, suppose $\omega \neq 0$ and let θ be the number found in Theorem 2.1. For $1 \leq j \leq n$, we have by (2.5)

$$u_j = \frac{1}{\rho c \sin\theta}\left(c u_1 \rho^j \sin j\theta - \alpha u_n \rho^{j-1}\sin(j-1)\theta\right), \quad j = 1, \ldots, n. \tag{3.1}$$

In the case when $\sin n\theta \neq 0$, we may find u which are simpler in form. Indeed, suppose $\sin n\theta \neq 0$, then from (2.8),

$$u_1 = \frac{(\rho c \sin\theta + \alpha\rho^{n-1}\sin(n-1)\theta)}{c\rho^n \sin n\theta} u_n \tag{3.2}$$

and from (2.9),

$$\alpha u_n = \frac{(c\rho^{n+1}\sin(n+1)\theta - \beta\rho\sin\theta)}{\rho^n \sin n\theta} u_1. \tag{3.3}$$

Substituting (3.2) into (3.1), we have

$$\begin{aligned} u_j &= \frac{1}{\rho c \sin\theta}\left(\frac{\rho^j \sin j\theta}{\rho^n \sin n\theta}(\rho c \sin\theta + \alpha\rho^{n-1}\sin(n-1)\theta)u_n - \alpha u_n \rho^{j-1}\sin(j-1)\theta\right) \\ &= \frac{u_n}{\rho c \sin\theta \sin n\theta}\left(\rho c \rho^{j-n}\sin\theta\sin j\theta + \alpha\rho^{j-1}\sin(n-j)\theta\sin\theta\right) \end{aligned}$$

https://doi.org/10.1017/S1446181100003424 Published online by Cambridge University Press

for $j = 1, 2, \ldots, n$. By letting $u_n = \rho^n \sin n\theta$, we obtain

$$u_j = \rho^j \sin j\theta + \frac{\alpha}{a}\rho^{j+n} \sin(n-j)\theta, \quad j = 1, \ldots, n, \tag{3.4}$$

which defines an eigenvector corresponding to λ (if θ is found). Similarly, substituting (3.3) into (3.1), we may obtain

$$u_j = \rho^j \sin(n+1-j)\theta + \frac{\beta}{c}\rho^{j-n} \sin(j-1)\theta, \quad j = 1, \ldots, n. \tag{3.5}$$

Suppose $\omega = 0$. Then we have by (2.13)

$$u_j = \frac{1}{\widetilde{\rho} c}\left(c u_1 j \widetilde{\rho}^j - \alpha u_n (j-1)\widetilde{\rho}^{j-1}\right), \quad j = 1, \ldots, n.$$

In view of (2.14) and (2.15), a similar argument leads to

$$u_j = \widetilde{\rho}^j j + \widetilde{\rho}^{j+n}\frac{\alpha}{a}(n-j), \quad j = 1, \ldots, n \tag{3.6}$$

and

$$u_j = \widetilde{\rho}^j (n+1-j) + \widetilde{\rho}^{j-n}\frac{\beta}{c}(j-1), \quad j = 1, \ldots, n. \tag{3.7}$$

4. Eigenvalues and eigenvectors of special Toeplitz matrices

Now we can apply the results of the previous sections to find the eigenvalues and the corresponding eigenvectors of several Toeplitz matrices of the form A_n. For motivation, consider the case where $\alpha = \beta = 0$. Then (2.16) is reduced to

$$\widetilde{\rho}^n ac(n+1) = 0$$

which is not possible so that, in view of Theorem 2.2, $\omega \neq 0$ and λ cannot be of the form $b \pm 2\rho c$. In view of Theorem 2.1, (2.10) must hold for some $\theta \notin \pi\mathbb{Z}$, or, $\sin(n+1)\theta = 0$ for some $\theta \notin \pi\mathbb{Z}$. Consequently, $\theta = k\pi/(n+1)$ for some $k \in \mathbb{Z}$ and $\theta \notin \pi\mathbb{Z}$. An eigenvalue λ_k is then necessarily of the form

$$\lambda_k = b + 2\rho c \cos\frac{k\pi}{n+1}, \quad k = 1, \ldots, n.$$

A corresponding eigenvector, from (3.4), is given by

$$u_j^{(k)} = \rho^j \sin\frac{jk\pi}{n+1}, \quad j = 1, \ldots, n, \tag{4.1}$$

which has also been obtained in [10] and elsewhere. Finally, by reversing the arguments that lead to Theorem 2.1, we see that each λ_k is an eigenvalue of A_n and the corresponding vector $u^{(k)} = (u_1^{(k)}, \ldots, u_n^{(k)})^\dagger$ defined by (4.1) is a corresponding eigenvector.

By similar ideas, we may now derive the eigenvalues and the corresponding eigenvectors for matrices of the form A_n.

4.1. The case where $a = c = \pm\alpha$ and $\beta = 0$, or $\alpha = 0$ and $a = c = \pm\beta$ We will use $[x]$ to denote the integral part of $x \in \mathbb{R}$. Note that $[(k-1)/2] + [(k+2)/2] = k$ for any positive integer k.

THEOREM 4.1. *Suppose $\alpha = a = c$ and $\beta = 0$, or, $\alpha = 0$ and $\beta = c = a$. Then the eigenvalues of A_n are given by*

$$\lambda_k = b + 2a\cos\frac{2k\pi}{n}, \quad k = 1, 2, \dots, [(n-1)/2] \tag{4.2}$$

and

$$\lambda_{m+[(n-1)/2]} = b + 2a\cos\frac{(2m-1)\pi}{n+2}, \quad m = 1, 2, \dots, [(n+2)/2]. \tag{4.3}$$

The eigenvectors corresponding to (4.2) *and* (4.3) *are given by*

$$u_j^{(k)} = \sin\frac{2k(n+1-j)\pi}{n}, \quad j = 1, 2, \dots, n \tag{4.4}$$

and

$$u_j^{(m+[(n-1)/2])} = \sin\frac{(2m-1)(n+1-j)\pi}{n+2}, \quad j = 1, 2, \dots, n \tag{4.5}$$

respectively for $\alpha = a = c$ and $\beta = 0$; and

$$u_j^{(k)} = \sin\frac{2kj\pi}{n} \quad \textit{and} \quad u_j^{(m+[(n-1)/2])} = \sin\frac{(2m-1)j\pi}{n+2}, \quad j = 1, 2, \dots, n$$

respectively for $\alpha = 0$ and $\beta = c = a$.

