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$bvca(\Sigma, X)$ REVISITED

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Abstract. Assuming that (Ω, Σ) is a measurable space and X is a Banach space we provide a quite general sufficient condition on X for $bvca(\Sigma, X)$ (the Banach space of all X-valued countably additive measures of bounded variation equipped with the variation norm) to contain a copy of c_0 if and only if X does. Some well-known results on this topic are straightforward consequences of our main theorem.

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1. Preliminaries. Throughout this paper X will be a Banach space over the field $\mathbb{K}$ of real or complex numbers. Our notation is standard [**3, 4**]. If (Ω, Σ) is a measurable space, $ca(\Sigma, X)$ denotes the Banach space over $\mathbb{K}$ of all X-valued countably additive measures F on Σ provided with the semivariation norm $\|F\|$ and $bvca(\Sigma, X)$ stand for the Banach space of all X-valued countably additive measures F of bounded variation on Σ equipped with the variation norm $|F|$. We represent by $ca^+(\Sigma)$ the set of all positive and finite measures defined on Σ. If (Ω, Σ, μ) is a finite measure space, recall that a weakly μ-measurable function $f : \Omega \to X$ is said to be Dunford integrable if $x^*f \in \mathcal{L}_1(\mu)$ for every $x^* \in X^*$. If f is Dunford integrable and $E \in \Sigma$ the map $x^* \mapsto \int_E x^*f \, d\mu$, denoted by $(D)\int_E f \, d\mu$, is a continuous linear form on X^*. If $(D)\int_E f \, d\mu \in X$ for each $E \in \Sigma$ then f is called Pettis integrable and one writes $(P)\int_E f \, d\mu$ instead of $(D)\int_E f \, d\mu$. A strongly μ-measurable function $f : \Omega \to X$ is said to be Bochner integrable if $\int_\Omega \|f(\omega)\| \, d\mu(\omega) < \infty$. As usual we denote by $L_1(\mu, X)$ the Banach space of all (equivalence classes of) μ-Bochner integrable functions equipped with the norm $\|f\|_1 = \int_\Omega \|f(\omega)\| \, d\mu(\omega)$. Recall that a series $\sum_{n=1}^\infty x_n$ in X is said to be weakly unconditionally Cauchy (wuC) if $\sum_{n=1}^\infty |x^*x_n| < \infty$ for each $x^* \in X^*$.

If each $\mu \in ca^+(\Sigma)$ is purely atomic, then $ca(\Sigma, X)$ contains a copy of c_0 or ℓ_∞ if and only if X contains, respectively, a copy of c_0 or ℓ_∞ [**5**]. Assuming that X has the Radon–Nikodym property with respect to each $\mu \in ca^+(\Sigma)$, then $bvca(\Sigma, X)$ contains a copy of c_0 or ℓ_∞ if and only if X does [**7**]. As a consequence, if each $\mu \in ca^+(\Sigma)$ is purely atomic then $bvca(\Sigma, X)$ contains a copy of c_0 or ℓ_∞ if and only if X contains, respectively, a copy of c_0 or ℓ_∞. If there exists a nonzero atomless measure $\mu \in ca^+(\Sigma)$, the latter statement is no longer true [**11**]. However, if the range space of the measures is a dual Banach space X^*, then $bvca(\Sigma, X^*)$ contains a copy of c_0 or ℓ_∞ if and only if X^* does [**10**].

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2. Banach spaces with property (M)**.** If (Ω, Σ, μ) is a complete probability space, we shall denote by $\mathcal{H}_1(\mu, X)$ the set of all those functions $f : \Omega \to X$ such that $\|f(\cdot)\| \in \mathcal{L}_1(\mu)$. We shall say that two functions $f, g \in \mathcal{H}_1(\mu, X)$ are μ-equivalent if there is a μ-zero set N such that $f(\omega) = g(\omega)$ for all $\omega \in \Omega \setminus N$ and we shall denote by $H_1(\mu, X)$ the set of all classes of μ-equivalent functions. By $\mathcal{L}_{w^*}^{\infty}(\mu, X^*)$ we shall design the linear space over $\mathbb{K}$ of all μ-essentially bounded functions $\varphi : \Omega \to X^*$ which are weak* measurable whereas $bvca_\mu(\Sigma, X)$ will represent the linear subspace of $bvca(\Sigma, X)$ of all those measures F for which there is some $a > 0$ (which depends on F) such that $\|F(E)\| \leq a\,\mu(E)$ for each $E \in \Sigma$, [**2**].

