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ON COHEN AND PRIKRY FORCING NOTIONS

TOM BENHAMOU AND MOTI GITIK

Abstract.
(1) We show that it is possible to add κ^+-Cohen subsets to κ with a Prikry forcing over κ. This answers a question from [9].
(2) A strengthening of non-Galvin property is introduced. It is shown to be consistent using a single measurable cardinal which improves a previous result by S. Garti, S. Shelah, and the first author [5].
(3) A situation with Extender-based Prikry forcings is examined. This relates to a question of H. Woodin.

§0. Introduction.

0.1. Intermediate models of the tree-Prikry forcing. In many mathematical theories, such as groups, vector spaces, topological spaces, and graphs, the study of submodels of a given model is indispensable to the understanding of the model and in some sense measures its complexity. In forcing theory, subforcings of a given forcing generate intermediate models to a generic extension by the forcing. Hence, the study of intermediate models is somehow parallel to the one regarding subforcings. There are numerous classification results in this spirit, for example, some forcing such as the Sacks forcing [34] and variants of the tree-Prikry forcing [25] do not have proper intermediate models. Other forcings such as the Cohen forcing [24], Random forcing [27], Prikry forcing [20], and Magidor forcing [6, 8] have intermediate models of the same type. A tree-Prikry forcing or its particular case, which will be central for us in this paper, the Prikry forcing with a non-normal ultrafilter can behave differently. For example, under suitable large cardinal assumptions, every κ-distributive forcing of cardinality κ is a projection of this forcing. Actually, more is true, under the assumption that κ is κ-compact there is a single Prikry-type forcing which absorbs all the κ-distributive forcings of cardinality κ (see [19]). In the absence of very large cardinals, the situation changes; indeed, Hayut and the authors [9] proved that if a certain $< \kappa$-strategically closed forcing of cardinality κ is a projection of the tree-Prikry forcing then it is consistent that there is a cardinal λ with high Mitchell order, namely $o(\lambda) > \lambda^+$. In [6], the authors proved that starting from a measurable cardinal (which is the minimal large cardinal assumption in the context of Prikry forcing) it is consistent that there is a (non-normal) ultrafilter U, such that the Prikry forcing with U projects onto the Cohen forcing $\text{Cohen}(\kappa, 1)$; this was improved later

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in [9] to a larger class of forcing notions called *Masterable forcings*. In the context of Prikry-type forcings, the existence of such embeddings and projections allows one to iterate distributive forcing notions on different cardinals (see [17, Section 6.4]).

It remained open whether it is possible to get more Cohen subsets of κ after forcing with the Prikry forcing with a κ-complete ultrafilter U over κ. This was asked explicitly in [9].

The basic difficulty is that the size of $\mathrm{Cohen}(\kappa,\kappa^+)$ is κ^+ and it is not hard to see (Proposition 2.9) that this cannot happen, if U has the Galvin property.

We formulate a certain strengthening of the negation of the Galvin property, show its consistency starting with a measurable cardinal, and finally apply it in order to construct an ultrafilter U such that the Prikry forcing (for a formal definition of the Prikry forcing with non-normal ultrafilter, see Definition 1.2) with it adds a generic subset to $\mathrm{Cohen}(\kappa,\kappa^+)$.

0.2. Extender-based Prikry forcing and a question of Woodin. Magidor and the second author developed the Extender-based Prikry forcing in [21] to violate the SCH under mild large cardinal assumptions. Later Merimovich [29, 30] presented a variation of this forcing which will be used in this paper.

H. Woodin asked[1] in the early 90s whether, assuming that there is no inner model with a strong cardinal, it is possible to have a model M in which $2^{\aleph_\omega} \geq \aleph_{\omega+3}$, GCH holds below $\aleph_\omega$, there is an inner model N such that $\kappa = (\aleph_\omega)^M$ is a measurable and $2^\kappa \geq (\aleph_{\omega+3})^M$.

A natural approach to tackle Woodin's question is to use the Extender-based Prikry with interleaved collapses forcing, defined by the second author and Magidor in [21]. This forcing collapses a measurable cardinal to $\aleph_\omega$ and simultaneously blows up the powerset of that measurable. Hence, if one can show that a generic extension by the Extender-based Prikry forcing has an intermediate model where κ stays measurable and 2^κ is large, this will provide a positive answer to Woodin's question. In this paper we show that this approach is doomed. More precisely, we address in general the question whether it is possible to add many subsets of κ $\langle x_\alpha \mid \alpha < \lambda\rangle$, $\lambda \geq \kappa^{++}$ with the Extender-based Prikry forcing over κ such that κ remains a regular cardinal in $V[\langle x_\alpha \mid \alpha < \lambda\rangle]$. We give a negative answer to this question with respect to the Extender-based Prikry forcing as defined in [21] and the Merimovich version of the forcing presented in [30, 31]. In particular, as a consequence of our results (Theorems 4.5 and 4.6), the Extender-based Prikry forcing cannot be used to answer Woodin's question.

0.3. The Galvin property. F. Galvin [2], in the 70s, showed that if $\kappa^{<\kappa} = \kappa$ and F is a normal filter over κ then the following combinatorial property holds:

$$\text{For every } \{X_i \mid i < \kappa^+\} \subseteq F \text{ there is } I \subseteq [\kappa^+]^\kappa \text{ such that } \cap_{i \in I} X_i \in F.$$

We denote this statement by $Gal(F,\kappa,\kappa^+)$. In particular, this holds for the club filter Cub_κ as it is a normal filter over a cardinal κ.

In [1], Abraham and Shelah constructed a model where $Gal(Cub_{\kappa^+},\kappa^+,\kappa^{++})$ fails for a regular κ. Garti [13, 14] and later together with the first author and

[1] We would like to thank Mohammad Golshani for reminding us of the exact formulation of Woodin's question.

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Poveda [4] continued the investigation of the Galvin property for the club filter. The Galvin property for κ-complete ultrafilters over a measurable cardinal κ was used recently in [7, 18]. The question of failure of the Galvin property for such ultrafilters was shown to be independent. Namely, in [7] the authors observed that in $L[U]$ every κ-complete ultrafilter has the Galvin property, and Garti, Shelah, and the first author, starting with a supercompact cardinal, produced a model with a κ-complete ultrafilter which contains Cub_κ and fails to satisfy the Galvin property.

In Section 2, we isolate a property of sequences we call a *strong witness for the failure of Galvin's property* which implies in particular the failure of Galvin's property. This property is used in Theorem 2.6, where we start from a single measurable cardinal, and construct a model with an ultrafilter which fails to satisfy the Galvin property. This improves the initial large cardinal assumption of [5].

Later in Theorem 2.10, we were able to slightly modify the construction of Theorem 2.6, construct an ultrafilter W and a strong witness for the failure of the Galvin property for it, which serves to glue together initial segments of functions, and obtain κ^+-mutually generic Cohen function on κ. This idea is generalized to longer sequences (and in turn to more Cohen functions) in Theorems 3.1 and 3.3.

Our main results are:

THEOREM 2.6. *Assume GCH and let κ be measurable in V. Then there is a cofinality preserving forcing extension V^* in which there is a κ-complete ultrafilter W over κ which concentrates on regulars, extends Cub_κ, and has a strong witness for the failure of Galvin's property.*

THEOREM 2.10. *Assume GCH and that κ is a measurable cardinal in V. Then there is a cofinality preserving forcing extension V^* in which GCH still holds, and there is a κ-complete ultrafilter $U^* \in V^*$ over κ such that forcing with Prikry forcing $Prikry(U^*)$ introduces a V^*-generic filter for $Cohen^{V^*}(\kappa, \kappa^+)$.*

THEOREM 3.1. *Assume GCH and that there is a (κ, κ^{++})-extender over κ in V. Then there is a cofinality preserving forcing extension V^* such that $V^* \models 2^\kappa = \kappa^{++}$, in V^* there is a κ-complete ultrafilter W over κ which concentrates on regulars, extends Cub_κ, and has a strong witness of length κ^{++} for the failure of Galvin's property.*

THEOREM 3.3. *Assume GCH and that E is a (κ, κ^{++})-extender in V. Then there is a cofinality preserving forcing extension V^* in which $2^\kappa = \kappa^{++}$ and a non-Galvin ultrafilter $W \in V^*$ such that forcing with* $\mathrm{Prikry}(W)$ *introduces a V^*-generic filter for $Cohen^{V^*}(\kappa, \kappa^{++})$-generic filter.*

THEOREM 4.5. *Let $\mathcal{P}_E$ be the Extender-based Prikry forcing of* [21], *and $G \subseteq \mathcal{P}$ be a generic. Suppose that $A \in V[G] \setminus V$ is a subset of κ. Then κ changes its cofinality to ω in $V[A]$.*

THEOREM 4.6. *Assume GCH, let E be an extender over κ, and let $\mathbb{P}_E$ be the Merimovich version of the Extender-based Prikry forcing of* [29–31]. *Let G be a generic subset of $\mathbb{P}_E$ and let $\langle A_\alpha \mid \alpha < \kappa^{++}\rangle$ be different subsets of κ in $V[G]$. Then there is $I \subseteq \kappa^{++}, I \in V, |I| = \kappa$ such that κ is a singular cardinal of cofinality ω in $V[\langle A_\alpha \mid \alpha \in I\rangle]$. In particular, there is no intermediate model of $V[G]$ where κ is measurable and $2^\kappa > \kappa^+$.*

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This paper is organized as follows:

- Section 1: We provide the basic definitions and background for this paper.
- Section 2: We prove Theorems 2.6 and 2.10.
- Section 3: We prove Theorems 3.1 and 3.3.
- Section 4: We prove Theorems 4.5 and 4.6.

§1. Basics.

1.1. The forcing notions. In our notations $p \leq q$ means that q is stronger than p. We assume that the reader is familiar with the forcing method and iterated forcing. Most of our notations are inspired by [12, 17] where we refer the reader for more information regarding forcing and iterations. Let us present the definitions of the forcing we intend to use:

Definition 1.1. The forcing adding λ-many Cohen functions to κ denoted by $\mathrm{Cohen}(\kappa, \lambda)$ consists of all partial functions $f : \kappa \times \lambda \to \{0, 1\}$ such that $|f| < \kappa$. The order is defined by $f \leq g$ iff $f \subseteq g$.

Definition 1.2. Let U be a κ-complete non-trivial ultrafilter over κ and let $\pi : \kappa \to \kappa$ be the function representing κ in the $\mathrm{Ult}(V, U)$. The Prikry forcing with U, denoted by $\mathrm{Prikry}(U)$, consists of all sequences $\langle \alpha_1, ..., \alpha_n, A\rangle$ such that:

(1) $\langle \alpha_1, ..., \alpha_n\rangle$ is a π-increasing sequence of ordinals below κ, i.e., for every $1 \leq i < n$, $\alpha_i < \pi(\alpha_{i+1})$,
(2) $A \in U$, $\pi(\min(A)) > \alpha_n$.