PROOF. Suppose $\alpha = a = c$ and $\beta = 0$, or $\alpha = 0$ and $\beta = c = a$. Then (2.16) is reduced to $(\pm 1)^n(n+1) = 1$. This relation cannot be valid, and hence, in view of Theorem 2.2, $\omega = 0$ does not hold. In view of Theorem 2.1, (2.10) holds for some $\theta \notin \pi\mathbb{Z}$, or

$$\sin\frac{n\theta}{2}\cos\left(\frac{n+2}{2}\right)\theta = 0, \quad \theta \notin \pi\mathbb{Z}.$$

Hence (a) $\sin(n\theta/2) = 0$ or (b) $\cos((n+2)/2)\theta = 0$ for some $\theta \notin \pi\mathbb{Z}$. In case (a), we have

$$\theta = 2k\pi/n, \quad \theta \notin \pi\mathbb{Z},\ k \in \mathbb{Z}, \tag{4.6}$$

so that an eigenvalue must be of the form

$$\lambda_k = b + 2a\cos(2k\pi/n), \quad k = 1, \dots, [(n-1)/2].$$

Similarly, in case (b), we have

$$\theta = \frac{(2m-1)\pi}{n+2}, \quad \theta \notin \pi\mathbb{Z},\ m \in \mathbb{Z}, \tag{4.7}$$

so that an eigenvalue must be of the form

$$\lambda_{m+[(n-1)/2]} = b + 2a\cos\frac{(2m-1)\pi}{n+2}, \quad m = 1, \ldots, [(n+2)/2].$$

The corresponding eigenvectors may be obtained as follows. For $\alpha = a = c$ and $\beta = 0$, since $\beta \neq \pm c\rho^n$, by Corollary 2.3 (iv) $u_n \neq 0$, while u_1 may or may not be 0. If $u_1 = 0$, then $\sin n\theta = 0$. By (3.1)

$$u_j = \frac{-u_n}{\sin\theta}\sin(j-1)\theta.$$

Since $\sin(n+1-j)\theta = \sin n\theta\cos(j-1)\theta - \cos n\theta\sin(j-1)\theta = \mp\sin(j-1)\theta$ when $\sin n\theta = 0$, we may write $u_j = (\pm u_n/\sin\theta)\sin(n+1-j)\theta$. Letting $\pm u_n = \sin\theta$, we have

$$u_j = \sin(n+1-j)\theta, \quad j = 1, 2, \ldots, n. \tag{4.8}$$

If $u_1 \neq 0$, then by Corollary 2.3 (vi), $\sin n\theta \neq 0$. Hence we may apply (3.5), which leads to the same result (4.8) since $\rho = 1$ and $\beta = 0$. By substituting θ given by (4.6) or (4.7) into (4.8), we obtain the desired results (4.4) and (4.5).

For $\alpha = 0$ and $\beta = c = a$, a similar argument leads to $u_j = \sin j\theta$, $j = 1, \ldots, n$. By substituting θ given by (4.6) or (4.7), we obtain the desired results.

Once we have found the eigenvalues and their corresponding eigenvectors, we may reverse the arguments leading to Theorem 2.1 and verify that they are indeed the true eigenvalues and associated eigenvectors of A_n. The proof is complete. □

We may follow the same arguments to show the following.

Suppose $-\alpha = a = c$ and $\beta = 0$, or $\alpha = 0$ and $-\beta = c = a$. Then the eigenvalues of A_n are given by

$$\lambda_k = b + 2a\cos\frac{2k\pi}{n+2}, \quad k = 1, 2, \ldots, [(n+1)/2] \tag{4.9}$$

and

$$\lambda_{m+[(n+1)/2]} = b + 2a\cos\frac{(2m-1)\pi}{n}, \quad m = 1, 2, \ldots, [n/2]. \tag{4.10}$$

The eigenvectors corresponding to (4.9) and (4.10) are given by

$$u_j^{(k)} = \sin\frac{2k(n+1-j)\pi}{n+2}, \quad j = 1, 2, \ldots, n$$

and

$$u_j^{(m+[(n+1)/2])} = \sin\frac{(2m-1)(n+1-j)\pi}{n}, \quad j = 1, 2, \ldots, n,$$

respectively for $-\alpha = a = c$ and $\beta = 0$, and

$$u_j^{(k)} = \sin \frac{2kj\pi}{n+2}, \quad j = 1, 2, \dots, n$$

and

$$u_j^{(m+[(n+1)/2])} = \sin \frac{(2m-1)j\pi}{n}, \quad j = 1, 2, \dots, n$$

respectively for $\alpha = 0$ and $-\beta = c = a$.

4.2. The case where $\alpha = -\beta = a = c$, or $-\alpha = \beta = a = c$

THEOREM 4.2. *Suppose $\alpha = -\beta = a = c$, or $-\alpha = \beta = a = c$. Then the eigenvalues of A_n are given by*

$$\lambda_k = \begin{cases} b + 2a\cos(k\pi/n), & k = 1, 2, \dots, n-1, \\ b, & k = n. \end{cases} \tag{4.11}$$

The eigenvectors corresponding to (4.11) *are given by*

$$u_j^{(k)} = \begin{cases} \sin(jk\pi/n), & k \text{ odd}, \\ \sin((j-1)k\pi/n), & k \text{ even}, \end{cases} \tag{4.12}$$

and

$$u_j^{(n)} = \begin{cases} \sin(j\pi/2) + (\alpha/a)\sin((n-j)\pi/2), & n \text{ odd}, \\ \sin(j\pi/2), & n = 6, 10, 14, \dots \\ \sin((j-1)\pi/2), & n = 4, 8, 12, \dots \end{cases} \tag{4.13}$$

respectively for $\alpha = -\beta = a = c$, $j = 1, 2, \dots, n$. For $-\alpha = \beta = a = c$, only the odd-even relation for k in (4.12) *should be interchanged.*

PROOF. Suppose $\alpha = -\beta = a = c$, or $-\alpha = \beta = a = c$, then $\rho = 1$, and (2.16) is reduced to $(\pm 1)^n 2n = 0$. This relation cannot be valid so that $\omega = 0$ does not hold. By Theorem 2.1, (2.10) holds for some $\theta \notin \pi\mathbb{Z}$, or $\sin n\theta \cos\theta = 0$, $\theta \notin \pi\mathbb{Z}$. In the case where $\sin n\theta = 0$ for some $\theta \notin \pi\mathbb{Z}$, we have $\theta = (k\pi/n) \notin \pi\mathbb{Z}$, $k \in \mathbb{Z}$, and the eigenvalue must be of the form $\lambda_k = b + 2a\cos(k\pi/n)$, $k = 1, 2, \dots, n-1$. In the case where $\cos\theta = 0$, we have $\lambda_n = b$.