Definition 2.1. We say that a Banach space X has property (M) with respect to a measurable space (Ω, Σ) if given a complete probability measure $\mu : \Sigma \to [0, 1]$ there is a map $T_\mu : bvca_\mu(\Sigma, X) \to H_1(\mu, X)$ with linear range which is linear as a map into its range and for each $F \in bvca_\mu(\Sigma, X)$ it holds that

$$|F| = \int_\Omega \|f(\omega)\|\, d\mu(\omega)$$

for each function $f \in T_\mu(F)$. If X has property (M) with respect to every measurable space (Ω, Σ), then we shall say that X has property (M).

Proposition 2.1. *Each dual Banach space* X^* *has property* (M)*.*

Proof. If (Ω, Σ, μ) is a complete probability space, according to a well-known consequence of the lifting theorem [**2**, Theorem 1.5.2] there is a linear injective map $S_\mu : bvca_\mu(\Sigma, X^*) \to \mathcal{L}_{w^*}^{\infty}(\mu, X^*)$ such that for each $F \in bvca_\mu(\Sigma, X^*)$ the function $f = S_\mu(F)$ satisfies:

(1) For each $E \in \Sigma$ and $x \in X$ one has

$$F(E)\,x = \int_E f(\omega)\, x\, d\mu(\omega).$$

(2) The function $\omega \to \|f(\omega)\|$ is measurable, belongs to $\mathcal{L}_1(\mu)$ and

$$|F|(E) = \int_E \|f(\omega)\|\, d\mu(\omega)$$

for each $E \in \Sigma$.

Since $S_\mu(bvca_\mu(\Sigma, X^*))$ is a linear subspace of $\mathcal{L}_{w^*}^{\infty}(\mu, X^*)$ contained in $\mathcal{H}_1(\mu, X^*)$, if Q denotes the quotient map from $\mathcal{H}_1(\mu, X^*)$ onto $H_1(\mu, X^*)$ which maps $f \in \mathcal{H}_1(\mu, X^*)$ into the class $\widehat{f}$ of all those functions of $\mathcal{H}_1(\mu, X^*)$ which are μ-equivalent to f, the map $T_\mu := Q \circ S_\mu$ which carries F into the class $\widehat{f}$ satisfies the required conditions. □

Proposition 2.2. *If* (Ω, Σ) *is a measurable space such that* X *has the Radon–Nikodym property with respect to each* $\mu \in ca^+(\Sigma)$*, then* X *has property* (M) *with respect to* (Ω, Σ)*. If* X *has the Radon–Nikodym property, then* X *has property* (M)*.*

Proof. Let us assume that X has the Radon–Nikodym property with respect to each complete measure space $(\Omega, \Sigma, \lambda)$ with $\lambda \in ca^+(\Sigma)$. Let μ be a complete probability on Σ and let $F \in bvca_\mu(\Sigma, X)$. Since X has the Radon–Nikodym property with respect to the complete probability space (Ω, Σ, μ) and $F \ll \mu$, there is a unique $\widetilde{f} \in L_1(\mu, X)$

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such that

$$F(E) = (B)\int_E f\,d\mu$$

for each $f \in \widetilde{f}$, so that

$$|F|(E) = \int_E \|f(\omega)\|\,d\mu(\omega)$$

for each $f \in \widetilde{f}$. Since $\mathcal{L}_1(\mu, X) \subseteq \mathcal{H}_1(\mu, X)$, if $\widehat{f}$ denotes the class in $H_1(\mu, X)$ defined by a representative $f \in \widetilde{f}$, the map $T_\mu : bvca_\mu(\Sigma, X) \to H_1(\mu, X)$ defined by $T_\mu(F) = \widehat{f}$ is linear into its range and satisfies the required conditions. □

Proposition 2.3. *Let (Ω, Σ) be a measure space with $\Sigma = 2^\Omega$. If B_{X^*} (the closed unit ball of X^*) is weak* sequentially dense in $B_{X^{***}}$ (the closed unit ball of X^{***}) and X is norm-one complemented in X^{**}, then X has property (M) with respect to (Ω, Σ).*

