ON A THEOREM OF BRUDNO OVER
NON-ARCHIMEDEAN FIELDS

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A classical theorem of Brudno, dealing with the consistency of
summability with regular matrices is shown by example not to hold
over a non-archimedian field.

1.

Following Monna [1], attempts have been made in recent times to study
different summability methods over non-archimedian fields which are
complete in the metric of valuation. In all such attempts, as in [3], [4],
significant differences in contrast to the classical case have been
obtained. The object of the present short note is to prove by an example
that the classical theorem of Brudno [2] dealing with the consistency of
regular matrices is not true in general in the non-archimedean case. In
§2, we shall describe the necessary preliminaries, whereas in §3, we shall
establish our claim.

2.

Let $K$ be a non-archimedian field which is complete under the metric
of valuation denoted by $| |$. We note that the valuation $| |$ is non-
archimedian if and only if $|n| < 1$ for every integer $n$ considered as an
element of $K$. Thus, in a field with non-trivial non-archimedian
valuation, the sequence $\{1, 2, 3, \ldots\} = \{n\}$ is a bounded sequence in the
metric of valuation.

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Let $A = (a_{np})$, $n, p = 1, 2, 3, \ldots$, be a matrix defined over such a field. For $n = 1, 2, 3, \ldots$, let us write

$$y_n = \sum_{p=1}^{\infty} a_{np} x_p .$$

For every sequence $x = \{x_n\}$ defined over $K$, let $\{y_n\}$ be convergent for each $n$. $y_n$ is called the $A$-transform of $x$. If $y_n \to y$ as $n \to \infty$ in the metric of valuation, then $x$ is said to be $A$-summable to $y$. $A$ is said to be convergence preserving if $\lim y_n$ exists for every convergent sequence $x$. $A$ is called regular if in addition $\lim y_n = \lim x_n$. Such regular matrices are also known as Toeplitz matrices. The theorem given below is practically contained in [1].

**Theorem (Monna).** A matrix $A = (a_{np})$ is a regular matrix defined over $K$ if and only if $\sup_{n,p} |a_{np}| \leq M$ where $M$ is a constant, $\lim_{n \to \infty} a_{np} = 0$ for every fixed $p$, $\sum_{p=1}^{\infty} a_{np} A_{np} = 1 + A$ as $n \to \infty$.

The following is the classical Brudno theorem on a regular matrix for which a simple proof was given by Petersen [2].

**Theorem (Petersen).** Let every bounded sequence summable by a Toeplitz matrix $A$ also be summable by a Toeplitz matrix $B$. Then it is summable to the same value by $B$ as by $A$.

Petersen [2] established this theorem by showing that if two regular matrix methods $A = (a_{mn})$ and $B = (b_{mn})$ sum bounded sequence $\{s_n\}$ to different sums, then there exists a bounded sequence which is summed by $A$ but not by $B$.

3.

In this section we shall give examples of two regular matrices $A$ and $B$ over $K$ such that every bounded sequence summed by $A$ is also summed by $B$ and show that there exists a bounded sequence summable by these two regular matrices to two different sums.
Let \( A = (a_{np}) \) and \( B = (b_{np}) \) be defined as follows:

\[
a_{np} = \begin{cases} 
  n + 1 & \text{when } p = n, \\
  -n & \text{when } p = n+1, \\
  0 & \text{for all other values of } n \text{ and } p;
\end{cases}
\]

\[
b_{np} = \begin{cases} 
  n + 2 & \text{when } p = n, \\
  -(n+1) & \text{when } p = n+1, \\
  0 & \text{for all other values of } n \text{ and } p.
\end{cases}
\]

The matrix \( A \) satisfies the conditions of the theorem of Monna given in §2 as seen below.

(i) Since \(|n+1| = \max(|n|, 1)\) and \(|n| < 1\), we have \(|n+1| = 1\). Hence we have from this \( \sup_{n,p} |a_{np}| \leq \sup(|n+1|, |n|) = 1 \).

(ii) Since each column of \( A \) contains infinitely many zeros and \(|n+1| = 1\) and \(|n| < 1\), \( a_{np} \to 0 \) as \( n \to \infty \).

(iii) \( \sum_{p=1}^{\infty} a_{np} = n + l - n = 1 + l \) as \( n \to \infty \).

Hence \( A = (a_{np}) \) is a regular matrix. In a similar manner, we can verify that \( B \) is also a regular matrix over \( K \).

As a next step, we shall show that every bounded sequence summed by \( A \) is also summed by \( B \). For this let \( \{x_n\} \) be any bounded sequence. If \( y_n \) is the \( A \)-transform of \( x_n \), then we have \( y_n = (n+1)x_n - nx_{n+1} \). If \( y'_n \) is the \( B \)-transform of \( x_n \), then

\[
y'_n = (n+2)x_n - (n+1)x_{n+1}.
\]

The relation between \( y_n \) and \( y'_n \) is easily seen to be

\[
y'_n = y_n + (x_n - x_{n+1}).
\]

Hence \( |y'_n| \leq \max\{|y_n|, |x_n - x_{n+1}|\} \leq |y_n| + \lambda \) where \( |x_n| \leq \lambda \) for all \( n \), \( \lambda \) being a constant. Thus we have \( |y'_n| \leq |y_n| + \lambda \).

If \( \{y_n\} \) is convergent, then \( \{y'_n\} \) is also convergent. Thus if
\( \{x_n\} \) is summable by \( A \), then it is summable by \( B \) also. This shows that
the bounded convergence field of \( A \) is contained in the bounded
convergence field of \( B \).

We establish our claim by showing that there exists a bounded sequence
summable by these two regular matrices to two different sums. For this
consider the bounded sequence \( N = \{n\} = \{1, 2, 3, \ldots\} \) in \( K \). The
\( A \)-transform of the sequence \( N \) gives rise to the sequence
\( \{y_n\} = \{0, 0, 0, 0, \ldots\} \). So \( N \) is \( A \)-summable to 0. The \( B \)-transform
of the sequence \( N \) gives rise to the sequence \( \{y'_n\} = \{-1, -1, -1, \ldots\} \).
So \( N \) is \( B \)-summable to -1. Hence given two regular methods \( A \) and \( B \)
defined over \( K \) such that every bounded sequence summed by \( A \) is also
summed by \( B \), there exists a bounded sequence \( N = \{n\} \) summable by \( A \)
and \( B \) to two different sums which cannot happen in the case of the
classical Brudno's theorem. This establishes our claim.

References


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