The corresponding eigenvectors may be found as follows. For $k = 1, \dots, n-1$, suppose $\alpha = -\beta = a = c$. Since $\sin n\theta = \sin k\pi = 0$ and $\cos n\theta = \cos k\pi = -1$ if k is odd and $+1$ if k is even, we have

$$\begin{aligned} &c\sin\theta + \alpha\sin(n-1)\theta = a\sin\theta(1-\cos n\theta) = 2a\sin\theta \neq 0, \quad k \text{ odd}, \quad \text{and} \\ &c\sin(n+1)\theta - \beta\sin\theta = c(\cos n\theta + 1)\sin\theta = 2a\sin\theta \neq 0, \quad k \text{ even}. \end{aligned}$$

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Hence if k is odd, by (2.8) $u_n = 0$ and in view of (3.1) an eigenvector must be of the form

$$u_j = \sin j\theta. \tag{4.14}$$

If k is even, then by (2.9) $u_1 = 0$ and an eigenvector must be of the form

$$u_j = \sin(j-1)\theta. \tag{4.15}$$

By substituting $\theta = k\pi/n$, we have (4.12). Suppose $-\alpha = \beta = a = c$, then (4.14) is for even k and (4.15) is for odd k.

For $k = n$, we have $\theta = \pi/2$ and $\sin n\theta = \sin(n\pi/2) = \pm 1 \neq 0$ if n is odd. We may apply (3.4) to obtain

$$u_j^{(n)} = \sin\frac{j\pi}{2} + \frac{\alpha}{a}\sin\frac{(n-j)\pi}{2}, \quad j = 1, 2, \ldots, n.$$

If n is even, then $\sin n\theta = 0$ and $\cos n\theta = -1$ if $n = 6, 10, \ldots$ and $+1$ if $n = 4, 8, \ldots$. Hence by a similar argument as for $k = 1, 2, \ldots, n-1$, we have for $j = 1, \ldots, n$

$$u_j^{(n)} = \begin{cases} \sin(j\pi/2), & \text{for } n = 6, 10, 14, \ldots, \\ \sin((j-1)\pi/2), & \text{for } n = 4, 8, 12, \ldots. \end{cases}$$

The proof is complete. □

4.3. The case where $\alpha = \pm a$ and $\beta = \pm c$ In the case where $\alpha = a$ and $\beta = c$, A_n is the well-known circulant matrix [3]. There are many results [3, 5–9] concerning the eigenvalues and inverses of such matrices. However, most of them are algorithmic in nature. Here we will derive explicit formulas for the case where $\alpha = -a$ and $\beta = -c$ based on our theorems, while those for $\alpha = a$ and $\beta = c$ will be listed only since they are already known.

THEOREM 4.3. *Suppose $\alpha = -a$ and $\beta = -c$ in the matrix A_n. Then the eigenvalues and the corresponding eigenvectors of A_n are given by*

$$\begin{aligned}\lambda_k = b &+ (a+c)\cos\frac{(2k-1)\pi}{n} \\ &+ i(a-c)\sin\frac{(2k-1)\pi}{n}, \quad k = 1, 2, \ldots, n \end{aligned} \tag{4.16}$$

and

$$u_j^{(k)} = e^{-ij(2k-1)\pi/n}, \quad j = 1, 2, \ldots, n \tag{4.17}$$

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respectively. If $a = c$, an alternative formula for the eigenvectors is also given by

$$u_j^{(k)} = c_1 \cos \frac{(2k-1)j\pi}{n} + c_2 \sin \frac{(2k-1)j\pi}{n}, \quad j = 1, 2, \ldots, n, \tag{4.18}$$

where c_1 and c_2 are two independent constants not both equal to 0. In particular, if we take $c_1 = 1$ and $c_2 = -i$, we have (4.17) as its special case.

PROOF. Suppose the conditions in Theorem 2.1 hold. Then we must have $\lambda = b + 2\rho c \cos\theta$, where θ is some number that satisfies

$$\rho^n\big(\sin(n+1)\theta - \sin(n-1)\theta\big) + (\rho^{2n} + 1)\sin\theta = 0, \quad \theta \notin \pi\mathbb{Z},$$

or, since $\sin\theta \neq 0$, $\rho^{2n} + 2\rho^n \cos n\theta + 1 = 0$, $\theta \notin \pi\mathbb{Z}$. This yields

$$\rho^n = -\cos n\theta \pm i \sin n\theta = -e^{\mp in\theta}. \tag{4.19}$$

Let $\rho^n = -e^{in\theta}$, then since $e^{\pm i(2k-1)\pi} = -1$, we may write $\rho^n = e^{in\theta}e^{-i(2k-1)\pi}$, so that

$$e^{i\theta} = \rho e^{i(2k-1)\pi/n}, \quad e^{-i\theta} = \rho^{-1}e^{-i(2k-1)\pi/n} \tag{4.20}$$

and

$$\begin{aligned}\cos\theta &= \frac{1}{2}\left(\rho e^{i(2k-1)\pi/n} + \rho^{-1}e^{-i(2k-1)\pi/n}\right)\\ &= \frac{1}{2}\left[\left(\rho + \frac{1}{\rho}\right)\cos\frac{(2k-1)\pi}{n} + i\left(\rho - \frac{1}{\rho}\right)\sin\frac{(2k-1)\pi}{n}\right].\end{aligned}$$

Note that if $a \neq c$, then $\rho \neq 1$ and $\cos\theta \neq \pm 1$ for any $k \in \mathbb{Z}$. By noting that $\rho^2 c = a$ and that $\sin x$ and $\cos x$ are periodic functions, we have finally for $k = 1, \ldots, n$

$$\lambda_k = b + 2\rho c\cos\theta = b + (a+c)\cos\frac{(2k-1)\pi}{n} + i(a-c)\sin\frac{(2k-1)\pi}{n},$$

which is (4.16). If $a = c$, then $\rho = 1$ and we have $\cos\theta = \cos((2k-1)\pi/n) \neq \pm 1$ so that $k \neq (n+1)/2$ in (4.16). But then we have Theorem 2.2.

Suppose $\omega = 0$ and the conditions in Theorem 2.2 hold. Then (2.16) is valid for $\alpha = -a$ and $\beta = -c$. Thus

$$\widetilde{\rho}^{2n} + 2\widetilde{\rho}^n + 1 = 0, \tag{4.21}$$

which holds if $a = c$, $\widetilde{\rho} = -1$ and n is odd. Furthermore, under these conditions, the eigenvalue must be of the form $\lambda = b - 2c$, which can also be written as $\lambda_{(n+1)/2}$ in (4.16). Hence (4.16) holds regardless of $a = c$ or $a \neq c$.

In case the negative sign in $e^{\mp in\theta}$ holds, then $\rho^n = -e^{-in\theta} = e^{-in\theta}e^{-i(2k-1)\pi}$, and it is easily seen that we may get the same result (4.16).