Proof. Let μ be a complete probability on Σ. The following argument is based on the proof of [**8**, Theorem 1.1]. By the lifting theorem there is a linear injective map $R_\mu : bvca_\mu(\Sigma, X^{**}) \to \mathcal{L}^\infty_{w^*}(\mu, X^{**})$ such that if $f := R_\mu(F)$ then

(1) $x^*F(E) = \int_E f(\omega)x^*\,d\mu(\omega)$ for each $x^* \in X^*$ and $E \in \Sigma$, and

(2) $|F|(E) = \int_E \|f(\omega)\|\,d\mu(\omega)$ for each $E \in \Sigma$.

Since B_{X^*} is weak* sequentially dense in $B_{X^{***}}$, given $x^{***} \in B_{X^{***}}$ there is a sequence $\{x_n^*\}$ in B_{X^*} that converges to x^{***} under the weak* topology of $B_{X^{***}}$. Then, choosing a fixed $F \in bvca_\mu(\Sigma, X)$ and setting $f := R_\mu(F)$, it follows that $f(\omega)x_n^* \to x^{***}f(\omega)$ for each $\omega \in \Omega$. Since $|f(\omega)x_n^*| \le \|f(\omega)\|$ for μ-almost all $\omega \in \Omega$ then $x^{***}f \in \mathcal{L}_1(\mu)$ so that $f : \Omega \to X^{**}$ is Dunford integrable in Ω. Moreover, the dominated convergence theorem and condition 1 above imply that

$$x^{***}F(E) = \int_E x^{***}f(\omega)\,d\mu(\omega)$$

for each $E \in \Sigma$. This guarantees that the function $f : \Omega \to X^{**}$ is Pettis integrable and that $F(E) = (P)\int_E f\,d\mu$ for each $E \in \Sigma$. On the other hand, since $x^{***}f \in \mathcal{L}_1(\mu)$ for each $x^{***} \in X^{***}$, if S is a norm-one linear projection form X^{**} onto X then

$$\int_E x^*(S \circ f)(\omega)\,d\mu(\omega) = \langle S^*x^*, F(E)\rangle = \langle x^*, S(F(E))\rangle = x^*F(E)$$

for each $x^* \in X^*$. This establishes that $S \circ f : \Omega \to X$ is Pettis integrable and that

$$F(E) = (P)\int_E (S \circ f)(\omega)\,d\mu(\omega)$$

for all $E \in \Sigma$. Since $\omega \mapsto \|(S \circ f)(\omega)\|$ is μ-measurable because $\Sigma = 2^\Omega$, it follows that

$$|F| \le \int_\Omega \|(S \circ f)(\omega)\|\,d\mu(\omega). \tag{2.1}$$

But the fact that $\|S\| = 1$ and condition 2 yield

$$\int_\Omega \|(S \circ f)(\omega)\|\,d\mu(\omega) \le \int_\Omega \|f(\omega)\|\,d\mu(\omega) = |F|. \tag{2.2}$$

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Form (2.1) and (2.2) we conclude that

$$|F| = \int_{\Omega} \|(S \circ f)(\omega)\| \, d\mu(\omega). \tag{2.3}$$

Since $(S \circ R_\mu)(bvca(\Sigma, X))$ is a linear space contained in $\mathcal{H}_1(\mu, X)$, if Q denotes the quotient map from $\mathcal{H}_1(\mu, X)$ onto $H_1(\mu, X)$, the map $T_\mu := Q \circ S \circ R_\mu$ which carries F into the class $\widehat{h}$ in $\mathcal{H}_1(\mu, X)$ given by $h = (S \circ R_\mu)(F)$ is as required. □

3. Main theorem and its consequences.

THEOREM 3.1. *If X has property (M) with respect to a measurable space (Ω, Σ), then $bvca(\Sigma, X)$ contains a copy of c_0 if and only if X does.*