The order is defined by $\langle \alpha_1, ..., \alpha_n, A\rangle \leq \langle \beta_1, ..., \beta_m, B\rangle$ iff:

(1) $n \leq m$ and for every $i \leq n$, $\alpha_i = \beta_i$,
(2) for every $n < i \leq m$, $\beta_i \in A$,
(3) $B \subseteq A$.

If $n = m$ we say that q directly extends p and denote it by $p \leq^* q$.

If U is normal then we can take $\pi = id$ and the forcing $\mathrm{Prikry}(U)$ is the standard Prikry forcing. The requirement that the sequence is π-increasing ensures that the forcing $\mathrm{Prikry}(U)$ is forcing equivalent to the tree-Prikry forcing defined in [17]. Also, it enables to define a diagonal intersection suitable for the non-normal case, namely, for $\{A_i \mid i < \kappa\} \subseteq U$ define

$$\Delta^*_{i<\kappa} A_i := \{\alpha < \kappa \mid \forall i < \pi(\alpha). \alpha \in A_i\}.$$

This kind of diagonal intersection instead of the standard one is used to prove the Prikry property of $\mathrm{Prikry}(U)$.

Later we will need the easy direction of the Mathias criterion [28] for Prikry-generic sequences, and the proof can be found in [3, Corollary 4.22]:

Lemma 1.3. *Let $G \subseteq \mathrm{Prikry}(U)$ be a generic filter producing a Prikry sequence $\{c_n \mid n < \omega\}$. Then for every $A \in U$, there is $N < \omega$ such that for every $n \geq N$, $c_n \in A$.*

For more information regarding the tree-Prikry forcing see [17] or [3]. In the following, we define the notion of *lottery sum*. The terminology "lottery sum" is due to Hamkins, although the concept of the lottery sum of partial orderings has been

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around for quite some time and has been referred to, for example, as "disjoint sum of partial orderings":

Definition 1.4. Let $\mathbb{P}_0, \mathbb{P}_1$ be two forcing notions. The *lottery sum* of $\mathbb{P}_0$ and $\mathbb{P}_1$ denoted by $\mathrm{LOTT}(\mathbb{P}_0, \mathbb{P}_1)$ is the forcing whose underlining set is $\mathbb{P}_0 \times \{0\} \cup \mathbb{P}_1 \times \{1\}$ and the order is define by $\langle p, i\rangle \leq \langle p', j\rangle$ iff $i = j$ and $p \leq_{\mathbb{P}_i} p'$.

The forcing $\mathrm{LOTT}(\mathbb{P}_0, \mathbb{P}_1)$ generically chooses $\mathbb{P}_0$ or $\mathbb{P}_1$ and adds a V-generic filter for it. As Hamkins observed in [22], iterating such forcing notions leaves a certain amount of freedom when lifting ground model embeddings; this will be exploited in most of our construction.

In Section 4 we will discuss the Extender-based Prikry forcing which was originally defined by Magidor and the second author in [21]. A more recent variation of it is due to Merimovich [29–31].

Let us present the two versions. Let E be a (κ, λ)-extender and $j = j_E : V \to M_E \simeq Ult(V, E)$ the natural elementary embedding (see [23] for the definition of extenders and related constructions) and suppose that $f_\lambda : \kappa \to \kappa$ is a function such that $j(f_\lambda)(\kappa) = \lambda$ (our result uses $\lambda = \kappa^{++}$ and we can simply take $f_\lambda(\nu) = \nu^{++}$). Let us first present the *Merimovich version* of the Extender-based Prikry forcing.

For each set of cardinality $\leq \kappa$, $d \in [\lambda \setminus \kappa]^{\leq\kappa}$ with $\kappa \in d$. Define

$$E(d) = \{X \in V_\kappa \mid (j \restriction d)^{-1} \in j(X)\}.$$

If $A \in E(d)$ we can assume that for every $\nu, \mu \in A$, $\nu : d \to \kappa$ is order preserving, $\kappa \in \mathrm{dom}(\nu)$, $|\nu| \leq \nu(\kappa)$, $\nu(\kappa) = \mu(\kappa) \to \mathrm{dom}(\nu) = \mathrm{dom}(\mu)$. Merimovich calls such a set a *good set*.

Definition 1.5. The *conditions of* $\mathbb{P}_E$ are pairs $p = \langle f^p, A^p\rangle$ such that:

(1) $f^p : d \to [\kappa]^{<\omega}$ is the "Cohen Part" of the condition, $d \in [\lambda \setminus \kappa]^{<\omega}$, $\kappa \in d$.
(2) $A^p \in E(d)$ is a good set.
(3) For every $\nu \in A^p$ and $\alpha \in \mathrm{dom}(\nu)$, $\max(f^p(\alpha)) < \nu(\kappa)$.

The *order of* $\mathbb{P}_E$ is defined in two steps: a direct extension is defined by $\langle f, A\rangle \leq^* \langle g, B\rangle$ if:

(1) $f \subseteq g$.
(2) $B \restriction \mathrm{dom}(f) := \{\nu \restriction \mathrm{dom}(\nu) \cap \mathrm{dom}(f) \mid \nu \in B\} \subseteq A$.

A one-point extension of $p = \langle f, A\rangle$ for $\nu \in A$ is defined by $p^\frown \nu = \langle g, B\rangle$ where:

(1) $\mathrm{dom}(g) = \mathrm{dom}(f)$.
(2) For every $\alpha \in \mathrm{dom}(g)$,

$$g(\alpha) = \begin{cases} f(\alpha)^\frown \nu(\alpha), & \alpha \in \mathrm{dom}(\nu), \\ f(\alpha), & \text{else.} \end{cases}$$

(3) $B = \{\mu \in A \mid \sup_{\alpha\in\mathrm{dom}(\nu)}(\nu(\alpha) + 1) \leq \mu(\kappa)\}$.

An n-point extension $p^\frown \vec{\nu}$ is defined recursively by consecutive one-point extensions. A general extension is defined by $p \leq q$ iff for some $\vec{\nu} \in [A^p]^{<\omega}$, $p^\frown \vec{\nu} \leq^* q$.

As in [30], we will sometime replace the large set A in a condition $\langle f, A\rangle$ with a Tree T which is $E(\mathrm{dom}(f))$-fat.

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Let us now present the *original version* defined by Magidor and the second author from [21]. Define for every $\kappa \le \alpha < \lambda$:

$$U_\alpha := \{X \subseteq \kappa \mid \alpha \in j(X)\}.$$

These are P-point ultrafilters. For every $\alpha \le \beta < \lambda$ we define that $\alpha \le_E \beta$ if there is some $f : \kappa \to \kappa,\ j(f)(\beta) = \alpha$. This implies that f Rudin–Keisler projects U_β onto U_α. For every such pair $\alpha \le_E \beta$ fix such a projection $\pi_{\beta,\alpha}$ such that $\pi_{\alpha,\alpha} = id$. The projections to the normal measure U_κ have a uniform definition, $\pi_{\alpha,\kappa}(\nu) = \nu^0$ where ν^0 is the maximal inaccessible $\nu^* \le \nu$ such that $f_\lambda \restriction \nu^* : \nu^* \to \nu^*$, $f_\lambda(\nu^*) > \nu$, and $\pi_{\alpha,\kappa}(\nu) = 0$ if there is no such ν^*. Suppose that the system $\langle U_\alpha, \pi_{\alpha,\beta} \mid \alpha \le \beta < \lambda, \alpha \le_E \beta\rangle$ is a *nice system* (see [21] or [17, Discussion after Lemma 3.5]). Let us say that ν is *permitted* for $\nu_0, ..., \nu_n$ is $\nu^0 > \max_{i=0,...,n} \nu_i^0$.

DEFINITION 1.6. *The conditions* of the forcing $\mathcal{P}_E$ are pairs $p = \langle f, T\rangle$ such that:

(1) $f : \lambda \setminus \kappa \to [\kappa]^{<\omega}$, $\kappa \in \mathrm{dom}(f)$, $|f| \le \kappa$.
(2) For each $\alpha \in \mathrm{Supp}(p) := \mathrm{dom}(f)$, $\pi''_{\alpha,\kappa} f(\alpha)$ is a finite increasing sequence.
(3) The domain of f has a $\le_E$-maximal element $mc(p) := \alpha = \max(\mathrm{Supp}(p))$.
(4) $\pi''_{mc(p),\kappa} f(mc(p)) = f(\kappa)$.
(5) For every $\gamma \in \mathrm{Supp}(p)$, $\pi_{mc(p),\gamma}(\max(f(mc(p)))$ is not permitted to $f(\gamma)$.
(6) T is a $U_{mc(p)}$-splitting tree with stem $f(mc(p))$, namely, for $s \in T$, either $s \le t$, or $s \ge t$ and $\mathrm{Succ}_T(s) := \{\alpha < \kappa \mid s^\frown\alpha \in T\} \in U_{mc(p)}$.
(7) For every $\nu \in \mathrm{Succ}_T(f(mc(p)))$,

$$|\{\gamma \in \mathrm{Supp}(p) \mid \nu \text{ is permitted to } f(\gamma)\}| \le \nu^0.$$

The order is defined $p \le q$ if:

(1) $\mathrm{Supp}(p) \subseteq \mathrm{Supp}(q)$.
(2) For $\gamma \in \mathrm{Supp}(p)$, $f^q(\gamma)$ is an end-extension of $f^p(\gamma)$.
(3) $f^q(mc(p)) \in T^p$.
(4) For $\gamma \in \mathrm{Supp}(p)$, $f^q(\gamma) \setminus f^p(\gamma) = \pi''_{mc(p),\gamma} f^q(mc(p)) \setminus f^p(mc(p)) \restriction (i+1)$, where i is maximal such that $f^q(mc(p))$ is not permitted for $f^p(\gamma)$.
(5) $\pi''_{mc(q),mc(p)} T^q \subseteq T^p$.
(6) For every $\gamma \in \mathrm{Supp}(p)$, and $\nu \in \mathrm{Succ}_{T^q}(f^q(mc(q)))$, such that ν is permitted for $f^q(\gamma)$ (so by condition (7) there are only ν^0-many such γ's) then $\pi_{mc(q),\gamma}(\nu) = \pi_{mc(p),\gamma}(\pi_{mc(q),mc(p)}(\nu))$.