To find the corresponding eigenvectors, we first consider the case $a \neq c$. Then $\rho \neq 1$ so that $\beta \neq \pm c\rho^n$ and $\alpha \neq \pm a\rho^{-n}$. By Corollary 2.3 (viii), $u_1, u_2 \neq 0$ and $\sin n\theta \neq 0$, so we may apply (3.4). Since by (4.20) $e^{\pm ij\theta} = \rho^{\pm j}e^{\pm ij(2k-1)\pi/n}$, we have

$$e^{\pm i(n-j)\theta} = \rho^{\pm(n-j)}e^{\pm i(n-j)(2k-1)\pi/n} = -\rho^{\pm(n-j)}e^{\mp ij(2k-1)\pi/n}.$$

By substituting this and $\alpha = -a$ into (3.4),

$$\begin{aligned} u_j^{(k)} &= \frac{1}{2i}\left(\rho^j\left(\rho^j e^{ij(2k-1)\pi/n} - \rho^{-j}e^{-ij(2k-1)\pi/n}\right)\right. \\ &\quad \left. +\rho^{j+n}\left(\rho^{n-j}e^{-ij(2k-1)\pi/n} - \rho^{-n+j}e^{ij(2k-1)\pi/n}\right)\right) \\ &= \frac{1}{2i}(\rho^{2n} - 1)e^{-ij(2k-1)\pi/n}. \end{aligned}$$

By dropping the constant factor $(\rho^{2n} - 1)/2i$, we obtain (4.17). Next suppose $a = c$. By (4.19), $\rho = 1$ implies $\cos n\theta = -1$ and $\sin n\theta = 0$. The former implies $\theta = (2k-1)\pi/n \notin \pi\mathbb{Z}$ so that $k \neq (n+1)/2$, the latter implies either one of the u_1 or u_n may be zero. If $u_n = 0$, then by (3.1) an eigenvector must be of the form $u_j^{(k)} = \sin j\theta$. If $u_1 = 0$, then an eigenvector must be of the form $u_j^{(k)} = \sin(j-1)\theta$. Hence the linear combination

$$u_j^{(k)} = k_1 \sin j\theta + k_2 \sin(j-1)\theta = c_1 \cos j\theta + c_2 \sin j\theta$$

is an eigenvector of A_n corresponding to λ_k. After substituting $\theta = (2k-1)\pi/n$, we have (4.18) for $a = c$ and $k \neq (n+1)/2$.

For $a = c$ and $k = (n+1)/2$, then $\theta = \pi$, which implies $\omega = 0$, and we already have $\widetilde{\rho}^n = \widetilde{\rho} = -1$ from (4.21), hence we may apply either (3.6) or (3.7) to obtain

$$u_j = (-1)^j n, \quad j = 1, \dots, n,$$

which is of the form $u_j^{(n+1)/2}$ in (4.18). Hence (4.18) is valid for $a = c$ and $k = 1, \dots, n$. The proof is complete. □

Now we may follow the same arguments to show the following: Suppose $\alpha = a$ and $\beta = c$ in the matrix A_n, then the eigenvalues and the corresponding eigenvectors of A_n are given by

$$\lambda_k = b + (a+c)\cos\frac{2k\pi}{n} + i(a-c)\sin\frac{2k\pi}{n}, \quad k = 1, 2, \dots, n$$

and

$$u_j^{(k)} = e^{-i\frac{2jk\pi}{n}}, \ j = 1, 2, \dots, n \tag{4.22}$$

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respectively. If $a = c$, an alternative formula for the eigenvectors is also given by [2]

$$u_j^{(k)} = c_1 \cos \frac{2jk\pi}{n} + c_2 \sin \frac{2jk\pi}{n}, \quad j = 1, 2, \ldots, n,$$

where c_1 and c_2 are two independent constants not both equal to 0. In particular, if we take $c_1 = 1$ and $c_2 = -i$, we have (4.22) as its special case.

5. Necessary conditions for the inverse

The method used in the previous sections may also be used to find the inverse of the matrix A_n under the condition $ac \neq 0$. Let the (unique) inverse of A_n, if it exists, be denoted by

$$G_n = \left(g^{(1)} \mid g^{(2)} \mid \cdots \mid g^{(n)}\right) = \begin{pmatrix} g_1^{(1)} & g_1^{(2)} & \cdots & g_1^{(n)} \\ g_2^{(1)} & g_2^{(2)} & \cdots & g_2^{(n)} \\ \cdots & \cdots & \cdots & \cdots \\ g_n^{(1)} & g_n^{(2)} & \cdots & g_n^{(n)} \end{pmatrix}_{n\times n}. \tag{5.1}$$

Then $A_n G_n = I_n$. We may view the numbers $g_1^{(k)}, g_2^{(k)}, \ldots, g_n^{(k)}$ respectively as the first, second, ..., and the n-th term of an infinite (complex) sequence $g^{(k)} = \{g_j^{(k)}\}_{j\in\mathbf{N}}$. Since $A_n G_n = I_n$ can be expanded as

$$\begin{aligned}
ag_0^{(k)} + bg_1^{(k)} + cg_2^{(k)} &= \hbar_1^k - \alpha g_n^{(k)}, \\
ag_1^{(k)} + bg_2^{(k)} + cg_3^{(k)} &= \hbar_2^k, \\
ag_2^{(k)} + bg_3^{(k)} + cg_4^{(k)} &= \hbar_3^k, \\
&\cdots\cdots\cdots \\
ag_{n-1}^{(k)} + bg_n^{(k)} + cg_{n+1}^{(k)} &= \hbar_n^k - \beta g_1^{(k)},
\end{aligned}$$

with $g_0^{(k)} = g_{n+1}^{(k)} = 0$, we have

$$ag_{j-1}^{(k)} + bg_j^{(k)} + cg_{j+1}^{(k)} = \hbar_j^k + f_j^{(k)}, \quad j = 1, 2, \ldots$$

where

$$f_j^{(k)} = \begin{cases} -\alpha g_n^{(k)} & j = 1, \\ -\beta g_1^{(k)} & j = n, \\ 0 & \text{otherwise.} \end{cases}$$

Since $c \neq 0$, we may obtain

$$g^{(k)} = \frac{(cg_1^{(k)} + \hbar^k + f^{(k)})\hbar}{a\hbar^2 + b\hbar + \bar{c}}. \tag{5.2}$$

Let $\eta_\pm = \left(-b \pm \sqrt{\xi}\right)/2a$ be the two roots of $az^2 + bz + c = 0$, where $\xi = b^2 - 4ac$. As in Section 2, there are two cases to be considered.