Proof. Let (Ω, Σ) be a measurable space, let $\{F_n\}$ denote a normalized basic sequence in $bvca(\Sigma, X)$ equivalent to the unit vector basis of c_0 and set $\mu := \sum_{n=1}^{\infty} 2^{-n} |F_n|$, so that $\|F_n(E)\| \leq 2^n \mu(E)$ for each $E \in \Sigma$ and $n \in \mathbb{N}$. By μ-completing the σ-algebra Σ and extending by zero the F_n if necessary we may assume μ to be complete. Clearly $\operatorname{span}(\{F_n\}) \subseteq bvca_\mu(\Sigma, X)$ and the fact that X has property (M) with respect to (Ω, Σ) provides a linear map T_μ from $\operatorname{span}(\{F_n\})$ into $H_1(\mu, X)$ such that

$$|F| = \int_{\Omega} \|f(\omega)\| \, d\mu(\omega) \tag{3.1}$$

for each $f \in T_\mu(F)$, with $F \in \operatorname{span}(\{F_n\})$.

For each $n \in \mathbb{N}$ pick a concrete representative $f_n \in T_\mu(F_n)$. Since the series $\sum_{n=1}^{\infty} F_n$ in $bvca(\Sigma, X)$ is wuC, there is $C > 0$ such that $|\sum_{i=1}^{n} \varepsilon_i f_i| < C$ for all finite set of signs ε_i. Using the fact that T_μ is a linear map into its range, then

$$\sum_{i=1}^{n} \varepsilon_i T_\mu(F_i) = T_\mu\left(\sum_{i=1}^{n} \varepsilon_i f_i\right) \tag{3.2}$$

for each $n \in \mathbb{N}$. Since $\sum_{i=1}^{n} \varepsilon_i f_i$ is a representative of the class $\sum_{i=1}^{n} \varepsilon_i T_\mu F_i$, equations (3.1) and (3.2) imply

$$\int_{\Omega} \left\| \sum_{i=1}^{n} \varepsilon_i f_i(\omega) \right\| d\mu(\omega) = \left| \sum_{i=1}^{n} \varepsilon_i f_i \right| < C \tag{3.3}$$

for each $\varepsilon_i \in \{-1, 1\}$, $1 \leq i \leq n$ and $n \in \mathbb{N}$.

Equation (3.3) along with Rosenthal's disjointification lemma (*cf.* [**2**, Lemma 1.2.1]) forces the sequence $\{\|f_n(\cdot)\|\}$ in $\mathcal{L}_1(\mu)$ to be uniformly integrable (this is almost contained in the proof of [**2**, Theorem 2.1.1] as well as in the first part of the proof of [**8**, Lemma 2.3]). Now, setting $A_1 = \{\omega \in \Omega : \overline{\lim}_{n\to\infty} \|f_n(\omega)\| > 0\}$, we claim that $\mu(A_1) > 0$. Indeed, otherwise $\lim_{n\to\infty} \|f_n(\omega)\| = 0$ for almost all $\omega \in \Omega$, and since the $\{\|f_n(\cdot)\|\}$ is uniformly integrable it follows from Vitali's lemma ([**9**, Exercise 13.38] or [**6**, IV.10 Theorem 9]) that

$$\lim_{n\to\infty} \int_{\Omega} \|f_n(\omega)\| \, d\mu(\omega) = 0,$$

contradicting that $\int_{\Omega} \|f_n(\omega)\| \, d\mu(\omega) = 1$ for each $n \in \mathbb{N}$.

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Denoting by Δ the product space $\{-1, 1\}^{\mathbb{N}}$, Γ the σ-algebra of subsets of Δ generated by the n-cylinders of Δ, $n = 1, 2, \ldots$ and ν the probability measure $\otimes_{i=1}^{\infty} \nu_i$ on Γ, where $\nu_i : 2^{\{-1,1\}} \to [0, 1]$ satisfies that $\nu_i(\emptyset) = 0$, $\nu_i(\{-1\}) = \nu_i(\{1\}) = 1/2$ and $\nu_i(\{-1, 1\}) = 1$ for each $i \in \mathbb{N}$, we may consider the μ-measurable map $\varphi_n : \Omega \to \mathbb{R}$ defined by

$$\varphi_n(\omega) = \int_\Delta \left\| \sum_{i=1}^{n} \varepsilon_i f_i(\omega) \right\| d\nu(\varepsilon)$$