1.2. Canonical functions. The main construction of this paper uses the notion of canonical functions:

DEFINITION 1.7. For every limit ordinal $\delta < \kappa^+$, fix a cofinal sequence $\bar{\delta} = \{\delta_i \mid i < cf(\delta)\}$. Let us define inductively functions $\tau_\alpha : \kappa \to \kappa$ for $\alpha < \kappa^+$:

$$\tau_0(x) = 0,$$

$$\tau_{\alpha+1}(x) = \tau_\alpha(x) + 1,$$

$$\text{For limit } \delta,\ \tau_\delta(x) = \sup_{y < \min(x, cf(\delta))} \tau_{\delta_y}(x).$$

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Proposition 1.8. *Let $\lambda \leq \kappa$ be a regular cardinal. Then*:
(1) *For every $\alpha < \beta < \lambda^+$, $\{\nu \mid \tau_\alpha(\nu) \geq \tau_\beta(\nu)\}$ is bounded in λ.*
(2) *For every any $\alpha < \lambda^+$, $\tau_\alpha : \lambda \to \lambda$.*
(3) *For every normal measure $\mathcal{V}$ on λ, and for every $\alpha < \lambda^+$, $[\tau_\alpha]_\mathcal{V} = \alpha$.*
(4) *If $\lambda < \kappa$, then for every β, $\tau_\beta(\lambda) < \lambda^+$.*

Proof. For (1), we prove inductively on $\beta < \lambda^+$ that for every $\alpha < \beta$, (1) holds. For $\beta = 0$ this is vacuous. The successor stage is also easy since for every x, $\tau_\beta(x) < \tau_{\beta+1}(x)$ so if $\alpha < \beta$ then by induction hypothesis there is $\xi < \lambda$ from which τ_β dominates τ_α, i.e., $\forall \nu \in (\xi, \lambda).\tau_\alpha(\nu) < \tau_\beta(\nu)$. It follows that for the same ξ, $\tau_\alpha(\nu) < \tau_{\beta+1}(\nu)$. As for limit points δ. Fix any $\alpha < \delta$, then there is $i < cf(\delta) \leq \lambda$ such that $\delta_i > \alpha$. By induction hypothesis there is $\xi_i < \lambda$ such that $\tau_{\delta_i}(\nu) > \tau_\alpha(\nu)$ for every $\nu \in (\xi_i, \lambda)$. Let $\xi^* := \max\{\xi_i, i\} + 1 < \lambda$. It follows that for every $\nu \in (\xi^*, \lambda)$, $\nu > i$, and hence

$$\tau_\delta(\nu) = \sup_{y < \min(\nu, cf(\delta))} \tau_{\delta_y}(\nu) \geq \tau_{\delta_i}(\nu) > \tau_\alpha(\nu).$$

Prove (2)–(4) by induction on $\alpha < \lambda^+$. For $\alpha = 0$ this is trivial. Suppose that (2)–(4) hold for α then clearly by induction hypothesis $\tau_{\alpha+1} : \lambda \to \lambda$, and $\tau_{\alpha+1}(\lambda) = \tau_\alpha(\lambda) + 1 < \lambda^+$, namely (2) and (4) follow. Also, $\lambda = \{\nu < \lambda \mid \tau_\alpha(\nu) + 1 = \tau_{\alpha+1}(\nu)\} \in \mathcal{V}$, ane hence by the Lós theorem and the induction hypothesis:

$$\alpha + 1 = [\tau_\alpha]_\mathcal{V} + 1 = [\tau_{\alpha+1}]_\mathcal{V}.$$

Suppose that $\delta < \lambda^+$ is limit, then by induction hypothesis, for every $x < \lambda$ and $y < \min(x, cf(\delta)) < \lambda$, $\tau_{\delta_y}(x) < \lambda$. It follows from the regularity of λ that

$$\tau_\delta(x) = \sup_{y < \min(x, cf(\delta))} \tau_{\delta_y}(x) < \lambda.$$

This concludes (2). Also, (4) follows similarly using the regularity of λ^+. As for (3), we use (1) to conclude that for every $\alpha < \delta$, $\{\nu < \lambda \mid \tau_\alpha(\nu) \geq \tau_\delta(\nu)\}$ is bounded. Hence by induction $\alpha = [\tau_\alpha]_\mathcal{V} < [\tau_\delta]_\mathcal{V}$. It follows that $\delta \leq [\tau_\delta]_\mathcal{V}$. For the other direction, suppose that $[f]_\mathcal{V} < [\tau_\delta]_\mathcal{V}$, then

$$E := \{x < \lambda \mid f(x) < \tau_\delta(x)\} \in \mathcal{V}.$$

By definition of τ_δ, for every $x \in E$, there is $y_x < \min(x, cf(\delta))$ such that $\tau_{\delta_{t_x}}(x) > f(x)$. The function $x \mapsto y_x$ is regressive, and by normality we conclude that there is $y^* < cf(\delta)$ and $E' \subseteq E$ such that for every $x \in E'$, $f(x) < \tau_{\delta_{y^*}}(x)$. Hence $[f]_\mathcal{V} < [\tau_{\delta_{y^*}}]_\mathcal{V} = \delta_{y^*} < \delta$ and in turn $\delta = [\tau_\delta]_\mathcal{V}$. ⊣

§2. The results where GCH holds.

2.1. Non-Galvin ultrafilter from optimal assumption. In [5], Garti, Shelah, and the first author constructed a model with a κ-complete ultrafilter which contains Cub_κ and fails to satisfy the Galvin property. The initial assumption was a supercompact cardinal and the construction went through adding slim Kurepa trees.

Here we present a different construction. Our initial assumption will be a measurable cardinal and the property obtained will be a certain strengthening of the negation of the Galvin property. It will be used further to produce many Cohens.

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Let us first present the stronger form of negation:

DEFINITION 2.1. Let U be a κ-complete ultrafilter non-normal over κ. We call a family $\{A_\alpha \mid \alpha < \kappa^+\} \subseteq U$ a *strong witness* for the failure of the Galvin property iff for every subfamily $\langle A_{\alpha_\xi} \mid \xi < \kappa\rangle$ of size κ the following holds:

$$\text{for every } \zeta, \kappa \leq \zeta < [id]_U,\ [id]_U \notin A'_{\alpha_\zeta},$$

$$\text{where } \langle A'_{\alpha_\zeta} \mid \zeta < j_U(\kappa)\rangle = j_U(\langle A_{\alpha_\xi} \mid \xi < \kappa\rangle).$$

REMARK 2.2. (1) Note that the interval $[\kappa, [id]_U)$ is non-empty since U is not normal.
(2) The family $\{A_\alpha \mid \alpha < \kappa^+\}$ witnesses the failure of the Galvin property for U.

PROOF. Since whenever $\langle A_{\alpha_\xi} \mid \xi < \kappa\rangle$ is a subfamily of size κ, then $\bigcap_{\xi<\kappa} A_{\alpha_\xi}$ is not in U. Otherwise, suppose that $\bigcap_{\xi<\kappa} A_{\alpha_\xi} = B \in U$. Then $[id]_U \in j_U(B)$, but $j_U(B) = \bigcap_{\zeta<j_U(\kappa)} A'_{\alpha_\zeta}$. However, $[id]_U \notin A'_{\alpha_\zeta}$, for every $\zeta, \kappa \leq \zeta < [id]_U$. Contradiction. ⊣

LEMMA 2.3. *Suppose that $\{A_\alpha \mid \alpha < \kappa^+\}$ is a strong witness for the failure of the Galvin property of the ultrafilter U over κ. Let $U^0 = \{X \subseteq \kappa \mid \kappa \in j_U(X)\}$ be a projection of U to a normal ultrafilter, $\nu \mapsto \pi_{nor}(\nu)$ a projection map, and $k : M_{U^0} \to M_U$ the corresponding elementary embedding. Assume that $crit(k) = j_{U^0}(\kappa) = [id]_U$. Then $[id]_U \notin B$, for every $B \in j_U(\{A_\alpha \mid \alpha < \kappa^+\})$ which is in* $\operatorname{rng}(k) \setminus \operatorname{rng}(j_U)$.

PROOF. Let B be as in the statement of the lemma. Pick $A' \subseteq j_{U^0}(\kappa)$ such that $k(A') = B$. Then $A' \notin \operatorname{rng}(j_{U^0})$, since otherwise its image B will be in the range of $j_U = k \circ j_{U^0}$. Denote by

$$\{A'_\nu \mid \nu < j_{U^0}(\kappa^+)\} = j_{U^0}(\{A_i \mid i < \kappa^+\}),$$

$$\{A''_\nu \mid \nu < j_U(\kappa^+)\} = j_U(\{A_i \mid i < \kappa^+\}).$$

Since U^0 is normal, there is $f : \kappa \to \kappa^+$ such that $A' = A'_{j_{U^0}(f)(\kappa)}$ and thus

$$B = k(A') = k(A'_{j_{U^0}(f)(\kappa)}) = A''_{j_U(f)(\kappa)}.$$

Since B is not in the range of k, f is not constant. Recall that $\{A_\alpha \mid \alpha < \kappa^+\}$ is a strong witness for U being non-Galvin ultrafilter over κ. Apply this to the family $\{A_{f(\nu)} \mid \nu < \kappa\}$. It follows that $[id]_U \notin A''_{j_U(f)(\kappa)} = B$. ⊣

Before proving the main result of this section we present two preservation theorems for being a strong witnesses for the failure of the Galvin property. These theorems are not used later and the reader can proceed directly to Theorem 2.6.

THEOREM 2.4. *Assume $2^\kappa = \kappa^+$. Suppose that the family $\{A_\alpha \mid \alpha < \kappa^+\}$ is a strong witness for U being a non-Galvin ultrafilter over κ. Let $U^0 = \{X \subseteq \kappa \mid \kappa \in j_U(X)\}$ be a projection of U to a normal ultrafilter, $\nu \mapsto \pi_{nor}(\nu)$ a projection map, and $k : M_{U^0} \to M_U$ the corresponding elementary embedding. Assume that $crit(k) = j_{U^0}(\kappa)$ and $[id]_U = j_{U^0}(\kappa)$.*

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Suppose that V^* is an extension of V in which all the embeddings j_{U^0}, j_U, k extend to an elementary embedding $j^{0*} : V^* \to M^{0*}$, $j^* : V^* \to M^*$, $k^* : M^{0*} \to M^*$. Define $U^* = \{X \subseteq \kappa \mid [id]_U \in j^*(X)\}$.

Then $\{A_\alpha \mid \alpha < \kappa^+\}$ is a strong witness that U^* is a non-Galvin ultrafilter over κ.