Case I. Suppose $\xi \neq 0$ so that η_+ and η_- are two different numbers. Since $\eta_+\eta_- = c/a \neq 0$, we may write $\eta_\pm = e^{\pm i\phi}/\rho$ for some ϕ in the strip $\{z \in \mathbb{C} | 0 \leq \operatorname{Re} z < 2\pi\}$, where $\rho = \sqrt{a/c}$ and

$$\cos\phi = -b/2\rho c. \tag{5.3}$$

We also have $\sin\phi = \sqrt{\xi}/(2i\rho c) \neq 0$.

By the method of partial fractions, we may write $g^{(k)}$ in the form

$$g^{(k)} = \frac{1}{\sqrt{\xi}}\left(\frac{1}{\eta_- - \hbar} - \frac{1}{\eta_+ - \hbar}\right)\left(cg_1^{(k)} + \hbar^k + f^{(k)}\right)\hbar,$$

which gives the j-th term of $g^{(k)}$:

$$\begin{aligned} g_j^{(k)} = \frac{2i}{\sqrt{\xi}} \Big\{ & cg_1^{(k)}\rho^j \sin j\phi - \alpha g_n^{(k)}\rho^{j-1}\sin(j-1)\phi \\ & + H(j-k)\rho^{j-k}\sin(j-k)\phi - H(j-n)\beta g_1^{(k)}\rho^{j-n}\sin(j-n)\phi \Big\} \end{aligned} \tag{5.4}$$

for $j \geq 1$. In particular,

$$\frac{\sqrt{\xi}}{2i} g_n^{(k)} = cg_1^{(k)}\rho^n \sin n\phi - \alpha g_n^{(k)}\rho^{n-1}\sin(n-1)\phi + \rho^{n-k}\sin(n-k)\phi$$

and

$$\begin{aligned} 0 = {} & cg_1^{(k)}\rho^{n+1}\sin(n+1)\phi - \alpha g_n^{(k)}\rho^n \sin n\phi \\ & - \beta g_1^{(k)}\rho \sin\phi + \rho^{n+1-k}\sin(n+1-k)\phi. \end{aligned}$$

If the inverse exists, then $g_1^{(k)}$ and $g_n^{(k)}$ form a *unique* solution pair and hence

$$\Delta = \begin{vmatrix} c\rho^n \sin n\phi & -\left(\sqrt{\xi}/2i + \alpha\rho^{n-1}\sin(n-1)\phi\right) \\ c\rho^{n+1}\sin(n+1)\phi - \beta\rho\sin\phi & -\alpha\rho^n \sin n\phi \end{vmatrix} \neq 0,$$

or

$$\Delta = \left(\rho^n(ac\sin(n+1)\phi - \alpha\beta\sin(n-1)\phi) - (\alpha c\rho^{2n} + a\beta)\sin\phi\right)\sin\phi \neq 0. \tag{5.5}$$

Furthermore, if $\Delta \neq 0$, then we have

$$g_1^{(k)} = \Delta_1/\Delta, \quad g_n^{(k)} = \Delta_n/\Delta, \tag{5.6}$$

where

$$\Delta_1 = -\alpha\rho^{2n-k}\sin(k-1)\phi\sin\phi - a\rho^{n-k}\sin(n+1-k)\phi\sin\phi \quad \text{and} \tag{5.7}$$

$$\Delta_n = -c\rho^{2n+1-k}\sin k\phi\sin\phi - \beta\rho^{n+1-k}\sin(n-k)\phi\sin\phi. \tag{5.8}$$

Case II. Suppose $\xi = 0$ so that $\eta_\pm$ are two equal roots. In this case $b^2 = 4ac$. Furthermore, from (5.2), we have

$$g^{(k)} = \frac{(cg_1^{(k)} + \hbar^k + f^{(k)})\hbar}{c(1 - 2(-b/2c)\hbar + (-b\hbar/2c)^2)}$$
$$= \frac{1}{\rho c}\frac{\rho\hbar}{(1-\rho\hbar)^2}(cg_1^{(k)} + \hbar^k + f^{(k)}) = \frac{1}{\rho c}\{j\rho^j\} * (cg_1^{(k)} + \hbar^k + f^{(k)}),$$

where $\rho = -b/2c$. The j-th term of $g^{(k)}$ is now

$$g_j^{(k)} = \frac{1}{\rho c}\left\{cg_1^{(k)} j\rho^j - \alpha g_n^{(k)}(j-1)\rho^{j-1} \right.$$
$$\left. + H(j-k)(j-k)\rho^{j-k} - H(j-n)\beta g_1^{(k)}(j-n)\rho^{j-n}\right\}. \tag{5.9}$$

In particular, $g_n^{(k)} = \left(cg_1^{(k)} n\rho^n - \alpha g_n^{(k)}(n-1)\rho^{n-1} + (n-k)\rho^{n-k}\right)/\rho c$ and

$$0 = \frac{1}{\rho c}\left(cg_1^{(k)}(n+1)\rho^{n+1} - \alpha g_n^{(k)} n\rho^n + (n+1-k)\rho^{n+1-k} - \beta g_1^{(k)}\rho\right).$$

If the inverse exists, then $g_1^{(k)}$ and $g_n^{(k)}$ form a *unique* solution pair and hence we must have

$$\Delta = -\alpha c\rho^{2n} + \rho^n\left(ac(n+1) - \alpha\beta(n-1)\right) - a\beta \neq 0, \tag{5.10}$$

and

$$g_1^{(k)} = \Delta_1/\Delta, \quad g_n^{(k)} = \Delta_n/\Delta, \tag{5.11}$$

where

$$\Delta_1 = -\rho^{2n-k}\alpha(k-1) - \rho^{n-k}a(n+1-k) \quad \text{and} \tag{5.12}$$
$$\Delta_n = -\rho^{2n+1-k}ck - \rho^{n+1-k}\beta(n-k). \tag{5.13}$$

THEOREM 5.1. *Let the inverse of the matrix A_n be denoted by*

$$G_n = \left(g^{(1)} \mid \cdots \mid g^{(n)}\right).$$

If $b^2 - 4ac \neq 0$, then the necessary and sufficient condition for the inverse to exist is that (5.5) *holds for some $\phi \in \{z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z < 2\pi\}$ that satisfies* (5.3). *Furthermore, if the inverse exists, then $g_j^{(k)}$, $2 \leq j \leq n-1$, are given by* (5.4), *while $g_1^{(k)}$ and $g_n^{(k)}$ are given by* (5.6). *If $b^2 - 4ac = 0$, then the necessary and sufficient condition for the inverse to exist is that* (5.10) *holds. Furthermore, if the inverse exists, then $g_j^{(k)}$, $2 \leq j \leq n-1$, are given by* (5.9), *while $g_1^{(k)}$ and $g_n^{(k)}$ are given by* (5.11).