for $n = 1, 2, \ldots$ Since

$$\int_\Delta \left\| \sum_{i=1}^{n} \varepsilon_i f_i(\omega) \right\| d\nu(\varepsilon) \le \int_\Delta \left\| \sum_{i=1}^{n+1} \varepsilon_i f_i(\omega) \right\| d\nu(\varepsilon)$$

for every $n \in \mathbb{N}$ and $\omega \in \Omega$, then $\{\varphi_n\}$ is a monotone increasing sequence of nonnegative functions. Thus (3.3) and Fubini's theorem yield $\sup_{n\in\mathbb{N}} \int_\Omega \varphi_n(\omega)\, d\mu(\omega) \le C$. Hence, by the monotone convergence theorem there exists a μ-null set $A_2 \in \Sigma$ such that $\sup_{n\in\mathbb{N}} \varphi_n(\omega) < \infty$ for each $\omega \in \Omega \setminus A_2$. Considering the set $A := A_1 \cap (\Omega \setminus A_2)$, it is obvious that $\mu(A) > 0$, hence $A \neq \emptyset$. Moreover, $\overline{\lim}_{n\to\infty} \|f_n(\omega)\| > 0$ and

$$\sup_{n\in\mathbb{N}} \int_\Delta \left\| \sum_{i=1}^{n} \varepsilon_i f_i(\omega) \right\| d\nu(\varepsilon) < \infty$$

for each $\omega \in A$. Choosing $\omega_0 \in A$ and a strictly increasing sequence of positive integers $\{n_i\}$ such that $\inf_{i\in\mathbb{N}} \|f_{n_i}(\omega_0)\| > 0$, setting $y_i := f_{n_i}(\omega_0)$ for each $i \in \mathbb{N}$ and using the properties of the measure space we conclude that

$$\sup_{n\in\mathbb{N}} \int_0^1 \left\| \sum_{i=1}^{n} r_i(t) y_i \right\| dt = \sup_{n\in\mathbb{N}} \int_\Delta \left\| \sum_{i=1}^{n} \varepsilon_i y_i \right\| d\nu(\varepsilon) < \infty,$$

where $\{r_i\}$ is the Rademacher sequence on [0, 1]. Since X is a normed space, Bourgain averaging theorem [**1**] (see also [**2**, Lemma 2.1.2]) provides a subsequence of $\{y_n\}$ which is a basic sequence in X equivalent to the unit vector basis of c_0. □

Remark 3.1. The sequence $\{f_n\}$ with $f_n \in T_\mu F_n$ constructed in the first part of the proof of Theorem 3.1 is such that for any finite sequence of scalars $\{a_1, \ldots, a_n\}$ the function $\omega \mapsto \|a_1 f_1(\omega) + \cdots + a_n f_n(\omega)\|$ is μ-measurable and there are two absolute constant $\alpha, \beta > 0$ with

$$\alpha \sup_{1\le i\le n} |a_i| \le \int_\Omega \left\| \sum_{i=1}^{n} a_i f_i(\omega) \right\| d\mu(\omega) \le \beta \sup_{1\le i\le n} |a_i|.$$

From these facts one can deduce the existence of some $\omega_0 \in \Omega$ and of certain subsequence of $\{f_n(\omega_0)\}$ which is a basic sequence in X equivalent to the unit vector basis of c_0 just in the same way as Theorem 2 is deduced from Theorem 1 in [**1**]. This provides an alternative argument to the proof of our main theorem.

Corollary 3.2. ([**10**]) *The space $bvca(\Sigma, X^*)$ contains a copy of c_0 if and only if X^* does.*

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Proof. This is a straightforward consequence of Proposition 2.1 and Theorem 3.1. □

COROLLARY 3.3. ([7]) *If X has the Radon–Nikodym property with respect to each $\mu \in ca^{+}(\Sigma)$, then $bvca(\Sigma, X)$ contains a copy of c_0 if and only if X does.*

Proof. This is a straightforward consequence of Proposition 2.2 and Theorem 3.1. □

COROLLARY 3.4. ([8]) *Assume that B_{X^*} is weak* sequentially dense in $B_{X^{***}}$ and that $\Sigma = 2^{\Omega}$. If X is norm-one complemented in X^{**}, then $bvca(\Sigma, X)$ contains a copy of c_0 if and only if X does.*

Proof. This is a straightforward consequence of Proposition 2.3 and Theorem 3.1. □

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