PROOF. Note that $(\kappa^+)^{V^*} = (\kappa^+)^V$. Just otherwise, $(\kappa^{++})^V$ will be $\leq (\kappa^+)^{V^*}$, and then, $j^*(\kappa) > (\kappa^{++})^V$. This is impossible, since j^* extends j_U. The rest follows from the previous lemma and the fact that $[\kappa, [id]_U) \subseteq \mathrm{rng}(k) \setminus \mathrm{rng}(j_U)$ since $crit(k) = j_{U^0}(\kappa) = [id]_U$. ⊣

THEOREM 2.5. *Assume $2^\kappa = \kappa^+$. Suppose that $\{A_\alpha \mid \alpha < \kappa^+\}$ is a strong witness for U being a non-Galvin ultrafilter over κ which contains Cub_κ and be a witnessing family.*

Let V^* be a κ-c.c. extension of V in which j_U extends to an elementary embedding $j^* : V^* \to M^*$, where M^* is a corresponding extension of M_U.

Define $U^* = \{X \subseteq \kappa \mid [id]_U \in j^*(X)\}$.

Then $\{A_\alpha \mid \alpha < \kappa^+\}$ is a strong witness that U^* is a non-Galvin ultrafilter over κ.

PROOF. Suppose now that $\langle A_{\alpha_\xi} \mid \xi < \kappa\rangle$ is a subfamily of $\{A_\alpha \mid \alpha < \kappa^+\}$ of size κ in V^*.

Work in V. Let $\underset{\sim}{\alpha}_\xi$ be a name of α_ξ. By κ-c.c., then for every $\xi < \kappa$ there will be $s_\xi \subseteq \kappa^+$ of cardinality less than κ, such that $\Vdash \underset{\sim}{\alpha}_\xi \in s_\xi$.

Let $S = \sup_{\xi<\kappa} s_\xi$. Enumerate $S = \langle \beta_i \mid i < \kappa\rangle$ such that we if $\beta_i \in s_\zeta$ and $\beta_j \in s_\mu$ where $\zeta < \mu$ then $i < j$, i.e., enumerate first s_0 then s_1 and so on, such that the resulting enumeration of S is of order-type κ. This is possible since each s_ζ has cardinality less than κ. Define

$$C = \{\nu < \kappa \mid \forall \xi < \nu(\sup(\gamma \mid \beta_\gamma \in s_\xi) < \nu)\}.$$

Clearly, C is a club. Hence $[id]_U \in j_U(C)$. Then, by elementarity, for every $\zeta < [id]_U$, and every $\beta_i \in s'_\zeta$, $i < [id]_U$.

Let us use the fact that the sequence $\langle A_\alpha \mid \alpha < \kappa^+\rangle$ is a strong witness for U being non-Galvin, hence $[id]_U \notin A'_{\beta_\zeta}$, for every $\kappa \leq \zeta < [id]_U$. Fix any $\kappa \leq \xi < [id]_U$, then by elementarity we have $\Vdash \underset{\sim}{\alpha}'_\xi \in s'_\xi$ in M_U. Therefore there is some $\gamma < \kappa$ such that $\alpha'_\xi = \beta_\gamma$. Clearly, $\gamma \geq \kappa$, and by the closure property of $[id]_U$, we conclude that $\gamma < [id]_U$. Hence, in M^*, $[id]_U \notin A'_{\beta'_\gamma} = A'_{\alpha'_\xi}$, as wanted. ⊣

THEOREM 2.6. *Assume GCH and let κ be measurable in V. Then there is a cofinality preserving forcing extension V^* in which there is a κ-complete ultrafilter W over κ which concentrates on regulars, extends Cub_κ, and has a strong witness for the failure of Galvin's property.*

PROOF. The forcing is simply adding for each inaccessible $\alpha \leq \kappa$, α^+-many Cohen functions to α. Namely, consider the Easton support iteration

$$\langle \mathcal{P}_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa+1, \beta \leq \kappa\rangle$$

such that for $\alpha \leq \kappa$, $\underset{\sim}{Q}_\alpha$ is trivial unless α is inaccessible, in which case it is a $\mathcal{P}_\alpha$-name for $\mathrm{Cohen}(\alpha, \alpha^+)$.

Let $G := G_\kappa * g_\kappa$ be V-generic for $\mathcal{P}_\kappa * \underset{\sim}{Q}_\kappa$. Denote $\langle f_{\kappa,\alpha} \mid \alpha < \kappa^+\rangle$ be the enumeration of the κ^+ Cohen functions added by g_κ. The idea is that the sets which are going to be a strong witness for the failure of the Galvin property are $\langle A_\alpha \mid \alpha < \kappa^+\rangle$, where

$$A_\alpha = \{\beta < \kappa \mid f_{\kappa,\alpha}(\beta) = 1\}.$$

The next step is to construct the measure for this witness by extending ground model embeddings to $V[G]$. Let $U \in V$ be a normal measure over κ and consider the second ultrapower by U and the corresponding commutative diagram

$$j_1 := j_U : V \to M_U =: M_1,\ j_2 := j_{U^2} : V \to M_{U^2} =: M_2$$

$$k : M_1 \to M_2,\ j_2 = k \circ j_1,$$

where k is simply the ultrapower embedding defined in M_U using the ultrafilter $j_1(U)$. Denote $\kappa_1 = j_1(\kappa)$ and $\kappa_2 = j_2(\kappa)$, then $k(\kappa_1) = \kappa_2$.

By Easton support and elementarity,

$$j_1(\mathcal{P}_\kappa * \underset{\sim}{Q}_\kappa) = \mathcal{P}_\kappa * \underset{\sim}{Q}_\kappa * \mathcal{P}_{(\kappa,\kappa_1)} * \underset{\sim}{Q}_{\kappa_1},$$

where $\mathcal{P}_{(\kappa,\kappa_1)} * \underset{\sim}{Q}_{\kappa_1}$ is the quotient forcing above κ, which is forcing equivalent to the continuation of the iteration above κ using the same recipe as $\mathcal{P}_\kappa$.

In $V[G]$, let us first construct an M-generic filter for $j_1(\mathcal{P}_\kappa * \underset{\sim}{Q}_\kappa)$. Take $G_\kappa * g_\kappa$ to be the generic up to κ including κ. Above κ, from the point of view of $V[G]$, we have κ^+-closure for $\mathcal{P}_{(\kappa,\kappa_1)}$. By GCH, and since j_1 is an ultrapower by a measure, there are only κ^+-many dense open subsets of this forcing to meet. Therefore we can construct in $V[G]$ by standard construction an $M_1[G]$-generic filter $G_{(\kappa,\kappa_1)}$ for $\mathcal{P}_{(\kappa,\kappa_1)}$. By $\kappa_1^+ - cc$ of Q_{κ_1}, we can find g'_{κ_1} which is $M_1[G * G_{(\kappa,\kappa_1)}]$-generic for Q_{κ_1}. We need to change the values of $g'_{\kappa_1} = \langle f'_{\kappa_1,\alpha} \mid \alpha < \kappa_1^+\rangle$ to $g_{\kappa_1} = \langle f_{\kappa_1,\alpha} \mid \alpha < \kappa_1^+\rangle$ such that for every $\alpha < \kappa^+$, $f_{\kappa_1,j_1(\alpha)} \restriction \kappa = f_{\kappa,\alpha}$. This will ensure that the Silver criterion to lift an elementary embedding holds, namely, $j_1'' G_\kappa * g \subseteq G_\kappa * g * G_{(\kappa,\kappa_1)} * g'_{\kappa_1}$. Also, we would like to tweak the values of $f_{\kappa_1,j_1(\alpha)}(\kappa)$ to ensure that the sets A_α are members of the ultrafilter generated by κ. By the definition of A_α, the way to do this is to set $f_{\kappa_1,j_1(\alpha)}(\kappa) = 1$.

Formally, for each condition $p \in \mathrm{Cohen}(\kappa_1, \kappa_1^+)^{M_1[G_\kappa * G * G_{(\kappa,\kappa_1)}]}$, define a function p^* with $\mathrm{dom}(p^*) = \mathrm{dom}(p)$ and for every $\langle\gamma, \alpha\rangle \in \mathrm{dom}(p^*)$,

$$p^*(\langle\gamma,\alpha\rangle) = \begin{cases} f_{\kappa,\beta}(\gamma), & \gamma < \kappa \wedge j_1(\beta) = \alpha, \\ 1, & \gamma = \kappa \wedge j_1(\beta) = \alpha, \\ p(\langle\gamma,\alpha\rangle), & \text{else.} \end{cases}$$

Let $g_{\kappa_1} := \{p^* \mid p \in g'_{\kappa_1}\}$. Clearly, the functions $\langle f_{\kappa_1,\alpha} \mid \alpha < \kappa_1^+\rangle$ derived from g_{κ_1} satisfy that $f_{\kappa_1,j_1(\beta)} \restriction \kappa = f_{\kappa,\beta}$ and $f_{\kappa_1,j_1(\beta)}(\kappa) = 1$ for every $\beta < \kappa^+$. It remains to show that g_{κ_1} is generic:

Lemma 2.7. *The filter* g_{κ_1} *is* $\mathrm{Cohen}(\kappa_1, \kappa_1^+)^{M_1[G_\kappa * G * G_{(\kappa,\kappa_1)}]}$*-generic filter over* $M_1[G_\kappa * g * G_{(\kappa,\kappa_1)}]$.