We remark that sufficient conditions for the existence of the inverse of A_n are added in the above result. This is valid since the above arguments can be reversed. We remark also that since $\cos z$ is 2π-periodic, the restriction $\phi \in \{z \in \mathbb{C} \mid 0 \le \operatorname{Re} z < 2\pi\}$ can be relaxed to $\phi \in \mathbb{C}$. Furthermore, since if $b^2 \neq 4ac$, then $\cos\phi \neq \pm 1$ and $\phi \notin \pi\mathbb{Z}$ automatically (*cf.* Theorem 2.1).

6. Inverses of some special Toeplitz matrices

We may now apply Theorem 5.1 for finding the inverses for several special Toeplitz matrices. For motivation, consider the case where $\alpha = \beta = 0$ in A_n. Let $g^{(k)}$ be the k-th column in the inverse G_n of A_n. If $b^2 = 4ac$, then by substituting $\alpha = \beta = 0$ into (5.10) to (5.12), we have $\Delta = \rho^n ac(n+1) \neq 0$ and $\Delta_1 = -\rho^{n-k} a(n+1-k)$. Substituting these into (5.9), we obtain

$$\begin{aligned} g_j^{(k)} &= \frac{1}{\rho c}\left(c g_1^{(k)} j \rho^j + H(j-k)\rho^{j-k}(j-k)\right) \\ &= \frac{-\rho^{j-k}}{\rho c(n+1)} \times \begin{cases} j(n+1-k), & j < k, \\ k(n+1-j), & j \ge k. \end{cases} \end{aligned}$$

After finding $g^{(k)}$, we may directly reverse the arguments leading to Theorem 5.1 and conclude that $G_n = \left(g^{(1)} \mid g^{(2)} \mid \cdots \mid g^{(n)}\right)$.

Now let us suppose that $b^2 \neq 4ac$. Suppose also that the inverse G_n of A_n exists and is of the form (5.1). Then substituting $\alpha = \beta = 0$ into (5.5)–(5.7), in view of Theorem 5.1, we necessarily have $\cos\phi = -b/2\rho c$, $\phi \in \mathbb{C}$,

$$\Delta = \rho^n ac \sin(n+1)\phi \sin\phi \neq 0 \quad \text{and} \quad \Delta_1 = -a\rho^{n-k}\sin(n+1-k)\phi\sin\phi.$$

Substituting these into (5.4), we have

$$\begin{aligned} g_j^{(k)} &= \frac{1}{\rho c \sin\phi}\left(\frac{-\rho^{j-k}\sin j\phi \sin(n+1-k)\phi}{\sin(n+1)\phi} + H(j-k)\rho^{j-k}\sin(j-k)\phi\right) \\ &= \frac{-\rho^{j-k}}{\rho c \sin\phi \sin(n+1)\phi} \times \begin{cases} \sin j\phi \sin(n+1-k)\phi, & j < k, \\ \sin k\phi \sin(n+1-j)\phi, & j \ge k. \end{cases} \end{aligned}$$

Once we have found $g_j^{(k)}$, then if $\sin(n+1)\phi \neq 0$, we may reverse the arguments leading to Theorem 5.1 and conclude that $\left(g^{(1)} \mid g^{(2)} \mid \cdots \mid g^{(n)}\right)$ is our desired inverse. On the other hand, if $\sin(n+1)\phi = 0$, then $\Delta = 0$ and by Theorem 5.1, the inverse of A_n does not exist.

6.1. The case where $\alpha = \pm a$ and $\beta = 0$

THEOREM 6.1. *Suppose $\alpha = a$ and $\beta = 0$ in the matrix A_n.*

(i) *Suppose $b^2 \neq 4ac$. Then the inverse G_n of A_n given by* (5.1) *exists if, and only if, $\cos\phi = -b/2\rho c$ for some $\phi \in \mathbb{C}$ and $\sin(n+1)\phi - \rho^n \sin\phi \neq 0$. Furthermore, if it exists, then*

$$g_j^{(k)} = \frac{-\rho^{j-k}}{\rho c \sin\phi\,(\sin(n+1)\phi - \rho^n \sin\phi)} \times \begin{cases} \sin j\phi \sin(n+1-k)\phi + \rho^n \sin(k-j)\phi \sin\phi, & j < k, \\ \sin k\phi \sin(n+1-j)\phi, & j \geq k. \end{cases} \tag{6.1}$$

(ii) *Suppose $b^2 = 4ac$. Then the inverse G_n of A_n given by* (5.1) *exists if, and only if, $n + 1 - \rho^n \neq 0$ and, if it exists,*

$$g_j^{(k)} = \frac{-\rho^{j-k}}{\rho c\,(n+1-\rho^n)} \times \begin{cases} j(n+1-k) + \rho^n(k-j), & j < k, \\ k(n+1-j), & j \geq k. \end{cases} \tag{6.2}$$

PROOF. Suppose the inverse of A_n exists and is of the form (5.1). If $b^2 \neq 4ac$, then substituting $\alpha = a$ and $\beta = 0$ into (5.5)–(5.8), we necessarily have $\cos\phi = -b/2\rho c$, $\phi \in \mathbb{C}$,

$$\begin{aligned} \Delta &= ac\rho^n\,(\sin(n+1)\phi - \rho^n \sin\phi)\sin\phi \neq 0, \\ \Delta_1 &= -a\rho^{n-k}\,(\rho^n \sin(k-1)\phi + \sin(n+1-k)\phi)\sin\phi \quad \text{and} \\ \Delta_n &= -c\rho^{2n+1-k} \sin k\phi \sin\phi. \end{aligned}$$

Substituting these into (5.4), we obtain

$$g_j^{(k)} = \frac{\rho^{j-k}}{\rho c \sin\phi}\left(\frac{\rho^n \sin(j-k)\phi \sin\phi - \sin(n+1-k)\phi \sin j\phi}{\sin(n+1)\phi - \rho^n \sin\phi} + H(j-k)\sin(j-k)\phi\right),$$

which is equivalent to (6.1).

Once we have found $g_j^{(k)}$, then if $\sin(n+1)\phi - \rho^n \sin\phi \neq 0$, we may verify that G_n is the inverse of A_n. On the other hand, if $\sin(n+1)\phi - \rho^n \sin\phi = 0$, then the inverse does not exist.