Proof. First let us prove that $g_{\kappa_1} \subseteq \mathrm{Cohen}(\kappa_1, \kappa_1^+)^{M_1[G_\kappa * G * G_{(\kappa,\kappa_1)}]}$. Indeed, $g'_{\kappa_1} \subseteq \mathrm{Cohen}(\kappa_1, \kappa_1^+)^{M_1[G_\kappa * G * G_{(\kappa,\kappa_1)}]}$ and for any $p \in g'_{\kappa_1}$,

$$M_1[G_\kappa * G * G_{(\kappa,\kappa_1)}] \models |p| < \kappa_1,$$

and hence $\mathrm{dom}(p)_{\leq\kappa} := \{\alpha \mid \exists\langle\gamma,\alpha\rangle \in \mathrm{dom}(p), \gamma \leq \kappa\}$ is bounded in κ_1^+ while $j_1''\kappa^+$ is unbounded. It follows that there is $\theta < \kappa^+$ such that

$$\mathrm{dom}(p)_{\leq\kappa} \cap j_1''\kappa^+ \subseteq j_1''\theta.$$

Hence from the V-perspective, $|\mathrm{dom}(p)_{\leq\kappa} \cap j_1''\kappa^+| \leq \kappa$. The difference between p and p^* is only on the coordinates of $\mathrm{dom}(p)_{\leq\kappa} \cap j_1''\kappa^+$ and by closure of $M_1[G_\kappa * g * G_{(\kappa,\kappa_1)}]$ to κ-sequences it follows that

$$p^* \in \mathrm{Cohen}(\kappa_1,\kappa_1^+)^{M_1[G_\kappa * G * G_{(\kappa,\kappa_1)}]},\ g_{\kappa_1} \subseteq \mathrm{Cohen}(\kappa_1,\kappa_1^+)^{M_1[G_\kappa * G * G_{(\kappa,\kappa_1)}]}.$$

To see that g_{κ_1} is generic over $M_1[G_\kappa * G * G_{(\kappa,\kappa_1)}]$, let $D \in M_1[G_\kappa * G * G_{(\kappa,\kappa_1)}]$ be dense open. In $M_1[G_\kappa * G * G_{(\kappa,\kappa_1)}]$, define D^* to consist of all conditions $p \in \mathrm{Cohen}(\kappa_1,\kappa_1^+)$. Such that

$$\forall q.\, \mathrm{dom}(q) = \mathrm{dom}(p) \wedge |\{x \mid p(x) \neq q(x)\}| \leq \kappa \rightarrow q \in D$$

then D^* is dense open. To see this, pick any $p \in \mathrm{Cohen}(\kappa_1,\kappa_1^+)^{M_1[G_\kappa * G * G_{(\kappa,\kappa_1)}]}$ and enumerate by $\langle q_r \mid r < \theta\rangle$ all the conditions q such that

$$\mathrm{dom}(q) = \mathrm{dom}(p) \wedge |\{x \mid p(x) \neq q(x)\}| \leq \kappa.$$

Note $\theta < \kappa_1$ since κ_1 is inaccessible in $M_U[G_\kappa * g * G_{(\kappa,\kappa_1)}]$. We define inductively and increasing sequence $\langle p_r \mid r < \theta\rangle$, and exploit the κ_1-closure of $\mathrm{Cohen}(\kappa_1,\kappa_1^+)$ to take care of limit stages. Define $p_0 = p$, and suppose that p_r is defined, let $p'_{r+1} := q_r \cup p_r \restriction (\mathrm{dom}(p_r) \setminus \mathrm{dom}(p))$, find $p'_{r+1} \leq t_{r+1} \in D$ which exists by density and set

$$p_{r+1} = p_r \restriction \mathrm{dom}(p) \cup t_{r+1} \restriction (\mathrm{dom}(t_{r+1}) \setminus \mathrm{dom}(p)).$$

Then $p_r \leq p_{r+1}$. Let

$$p^* := \cup_{r<\theta} p_r$$

then p^* has the property that for κ many changes of p^* from the domain of p stays inside D. Namely any q with $\mathrm{dom}(q) = \mathrm{dom}(p^*)$,

$$q \restriction (\mathrm{dom}(p^*) \setminus \mathrm{dom}(p)) = p^* \restriction (\mathrm{dom}(p^*) \setminus \mathrm{dom}(p))$$

and $|\{x \in \mathrm{dom}(p) \mid p(x) \neq q(x)\}| \leq \kappa$, $q \restriction \mathrm{dom}(p) = q_r$ for some r, therefore $q \geq t_{r+1} \in D$. Now we define inductively $\langle p^{(r)} \mid r < \kappa^+\rangle$, $p^{(0)} = p$ at limit we take union, and at successor step we take $p^{(r+1)} = (p^{(r)})^*$. We claim that $p_* := \cup_{r<\kappa^+} p^{(r)} \in D^*$. First note that $\kappa^+ < \kappa_1$, hence $|p_*| < \kappa_1$ (all the definition is inside $M_U[G_\kappa * g_\kappa * G_{(\kappa,\kappa_1)}]$). Let q be any condition with $\mathrm{dom}(q) = \mathrm{dom}(p^*)$ and denote by

$$I = \{x \in \mathrm{dom}(p_*) \mid q(x) \neq p_*(x)\}$$

and suppose that $|I| \leq \kappa$. Since $\mathrm{dom}(p_*) = \cup_{r<\kappa^+} \mathrm{dom}(p^{(r)})$ and $\mathrm{dom}(p^{(r)})$ is $\subseteq$-increasing, there is $j < \kappa^+$ such that $I \subseteq \mathrm{dom}(p^{(j)})$. The condition $q \restriction I$ is enumerated in the construction of $p^{(j+1)}$, hence $q \restriction \mathrm{dom}(p^{(j+1)}) \in D$ and since D is open, $q \in D$. This means that $p_* \in D^*$.

Finally, by genericity of g'_{κ_1}, we can find $p \in D^* \cap g'_{\kappa_1}$. By definition, $p^* \in g_{\kappa_1}$ and since $\mathrm{dom}(p^*) = \mathrm{dom}(p)$ and $|\{x \mid p(x) \neq p^*(x)\}| \leq \kappa$ it follows that $p^* \in D$. $\dashv$

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Denote by $H = G_\kappa * g_\kappa * G_{(\kappa,\kappa_1)} * g_{\kappa_1}$, then $j_1''G \subseteq H$. Let

$$j_1^* : V[G] \to M_1[H]$$

be the extended ultrapower and derive the normal ultrafilter over κ,

$$U_1 := \{X \subseteq \kappa \mid \kappa \in j_1^*(X)\}$$

then $U \subseteq U_1$ and $j_1^* = j_{U_1}$. Indeed let $k_1 : M_{U_1} \to M_1[H]$ be the usual factor map $k_1(j_{U_1}(f)(\kappa)) = j_1^*(f)(\kappa)$. We will prove that k_1 is onto and therefore $k_1 = id$. For every $A \in M_1[H]$, there is a name $\underset{\sim}{A} \in M_1$ such that $A = (\underset{\sim}{A})_H$. M_U is the ultrapower by U, hence there is $f \in V$ such that $j_1(f)(\kappa) = \underset{\sim}{A}$. By elementarity for every $\alpha < \kappa$, $f(\alpha)$ is a name. In $V[G]$ define $f^*(\alpha) = (f(\alpha))_G$, then by elementarity

$$k_1(j_{U_1}(f)(\kappa)) = j_1^*(f^*)(\kappa) = (j_1^*(f)(\kappa))_{j(G)} = (j_1(f)(\kappa))_H = (\underset{\sim}{A})_H = A.$$

Denote by $M_1^* = M_1[H]$ and consider $j_1^*(U_1) \in M_1^*$. Let us now define inside M_1^* an M_2-generic filter for

$$j_2(\mathcal{P}_\kappa * \underset{\sim}{Q}_\kappa) = \mathcal{P}_{\kappa_1} * \underset{\sim}{Q}_{\kappa_1} * \mathcal{P}_{(\kappa_1,\kappa_2)} * \underset{\sim}{Q}_{\kappa_2},$$

in a similar fashion as H was defined. First we take H to be the generic for $\mathcal{P}_{\kappa_1} * \underset{\sim}{Q}_{\kappa_1}$. Note that M_2 is closed under κ_1-sequences with respect to M_1. Therefore, from the M_1^*-point of view, $\mathcal{P}_{(\kappa_1,\kappa_2)} * \underset{\sim}{Q}_{\kappa_2}$ is κ_1^+-closed, and we can construct an $M_2[H]$-generic filter $G_{(\kappa_1,\kappa_2)} * g'_{\kappa_2} \in M_1^*$ for it. We change the values of g'_{κ_2} a bit differently from the way we changed the values of g'_{κ_1}. If $\alpha < \kappa_1^+$ is of the form $j_1(\beta)$ let $f_{\kappa_2,k(\alpha)}(\kappa_1) = 1$ (to guarantee that A_α's belong to the ultrafilter generated by κ_1) and if $\alpha \in \kappa_1^+ \setminus j_1''\kappa^+$ let[2] $f_{\kappa_2,k(\alpha)}(\kappa_1) = 0$. Also, we would like that $f_{\kappa_2,\kappa_1}(0) = \kappa$. Formally, for every $p \in \text{Cohen}(\kappa_2, \kappa_2^+)^{M_2[H*G_{(\kappa_1,\kappa_2)}]}$, define p^* to be a function with $\text{dom}(p) = \text{dom}(p^*)$ and for every $\langle \gamma, \alpha\rangle \in \text{dom}(p^*)$,

$$p^*(\langle \gamma, \alpha\rangle) = \begin{cases} f_{\kappa_1,\beta}(\gamma), & \gamma < \kappa_1 \wedge \alpha = k(\beta), \\ 1, & \gamma = \kappa_1 \wedge \alpha = k(j_1(\beta)), \\ 0, & \gamma = \kappa_1 \wedge \alpha = k(\beta), \beta \notin j_1''\kappa^+, \\ \kappa, & \gamma = 0 \wedge \alpha = \kappa_1, \\ p(\langle \gamma, \alpha\rangle), & \text{else.} \end{cases}$$

Denote by $g_{\kappa_2} = \{p^* \mid p \in g'_{\kappa_2}\} \in V[G]$ the resulting filter. It is important that for each $p \in g'_2$, the set

$$X_1 := j_2''\kappa^+ \cap \text{dom}(f)_{\leq \kappa_1} = \{j_2(\alpha) \mid \exists \langle \gamma, j_2(\alpha)\rangle \in \text{dom}(f), \gamma \leq \kappa_1\}$$

has size at most κ. This ensured that $X_1 \in M_1^*$. Also, $k''\kappa_1^+$ is unbounded in κ_2^+ and conditions in $\text{Cohen}(\kappa_2, \kappa_2^+)^{M_2[H*G_{(\kappa_1,\kappa_2)}]}$ have $M_2[H * G_{(\kappa_1,\kappa_2)}]$-cardinality less than κ_2, which guarantees that for each $p \in \text{Cohen}(\kappa_2, \kappa_2^+)$,

$$X_2 := k''\kappa_1^+ \cap \text{dom}(p)_{\leq \kappa_1}$$

[2] Recall that $k : M_1 \to M_2$ is the factor map satisfying $j_2 = k \circ j_1$ defined by $k([f]_U) = j_2(f)(\kappa)$.

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has size at most κ_1. Note that p^* is definable in M_1^* from the parameters $p, X_1, X_2 \in M_1^*$, and p^* differs from p at most on κ_1-many values. By the closure of $M_2[H * G_{(\kappa_1,\kappa_2)}]$ to κ_1-sequences from M_1^*,

$$p^* \in M_2[H * G_{(\kappa_1,\kappa_2)}] \text{ and } g_{\kappa_2} \subseteq \text{Cohen}(\kappa_2, \kappa_2^+)^{M_2[H*G_{(\kappa_1,\kappa_2)}]}.$$

The genericity argument of Lemma 2.7 extends to the models M_1 and $M_2[H * G_{(\kappa_1,\kappa_2)}]$, hence g_{κ_2} is $M_2[H * G_{(\kappa_1,\kappa_2)}]$-generic. Denote by $M_2^* = M_2[H * G_{(\kappa_1,\kappa_2)} * g_{\kappa_2}]$. It follows that k can be extended (in $V[G]$) to k^* and also j_2 to $j_2^* = k^* \circ j_1^* : V[G] \to M_2^*$. Finally, let

$$W := \{X \in P^{V[G]}(\kappa) \mid \kappa_1 \in j_2^*(X)\} \in V[G].$$

Let us prove that W witnesses the theorem:

CLAIM 2.8. *W is a κ-complete ultrafilter over κ such that:*

(1) $j_W = j_2^*$ *and* $[id]_W = \kappa_1$.
(2) $Cub_\kappa \subseteq W$.
(3) $\{\alpha < \kappa \mid cf(\alpha) = \alpha\} \in W$.
(4) $\langle A_\alpha \mid \alpha < \kappa^+\rangle$ *is a strong witness for the failure of the Galvin property.*

PROOF. To see (1), let us denote by $j_W : V[G] \to M_W$ the ultrapower embedding by W and $k_W : M_W \to M_2^*$ defined by $k_W([f]_W) = j_2^*(f)(\kappa_1)$ the factor map satisfying $k_W \circ j_W = j_2^*$. Let us argue that k_W is onto and therefore $k_W = id$ and $[id]_W = \kappa_1$. Indeed, let $A \in M_2^*$ then there is $\underset{\sim}{A} \in M_2$ such that $(\underset{\sim}{A})_{j_2^*(G)} = A$. Since $j_2 = j_{U^2}$ there is $h \in V$ such that $j_2(h)(\kappa, \kappa_1) = \underset{\sim}{A}$. Note that $\kappa = j_2^*(f_\kappa)_{\kappa_1}(0)$, hence define in $V[G]$, $h^*(\alpha) = (h(f_{\kappa,\alpha}(0), \alpha))_G$. We have that

$$k_W([h^*]_W) = j_2^*(h^*)(\kappa_1) = (j_2(h)(\kappa, \kappa_1))_{j_2^*(G)} = (\underset{\sim}{A})_{j_2^*(G)} = A.$$

To see (2), for every club $C \in Cub_\kappa$, $j_2^*(C)$ is closed and $j_1^*(C)$ is unbounded in κ_1. Since $crit(k^*) = \kappa_1$ and $j_2^*(C) = k^*(j_1^*(C))$ it follows that $j_2^*(C) \cap \kappa_1 = j_1^*(C)$, hence $j_2^*(C) \cap \kappa_1$ is unbounded in κ_1 which implies that $\kappa_1 \in j_2^*(C)$.

For (3), since $M_2^* \models cf(\kappa_1) = \kappa_1$, it follows that $\{\alpha \mid cf(\alpha) = \alpha\} \in W$. Finally, for every $\alpha < \kappa^+$,

$$j_2^*(A_\alpha) = \{\beta < \kappa_2 \mid f_{\kappa_2, j_2(\alpha)}(\beta) = 1\}.$$

Since $j_2(\alpha) = k(j_1(\alpha))$, by the definition of g_{κ_2}, $f_{\kappa_2, j_2(\alpha)}(\kappa_1) = 1$, thus $\kappa_1 \in j_2^*(A_\alpha)$, and by definition of W, $A_\alpha \in W$.

For (3), let $\{A_{\alpha_i} \mid i < \kappa\}$ be any subfamily of length κ and $\kappa \le \eta < [id]_W = \kappa_1$. Denote

$$j_2^*(\langle A_{\alpha_i} \mid i < \kappa\rangle) = \langle A^{(2)}_{\alpha_i^{(2)}} \mid i < \kappa_2\rangle, \ j_1^*(\langle A_{\alpha_i} \mid i < \kappa\rangle) = \langle A^{(1)}_{\alpha_i^{(1)}} \mid i < \kappa_1\rangle.$$

Since $\kappa \le \eta < \kappa_1$, then $\eta \notin j_1''\kappa^+$ and thus $\alpha_\eta^{(1)} \notin j_1''\kappa^+$. Also, $k(\alpha_\eta^{(1)}) = \alpha^{(2)}_{k(\eta)} = \alpha_\eta^{(2)}$. Hence by definition, $f_{\kappa_2, \alpha_\eta^{(2)}}(\kappa_1) = 0$, hence $\kappa_1 \notin A'_{\alpha_\eta^{(2)}}$. ⊣

⊣

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2.2. Adding κ^+-Cohen subsets to κ by Prikry forcing. In this section we will construct a model in which there is a κ-complete ultrafilter W such that forcing with $\mathrm{Prikry}(W)$ adds a generic for $\mathrm{Cohen}(\kappa,\kappa^+)$. Let us first observe that such an ultrafilter must fail to satisfy the Galvin property:

PROPOSITION 2.9. *If $Gal(U,\kappa,\kappa^+)$ holds then* $\mathrm{Prikry}(U)$ *does not add a V-generic filter for* $\mathrm{Cohen}(\kappa,\kappa^+)$.

PROOF. Suppose that $Gal(U,\kappa,\kappa^+)$ holds and let $G \subseteq \mathrm{Prikry}(U)$ be V-generic. By [18, Proposition 1.3] every set $A \in V[G]$ of size κ^+ contains a set $B \in V$ of cardinality κ. Toward a contradiction suppose that $H \in V[G]$ is a V-generic filter for $\mathrm{Cohen}(\kappa,\kappa^+)$. Code $H : \kappa \times \kappa^+ \to 2$ as $X \subseteq \kappa^+$, just pick a bijection ϕ from κ^+ to $\kappa^+ \times \kappa$, and let $X = \{\alpha < \kappa^+ \mid H(\phi(\alpha)) = 1\}$. The set X does not contain an old subset of cardinality κ; this is a contradiction. To see this, let $Y \in V$ such that $|Y| = \kappa$, proceed with a density argument: any condition $p \in \mathrm{Cohen}(\kappa,\kappa^+)$ has size $< \kappa$ and therefore can be extended to a condition p' such that for some $y \in Y$, $\phi(y) \in \mathrm{dom}(p')$ and $p'(\phi(y)) = 0$. ⊣

Hence the failure of the Galvin property is necessary.

THEOREM 2.10. *Assume GCH and that κ is a measurable cardinal in V. Then there is a cofinality preserving forcing extension V^* in which GCH still holds, and there is a κ-complete ultrafilter $U^* \in V^*$ over κ such that forcing with Prikry forcing $Pikry(U^*)$ introduces a V^*-generic filter for $Cohen^{V^*}(\kappa,\kappa^+)$.*

PROOF. The model V^* is obtained by iterating with Easton support the lottery sum of Cohen forcings for adding α^+-Cohen functions $\langle f_{\alpha\gamma} \mid \gamma < \alpha^+\rangle$ over α, and Cohen^2 for adding two blocks of α^+-Cohen functions

$$\langle f_{\alpha\gamma} \mid \gamma < \alpha^+\rangle, \langle h_{\alpha\gamma} \mid \gamma < \alpha^+\rangle.$$

More specifically, let

$$\langle \mathcal{P}_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa+1, \beta \leq \kappa\rangle$$

denotes the Easton support iteration, such that for each $\alpha < \kappa$, $\underset{\sim}{Q}_\alpha$ is the trivial forcing unless α is inaccessible in which case $\underset{\sim}{Q}_\alpha$ is a $\mathcal{P}_\alpha$-name for the lottery sum

$$\mathrm{LOTT}(\mathrm{Cohen}(\alpha,\alpha^+), \mathrm{Cohen}(\alpha,\alpha^+) \times \mathrm{Cohen}(\alpha,\alpha^+)).$$

At κ itself we let $\underset{\sim}{Q}_\kappa = \mathrm{Cohen}(\kappa,\kappa^+)$. Let $G_\kappa * F_\kappa$ be a V-generic subset of $P_\kappa * \underset{\sim}{Q}_\kappa$ and let $V^* = V[G_\kappa * F_\kappa]$. We denote by $F_\alpha := \langle f_{\alpha\gamma} \mid \gamma < \alpha^+\rangle$ the generic Cohen function if $\mathrm{Cohen}(\alpha,\alpha^+)$ was forced in G_κ and by

$$F_\alpha := \langle f_{\alpha\gamma} \mid \gamma < \alpha^+\rangle,\ H_\alpha := \langle h_{\alpha,\gamma} \mid \gamma < \alpha^+\rangle$$

if $\mathrm{Cohen}(\alpha,\alpha^+) \times \mathrm{Cohen}(\alpha,\alpha^+)$ was.

Let $U \in V$ be a normal ultrafilter, $j_1 := j_U : V \to M_U$ the corresponding elementary embedding, $\kappa_1 = j_1(\kappa)$, $k := j_{j_1(U)} : M_U \to M_{U^2}$, $j_2 = k \circ j_1$, and $\kappa_2 = j_2(\kappa)$. Let us extend j_1, k, j_2 in $V[G_\kappa * F_\kappa]$:

We first extend $j_1 : V \to M_U$ to $j_1^* : V[G_\kappa * F_\kappa] \to M_U[G_{\kappa_1} * F_{\kappa_1}]$. Do this by taking first $G_{\kappa_1} \cap P_\kappa = G_\kappa$, at κ we force with the lottery sum so we can choose to force only one block of Cohens and take F_κ as a generic. Then defining

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a master condition sequence, using the closure of the forcing above κ in M_U exploiting GCH to ensure that there are only κ^+-many dense sets to meet. This defines G_{κ_1}. As for F_{κ_1}, we first find an $M_U[G_{\kappa_1}]$-generic $F'_{\kappa_1} \times H'_{\kappa_1} \in V[G_\kappa * F_\kappa]$ again using GCH, closure of $M_U[G_{\kappa_1}]$ under κ-sequences and the closure of the forcing $(\mathrm{Cohen}(\kappa_1, \kappa_1^+)^2)^{M_U[G_{\kappa_1}]}$. Let us alter some values of F'_{κ_1} and H'_{κ_1} to define $F_{\kappa_1} = \langle f_{\kappa_1,\gamma} \mid \gamma < \kappa_1^+\rangle$ and $H_{\kappa_1} = \langle h_{\kappa_1,\gamma} \mid \gamma < \kappa_1^+\rangle$ such that for every $\alpha < \kappa_1^+$:

(1) $f_{\kappa_1, j_1(\alpha)} \restriction \kappa = h_{\kappa_1, j_1(\alpha)} \restriction \kappa = f_{\kappa,\alpha}$.
(2) $f_{\kappa_1, j_1(\alpha)}(\kappa) = \alpha$.