If $b^2 = 4ac$, then by substituting $\alpha = a$, $\beta = 0$ into (5.10) to (5.13), we necessarily have $\Delta = ac\rho^n(n+1-\rho^n) \neq 0$, $\Delta_1 = -a\rho^{n-k}(\rho^n(k-1) + (n+1-k))$ and $\Delta_n = -\rho^{2n+1-k}ck$. Substituting these into (5.9), we obtain

$$g_j^{(k)} = \frac{\rho^{j-k}}{\rho c}\left\{\frac{\rho^n(j-k) - j(n+1-k)}{n+1-\rho^n} + H(j-k)(j-k)\right\},$$

which is equivalent to (6.2). Once we have found $g_j^{(k)}$, then if $n+1-\rho^n \neq 0$, we may verify that G_n is the inverse of A_n. On the other hand, if $n+1-\rho^n = 0$, then the inverse does not exist. The proof is complete. □

Suppose $\alpha = -a$ and $\beta = 0$ in the matrix A_n. We may follow the same arguments to show the following. (i) Suppose $b^2 \neq 4ac$, then the inverse G_n of A_n given by (5.1) exists if, and only if, $\cos\phi = -b/2\rho c$ for some $\phi \in \mathbb{C}$ and $\sin(n+1)\phi + \rho^n \sin\phi \neq 0$. Furthermore, if it exists, then

$$g_j^{(k)} = \frac{-\rho^{j-k}}{\rho c \sin\phi\,(\sin(n+1)\phi + \rho^n \sin\phi)} \times \begin{cases} \sin j\phi \sin(n+1-k)\phi - \rho^n \sin(k-j)\phi \sin\phi, & j < k, \\ \sin k\phi \sin(n+1-j)\phi, & j \geq k. \end{cases} \tag{6.3}$$

(ii) Suppose $b^2 = 4ac$, then the inverse G_n of A_n given by (5.1) exists if, and only if, $n+1+\rho^n \neq 0$ and, if it exists,

$$g_j^{(k)} = \frac{-\rho^{j-k}}{\rho c\,(n+1+\rho^n)} \times \begin{cases} j(n+1-k) - \rho^n(k-j), & j < k, \\ k(n+1-j), & j \geq k. \end{cases}$$

As we have seen, the derivation of the explicit formulas from Theorem 5.1 are straightforward. Theoretically, we can obtain formulas for arbitrary α and β, though in most cases those formulas may be complicated in form. But at least we can obtain elegant formulas for some special combinations of α and β. Some of the results are presented below without proof for comparison and quick reference. The proof is simple and may be obtained in a way similar to that of Theorem 6.1.

6.2. The case where $\alpha = 0$ and $\beta = \pm c$

THEOREM 6.2. *Suppose $\alpha = 0$ and $\beta = c$ in the matrix A_n.*

(i) *Suppose $b^2 \neq 4ac$. Then the inverse G_n of A_n given by* (5.1) *exists if, and only if, $\cos\phi = -b/2\rho c$ for some $\phi \in \mathbb{C}$ and $\rho^n \sin(n+1)\phi - \sin\phi \neq 0$. Furthermore, if it exists, then*

$$g_j^{(k)} = \frac{-\rho^{j-k}}{\rho c \sin\phi\,(\sin(n+1)\phi - \rho^{-n} \sin\phi)} \times \begin{cases} \sin j\phi \sin(n+1-k)\phi, & j < k, \\ \sin k\phi \sin(n+1-j)\phi + \rho^{-n} \sin(j-k)\phi \sin\phi, & j \geq k. \end{cases}$$

(ii) *Suppose $b^2 = 4ac$. Then the inverse G_n of A_n given by* (5.1) *exists if, and only if, $\rho^n(n+1) - 1 \neq 0$ and, if it exists,*

$$g_j^{(k)} = \frac{-\rho^{j-k}}{\rho c\,(n+1-\rho^{-n})} \times \begin{cases} j(n+1-k), & j < k, \\ k(n+1-j) + \rho^{-n}(j-k), & j \geq k. \end{cases}$$

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Suppose $\alpha = 0$ and $\beta = -c$ in the matrix A_n.

(i) *Suppose $b^2 \neq 4ac$. Then the inverse G_n of A_n given by* (5.1) *exists if, and only if, $\cos\phi = -b/2\rho c$ for some $\phi \in \mathbb{C}$ and $\rho^n \sin(n+1)\phi + \sin\phi \neq 0$. Furthermore, if it exists, then*

$$g_j^{(k)} = \frac{-\rho^{j-k}}{\rho c \sin\phi\,(\sin(n+1)\phi + \rho^{-n}\sin\phi)} \times \begin{cases} \sin j\phi \sin(n+1-k)\phi, & j < k, \\ \sin k\phi \sin(n+1-j)\phi - \rho^{-n}\sin(j-k)\phi\sin\phi, & j \geq k. \end{cases}$$

(ii) *Suppose $b^2 = 4ac$. Then the inverse G_n of A_n given by* (5.1) *exists if, and only if, $\rho^n(n+1) + 1 \neq 0$ and, if it exists,*

$$g_j^{(k)} = \frac{-\rho^{j-k}}{\rho c\,(n+1+\rho^{-n})} \times \begin{cases} j(n+1-k), & j < k, \\ k(n+1-j) - \rho^{-n}(j-k), & j \geq k. \end{cases}$$

6.3. The case where $\alpha = \pm a$, $\beta = \pm c$

THEOREM 6.3. *Suppose $\alpha = a$ and $\beta = c$ in the matrix A_n.*

(i) *Suppose $b^2 \neq 4ac$. Then the inverse G_n of A_n given by* (5.1) *exists if, and only if, $\cos\phi = -b/2\rho c$ for some $\phi \in \mathbb{C}$ and $\rho^{2n} - 2\rho^n \cos n\phi + 1 \neq 0$. Furthermore, if it exists, then*

$$g_j^{(k)} = \frac{\rho^{j-k}}{\rho c \sin\phi\,(\rho^n - 2\cos n\phi + \rho^{-n})} \times \begin{cases} \rho^n \sin(k-j)\phi + \sin(n+j-k)\phi, & j < k, \\ \rho^{-n}\sin(j-k)\phi + \sin(n+k-j)\phi, & j \geq k. \end{cases}$$

In particular, when $a = c$,

$$g_j^{(k)} = \frac{\sin(|k-j|\phi) + \sin(n-|k-j|)\phi}{2a\sin\phi(1-\cos n\phi)}.$$

(ii) *If $b^2 = 4ac$ and $\rho^n \neq 1$, then the inverse G_n given by* (5.1) *exists and*

$$g_j^{(k)} = \frac{\rho^{j-k}}{\rho c\,(\rho^n - 2 + \rho^{-n})} \times \begin{cases} \rho^n(k-j) + (n+j-k), & j < k, \\ \rho^{-n}(j-k) + (n+k-j), & j \geq k. \end{cases}$$

In particular, when $b = 2a = 2c$ and n is odd, then $g_j^{(k)} = (-1)^{j-k}(n - 2|k-j|)/4a$. If $b = 2a = 2c$ and n is even, or $b = -2a = -2c$, then $\rho^n = 1$ and the matrix A_n is singular.

Suppose $\alpha = -a$ *and* $\beta = -c$ *in the matrix* A_n.

(i) *Suppose* $b^2 \neq 4ac$. *Then the inverse* G_n *of* A_n *given by* (5.1) *exists if, and only if,* $\cos\phi = -b/2\rho c$ *for some* $\phi \in \mathbb{C}$ *and* $\rho^{2n} + 2\rho^n \cos n\phi + 1 \neq 0$. *Furthermore, if it exists, then*

$$g_j^{(k)} = \frac{\rho^{j-k}}{\rho c \sin\phi\,(\rho^n + 2\cos n\phi + \rho^{-n})} \times \begin{cases} \rho^n \sin(k-j)\phi - \sin(n+j-k)\phi, & j < k, \\ \rho^{-n}\sin(j-k)\phi - \sin(n+k-j)\phi, & j \geq k. \end{cases}$$

In particular, when $a = c$,

$$g_j^{(k)} = \frac{\sin(|k-j|\phi) - \sin(n-|k-j|)\phi}{2a\sin\phi(1+\cos n\phi)}.$$

(ii) *If* $b^2 = 4ac$ *and* $\rho^n \neq -1$, *then the inverse* G_n *of* A_n *exists and*

$$g_j^{(k)} = \frac{\rho^{j-k}}{\rho c\,(\rho^n + 2 + \rho^{-n})} \times \begin{cases} \rho^n(k-j) - (n+j-k), & j < k, \\ \rho^{-n}(j-k) - (n+k-j), & j \geq k. \end{cases}$$

In particular, if $b = -2a = -2c$, *then*

$$g_j^{(k)} = (2|k-j| - n)/4a. \tag{6.4}$$

If $b = 2a = 2c$ *and* n *is even, then*

$$g_j^{(k)} = (-1)^{j-k}(n - 2|k-j|)/4a. \tag{6.5}$$

If $b = 2a = 2c$ *and* n *is odd, then* $\rho^n = -1$ *and the matrix* A_n *is singular.*

7. Examples

We give two applications of our explicit formulas.

EXAMPLE 1. In the synchronisation problems of artificial neural networks [11], we encounter the tridiagonal Toeplitz matrix with $a = c = 1$, $b = -2$ with the corners $\alpha = \beta = -1$. Theorem 4.3 gives the eigenvalues and eigenvectors for such matrices. As a numerical example, consider the following matrix:

$$A_4 = \begin{pmatrix} -2 & 1 & 0 & -1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ -1 & 0 & 1 & -2 \end{pmatrix}. \tag{7.1}$$

By Theorem 4.3, we have

$$\lambda_k = -2 + 2\cos\frac{(2k-1)\pi}{4}, \qquad k = 1, 2, 3, 4,$$

which gives $\lambda_1 = \lambda_4 = -2 + \sqrt{2}$ and $\lambda_2 = \lambda_3 = -2 - \sqrt{2}$. Since $a = c$, the eigenvectors may be obtained by either (4.17) or (4.18). By (4.17), we have

$$u^{(1)} = \left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}, -i, -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}, -1\right)^{\dagger}.$$

By (4.18), if we take $c_1 = 1$ and $c_2 = 0$, then $u^{(1)} = (\sqrt{2}/2, 0, -\sqrt{2}/2, -1)^{\dagger}$; if we take $c_1 = 0$ and $c_2 = 1$, then $u^{(1)} = (\sqrt{2}/2, 1, \sqrt{2}/2, 0)^{\dagger}$. It can be easily checked that they are the correct eigenvectors corresponding to λ_1.

We remark that our theorem is also applicable when $b = 2$. An inspection of (4.16) to (4.18) reveals that only λ_k depends on b. Thus the eigenvalues of the matrix

$$B_4 = \begin{pmatrix} 2 & 1 & 0 & -1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ -1 & 0 & 1 & 2 \end{pmatrix} \tag{7.2}$$

are given by $\lambda_1 = \lambda_4 = 2 + \sqrt{2}$ and $\lambda_2 = \lambda_3 = 2 - \sqrt{2}$, with the corresponding eigenvectors unchanged.

The inverses of (7.1) and (7.2) may be obtained by (6.4) and (6.5) in Theorem 6.3 as

$$A_4^{-1} = \frac{1}{2}\begin{pmatrix} -2 & -1 & 0 & 1 \\ -1 & -2 & -1 & 0 \\ 0 & -1 & -2 & -1 \\ 1 & 0 & -1 & -2 \end{pmatrix} \quad \text{and} \quad B_4^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 1 & 0 & -1 & 2 \end{pmatrix}$$

respectively.

EXAMPLE 2. Consider the following three-point boundary value problem [12] of the form

$$\begin{cases} \Delta^2 u_{k-1} + u_k = f(u_k), & k = 1, 2, \ldots, n, \\ u_0 = \alpha u_l, & u_{n+1} = 0. \end{cases} \tag{7.3}$$

In matrix form, this may be written as

$$A_n u = F(u),$$

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where $u = (u_1, \dots, u_n)^\dagger$, $F(u) = (f(u_1), \dots, f(u_n))^\dagger$ and

$$A_n = \begin{pmatrix} -1 & 1 & 0 & \cdots & \alpha & \cdots & 0 & 0 \\ 1 & -1 & 1 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & -1 \end{pmatrix}_{n\times n}.$$

A necessary condition for the system (7.3) to be solvable is that A_n is invertible. In the case where $l = n$ and $\alpha = \pm 1$, Theorem 6.1 offers some help. As a numerical example for $l = n = 6$ and $\alpha = -1$, since $\rho = 1$,

$$\phi = \cos^{-1}\frac{1}{2} = \frac{\pi}{3} \quad \text{and} \quad \left(\sin\frac{\pi}{3}\right)\left(\sin\frac{7\pi}{3} + \sin\frac{\pi}{3}\right) = \frac{3}{2} \neq 0,$$

we see that the inverse exists and is given by (6.3) as

$$g_j^{(k)} = \frac{-2}{3} \times \begin{cases} \sin\frac{j\pi}{3}\sin\frac{(7-k)\pi}{3} - \sin\frac{(k-j)\pi}{3}\sin\frac{\pi}{3} & j < k, \\ \sin\frac{k\pi}{3}\sin\frac{(7-j)\pi}{3} & j \geq k, \end{cases}$$

which yields

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & -1 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 0 & 2 & 2 & 0 & -2 & -2 \\ 1 & 1 & 2 & 1 & -1 & -2 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ -1 & -1 & 0 & 1 & 1 & 2 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

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