Formally, we change every pair of partial functions $p = \langle p_0, p_1\rangle \in F'_{\kappa_1} \times H'_{\kappa_1}$ to the pair of partial functions $p_* = \langle p_0^*, p_1^*\rangle$ such that $\mathrm{dom}(p_0^*) = \mathrm{dom}(p_0)$, $\mathrm{dom}(p_1^*) = \mathrm{dom}(p_1)$ and for every $\langle \alpha, \delta\rangle \in \mathrm{dom}(p_0)$:

$$p_0^*(\langle \alpha, \delta\rangle) = \begin{cases} f_{\kappa,\alpha_0}(\delta), & \exists \alpha_0 < \kappa^+ . \alpha = j_1(\alpha_0) \text{ and } \delta < \kappa, \\ \alpha_0, & \exists \alpha_0 < \kappa^+ . \alpha = j_1(\alpha_0) \text{ and } \delta = \kappa, \\ p_0(\langle \alpha, \delta\rangle), & \text{else.} \end{cases}$$

$$p_1^*(\langle \alpha, \delta\rangle) = \begin{cases} f_{\kappa,\alpha_0}(\delta), & \exists \alpha_0 < \kappa^+ . \alpha = j_1(\alpha_0) \text{ and } \delta < \kappa, \\ p_1(\langle \alpha, \delta\rangle), & \text{else.} \end{cases}$$

Note that for every $p_0, p_1 \subseteq \mathrm{Cohen}(\kappa_1, \kappa_1^+)^{M_U[G_{\kappa_1}]}$ we only change κ-many values as $M_U[G_{\kappa_1}] \models |\mathrm{dom}(p_0)|, |\mathrm{dom}(p_1)| < \kappa_1$, hence

$$|j_1'' \kappa^+ \cap \{\alpha \mid \exists \delta . \langle \alpha, \delta\rangle \in \mathrm{dom}(p_0)\}| \leq \kappa$$

since $j_1(\kappa^+) = \bigcup j_1''\kappa^+$, the same holds for p_1. It follows that

$$p^* \in (\mathrm{Cohen}(\kappa_1, \kappa_1^+)^2)^{M_U[G_{\kappa_1}]}.$$

Changing less than κ_1-many values of a generic for $\mathrm{Cohen}(\kappa_1, \kappa_1^+)^2$ does not impact the genericity. Hence $F_{\kappa_1} \times H_{\kappa_1} := \{p^* \mid p \in F'_{\kappa_1} \times H'_{\kappa_1}\} \in V[G_\kappa * F_\kappa]$ is still $M_U[G_{\kappa_1}]$-generic.

Since at κ we only force $\mathrm{Cohen}(\kappa, \kappa^+)$, in order to extend j_1 we only need a generic for $\mathrm{Cohen}(\kappa_1, \kappa_1^+)$ in the M_U-side. We constructed F_{κ_1} so that $j_1'' F_\kappa \subseteq F_{\kappa_1}$, hence $j_1'' G_\kappa * F_\kappa \subseteq G_{\kappa_1} * F_{\kappa_1}$ (H_{κ_1} will be used later). Thus in $V[G_\kappa * F_\kappa]$, we have extended $j_1 \subseteq j_1^* : V[G_\kappa * F_\kappa] \to M_U[G_{\kappa_1} * F_{\kappa_1}]$. Let us note that j_1^* is actually the elementary embedding derived from the normal measure $U \subseteq U^0 := \{X \in P^{V[G_\kappa * F_\kappa]}(\kappa) \mid \kappa \in j_1^*(X)\}$:

Clearly the function $k_0 : M_{U^0} \to M_U[G_{\kappa_1} * F_{\kappa_1}]$ defined by $k_0([f]_{U^0}) = j_1^*(f)(\kappa)$ is elementary. To see the $k_0 = id$ let us prove that k_0 is onto. Fix $A = (\underset{\sim}{A})_{G_{\kappa_1} * F_{\kappa_1}} \in M_U[G_{\kappa_1} * F_{\kappa_1}]$ and let $f \in V$ be such that $j_1(f)(\kappa) = \underset{\sim}{A}$ and define in $V[G_\kappa * F_\kappa]$ the function $f^*(x) = (f(f_{\kappa,\kappa}(x)))_{G_\kappa * F_\kappa}$. Then

$$k_0(j_{U^0}(f^*)(\kappa)) = j_1^*(f^*)(\kappa) = (j_1^*(f)(j_1^*(f_{\kappa,\kappa})(\kappa)))_{G_{\kappa_1} * F_{\kappa_1}} =$$

$$= (j_1(f)(\kappa))_{G_{\kappa_1} * F_{\kappa_1}} = (\underset{\sim}{A})_{G_{\kappa_1} * F_{\kappa_1}} = A.$$

Recall that we have constructed the function $H_{\kappa_1} \in V[G_\kappa * F_\kappa]$ such that $F_{\kappa_1} \times H_{\kappa_1}$ is $M_U[G_{\kappa_1}]$-generic for $\mathrm{Cohen}(\kappa_1, \kappa_1^+)^2$. Now we wish to extend $k : M_U \to M_{U^2}$ to $k^* : M_U[G_{\kappa_1} * F_{\kappa_1}] \to M_{U^2}[G_{\kappa_2} * F_{\kappa_2}]$ in $V[G_\kappa * F_\kappa]$. We do this by taking

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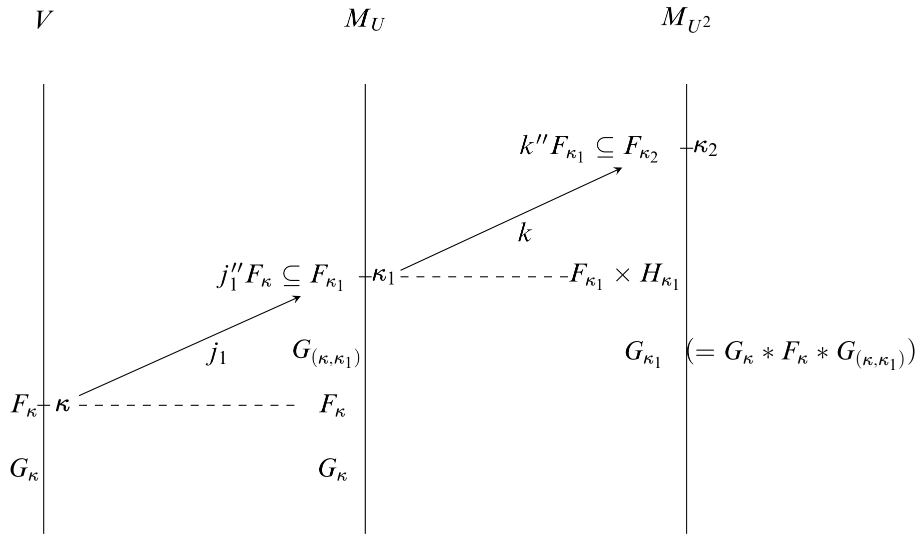
$G_{\kappa_2} \cap \kappa_1 = G_{\kappa_1}$, at κ_1 we force $\mathrm{Cohen}(\kappa_1, \kappa_1^+) \times \mathrm{Cohen}(\kappa_1, \kappa_1^+)$ putting the generic $F_{\kappa_1} \times H_{\kappa_1}$, then exploiting the closure and GCH to complete to a generic $G_{\kappa_2} * F'_{\kappa_2} \in V[G_\kappa * F_\kappa]$. Finally, we wish to modify some values of F'_{κ_2} to a generic $F_{\kappa_2} = \langle f_{\kappa_2,\gamma} \mid \gamma < \kappa_2^+\rangle$ so that for every $\alpha < \kappa_1^+$:

(1) $f_{\kappa_2,k(\alpha)} \restriction \kappa_1 = f_{\kappa_1,\alpha}$.
(2) For $\alpha \in j_1'' \kappa^+$, $f_{\kappa_2,k(\alpha)}(\kappa_1) = 1$.
(3) For $\alpha \in \kappa_1^+ \setminus j_1''\kappa^+$, $f_{\kappa_2,k(\alpha)}(\kappa_1) = 0$.
(4) $f_{\kappa_2,\kappa_1}(\kappa_1) = \kappa$.

Again, this is possible since we do not change too many values of F'_{κ_2}. At this point, let us emphasize that we do not use H_{κ_1} in the generic we have in the M_U-side [3]. The generic H_{κ_1} is used in the construction of the generic on the M_{U^2}-side where we can choose (due to the lottery sum) to force at κ_1 two copies of $\mathrm{Cohen}(\kappa_1, \kappa_1^+)$, of course, that at $\kappa_2 = j_2(\kappa)$ we are still obligated to force one copy of $\mathrm{Cohen}(\kappa_2 \kappa_2^+)$ which contains the point-wise image of F_{κ_1} under the factor map k.

Hence we extended in $V[G_\kappa * F_\kappa]$, $k \subseteq k^* : M_U[G_{\kappa_1} * F_{\kappa_1}] \to M_{U^2}[G_{\kappa_2} * F_{\kappa_2}]$.



Let $j_2^* = k^* \circ j_1^*$, $V^* = V[G_\kappa * F_\kappa]$, $M_1^* = M_U[G_{\kappa_1} * F_{\kappa_1}]$ and $M_2^* = M_{U^2}[G_{\kappa_2} * F_{\kappa_2}]$.

In V^*, define

$$U^* = \{X \subseteq \kappa \mid \kappa \in j_2^*(U)\},$$

$$W = \{X \subseteq \kappa \mid \kappa_1 \in j_2^*(X)\},$$

[3] Since over V, at κ we forced one copy of Cohen's, i.e., $\mathrm{Cohen}(\kappa, \kappa^+)$, over M_U we need to force only one copy of $\mathrm{Cohen}(\kappa_1, \kappa_1^+)$, thus we only need the generic F_{κ_1}.

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and for every $\alpha < \kappa^+$,

$$A_\alpha = \{\nu < \kappa \mid f_{\kappa,\alpha}(\nu) = 1\}.$$

Then as in Claim 2.8, we have that W is a κ-complete ultrafilter over κ such that:

(1) $j_1^* = j_U^*$, $j_2^* = j_W$ and $[id]_W = \kappa_1$.
(2) $\langle A_\alpha \mid \alpha < \kappa^+\rangle$ is a strong witness for W being non-Galvin.
(3) $Cub_\kappa \subseteq W$.
(4) $L_0 = \{\alpha < \kappa \mid \text{Cohen}(\alpha, \alpha^+) \times \text{Cohen}(\alpha, \alpha^+) \text{ was forced in } G_\kappa\} \in W$.

Also, recall that $j_2 : V \to M_2$ is also the ultrapower by $U \times U$ under the identification(isomorphism):

$$j_{U^2}(f)(\kappa, \kappa_1) = j_{2,1}(j_1(\nu \mapsto f(\nu, *))(\kappa))(\kappa_1).$$

Clearly, the projections $\pi_1, \pi_2 : \kappa \times \kappa \to \kappa$ on the first and second coordinates (resp. Rudin–Keisler) project U^2 on U. Also, $W \cap V = U^* \cap V = U$ and $U^* \leq_{R\text{–}K} W$ and the projection map is denoted by $\nu \mapsto \pi_{nor}(\nu)$.[4]

Let us prove that W witnesses the theorem:

THEOREM 2.11. *Let $H \subseteq$ Prikry(W) be a V^*-generic filter. There is $G^* \in V^*[H]$ which is V^*-generic for Cohen$(\kappa, \kappa^+)^{V^*}$.*

PROOF OF THEOREM 2.11. Let $\langle c_n \mid n < \omega\rangle$ be the W-Prikry sequence corresponding to H. Suppose without loss of generality that for every $n < \omega$, $c_n \in L_0$, this will hold from a certain point and the proof can be adjusted in a straightforward way. This guarantees that the generic $H_{c_n} = \langle h_{c_n,\gamma} \mid \gamma < \alpha^+\rangle$ for the second component of the generic we have in G_κ for $\text{Cohen}(c_n, c_n^+) \times \text{Cohen}(c_n, c_n^+)$ is defined for every $n < \omega$. The functions $h_{c_n,\gamma}$ will be used below to define the Cohen generic functions.

Define, for every $n < \omega$, the set

$$Z_n = \{\alpha < \kappa^+ \mid \{c_m \mid n \leq m < \omega\} \subseteq A_\alpha \text{ and } n \text{ is least possible}\}.$$

For every $\alpha < \kappa^+$, let n_α be the unique n such that $\alpha \in Z_n$. Let $\alpha < \kappa^+$, and define $f^*_\alpha : \kappa \to \kappa$ as follows:

Fix a sequence $\langle s_\alpha \mid \alpha < \kappa^+\rangle \in V^*$ of canonical functions in $\prod_{\nu<\kappa} \nu^+$:

$$f^*_\alpha \restriction c_{n_\alpha} = h_{c_{n_\alpha} s_\alpha(c_{n_\alpha})},$$

$$f^*_\alpha \restriction [c_{m-1}, c_m) = h_{c_m, s_\alpha(c_m)} \restriction [c_{m-1}, c_m), \text{for } m, n_\alpha < m < \omega.$$

Let us argue that $F = \langle f^*_\alpha \mid \alpha < \kappa^+\rangle$ induces a $\text{Cohen}(\kappa, \kappa^+)^{V^*}$ generic filter over V^*.

CLAIM 2.12. *Let $G^* = \{p \in \text{Cohen}(\kappa, \kappa^+)^{V^*} \mid p \subseteq F\}$, then G^* is a V^*-generic filter.*

Let $\mathcal{A} \in V^*$ be a maximal antichain in the forcing $\text{Cohen}(\kappa, \kappa^+)^{V^*}$. Note that since $\text{Cohen}(\kappa, \kappa^+)^{V^*}$ is κ-closed then

$$Cohen(\kappa, \kappa^+)^{V[G_\kappa]} = Cohen(\kappa, \kappa^+)^{V^*}.$$

[4]Explicitly, one can define in $V[G]$ the function $f(\alpha) = f_{\kappa,\alpha}(\alpha)$. Then $j_2^*(f)(\kappa_1) = f_{\kappa_2,\kappa_1}(\kappa_1) = \kappa$.

By κ^+-cc of the forcing $\mathcal{P}_{\kappa+1}$, there is $Y \subseteq \kappa^+$, $Y \in V$ such that $|Y| = \kappa$ and $\mathcal{A} \subseteq \mathrm{Cohen}(\kappa, Y)^{V^*}$. Also, since $|\mathcal{A}| = \kappa$, $\mathcal{A} \in V[G_\kappa * F_\kappa]$, there is $Z \subseteq \kappa^+$ such that $|Z| = \kappa$ such that $\mathcal{A} \in V[G_\kappa * F_\kappa \restriction Z]$. Without loss of generality assume that $Z = Y \in V$ (Otherwise just take the union). Let $V \ni \phi : \kappa \to Y$ be a bijection.

Claim 2.13. *There is an $\in$-increasing continuous chain $\langle N_\beta \mid \beta < \kappa\rangle$ of elementary submodels of H_χ such that:*

(1) $|N_\beta| < \kappa$.
(2) $G_\kappa, F_\kappa, \mathcal{A}, \phi, \langle s_\alpha \mid \alpha < \kappa^+\rangle \in N_0$.
(3) *$N_\beta \cap \kappa = \gamma_\beta$ is a cardinal $< \kappa$, $\gamma_{\beta+1}$ is regular.*
(4) *For every $\rho, \delta \in \phi''\gamma_\beta . \rho < \delta \to \forall \gamma_\beta \leq \mu < \kappa, s_\rho(\mu) < s_\delta(\mu)$.*
(5) *If γ_β is regular, then $N_\beta^{<\gamma_\beta} \subseteq N_\beta$. In particular* $\mathrm{Cohen}(\gamma_\beta, \phi''\gamma_\beta) = \mathrm{Cohen}(\kappa, Y) \cap N_\beta$.

Proof of Claim 2.13. Let us construct such a sequence inductively. Note that (4) follows from elementarity and (2). Requirements (1)–(5) are preserved at limit stages due to continuity. At successor stages, suppose we have constructed N_β, find an elementary submodel $N^0_{\beta+1}$ such that $N_\beta \subseteq N^0_{\beta+1}$, $\langle N_\alpha \mid \alpha < \beta\rangle \in N^0_{\beta+1}$, then we construct an auxiliary $\in$-increasing and continuous chain of elementary submodels $\langle N^\alpha_{\beta+1} \mid \alpha < \kappa\rangle$ as follows: $N^0_{\beta+1}$ is already defined. At limits we take the union and at successor let us take care of requirements 3 and 5. Let $\gamma'_\alpha = \sup(N^\alpha_{\beta+1} \cap \kappa) < \kappa$. Let $N^{\alpha+1}_{\beta+1}$ be an elementary submodel such that $N_\beta^{\alpha},{}^{<\gamma'_\alpha},\subseteq N^{\alpha+1}_{\beta+1}$ and $|N^{\alpha+1}_{\beta+1}| < \kappa$. Note that the sets

$$C_1 = \{\alpha < \kappa \mid N^\alpha_{\beta+1} \cap \kappa = \gamma'_\alpha \in \kappa\},$$

$$C_2 = \{\alpha \in C_1 \mid \text{if } \gamma_\alpha \text{ is regular then } N_\alpha^{<\gamma_\alpha} \subseteq N_\alpha\}$$

are clubs and also $\bar{C} = C_1 \cap C_2$ is. It follows that $\{\gamma'_\alpha \mid \alpha \in \bar{C}\}$ is a club and since κ is measurable, there is a $\alpha^* \in \bar{C}$ limit such that γ'_{α^*} is regular. Let $N_{\beta+1} = N^{\alpha^*}_{\beta+1}$, to conclude 2 since $\gamma_{\beta+1} = \gamma'_{\alpha^*}$ is regular. ⊣

Set

$$C = \{\beta < \kappa \mid \gamma_\beta = \beta\}.$$

This is club in κ since the sequence γ_β is continuous and since the set $\{\beta \mid \gamma_\beta = \beta\}$ is a club.

Claim 2.14. *Let*

$$E := \{\beta < \kappa \mid \forall \gamma \in \phi''\beta . \exists\, \delta < \beta^+ . f_{\kappa,\gamma} \restriction \beta = f_{\beta,\delta}\}.$$

Then $E \in W$.

Proof of Claim 2.14. By construction, for every $\alpha < \kappa_1^+$, $f_{\kappa_2, k(\alpha)} \restriction \kappa_1 = f_{\kappa_1,\alpha}$ and therefore for every $\alpha \in j_2^*(\phi)''\kappa_1$, there is $\nu < \kappa_1^+$ such that $\alpha = k^*(j_1^*(\phi))(\nu) = k^*(j_1^*(\phi)(\nu))$ and $j_1^*(\phi)(\nu) < \kappa_1^+$. Hence $f_{\kappa_2,\alpha} \restriction \kappa_1 = f_{\kappa_1,\beta}$ for some $\beta < \kappa_1^+$. Reflecting this we obtain the set $E \in W$. ⊣

To see that $G^* \cap \mathcal{A} \neq \emptyset$, we will need to catch a piece of $\mathcal{A}$ in the elementary submodels constructed and pick the Prikry points in the club C prepared:

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CLAIM 2.15. *For every $\nu_0 \in C \cap E$, there is $d = d^{\nu_0} \in N_{\nu_0} \cap \mathcal{A}$ such that d is extended by $\langle h_{\nu_0, s_\tau(\nu_0)} \mid \tau \in \phi''\nu_0 \rangle$.*

PROOF OF CLAIM 2.15. Fix any $\nu_0 \in C \cap E$. Consider the transitive collapse of $\pi : N_{\nu_0} \to N^*_{\nu_0}$. Then the critical point of $\pi^{-1} : N^*_{\nu_0} \to N_{\nu_0}$ is ν_0 and $\pi^{-1}(\nu_0) = \kappa$. Denote by $\overline{F}_\kappa = \pi(F_\kappa), \overline{\phi} = \pi(\phi)$. Denote $\overline{F}_\kappa = \langle \overline{f}_{\kappa,\gamma} \mid \gamma < \pi(\kappa^+) \rangle$. For every $\gamma \in \overline{\phi}''\nu_0$, there is some $\delta < \nu_0$ such that

$$\gamma = \pi(\phi)(\delta) = \pi(\phi(\delta)) \text{ and } \overline{f}_{\kappa,\gamma} = \pi(f_{\kappa,\phi(\delta)}).$$

Moreover, since $\nu_0 \in E$, $f_{\kappa,\phi(\delta)} \restriction \nu_0 = f_{\nu_0,\rho}$ for some $\rho < \nu_0{}^+$ and therefore $\overline{f}_{\kappa,\gamma} = f_{\nu_0,\rho}$. Recall that $\mathcal{A} = (\underset{\sim}{A})_{G_\kappa * F_\kappa \restriction Y}$, hence $\overline{\mathcal{A}} = (\underset{\sim}{A})_{G_{\nu_0} * \overline{F}_\kappa \restriction \overline{Y}}$. We conclude that for some subset $Z \subseteq \nu_0{}^+$,

$$\overline{\mathcal{A}} = (\underset{\sim}{A})_{G_{\nu_0} * F_{\nu_0} \restriction Z} \in V[G_{\nu_0} * F_{\nu_0} \restriction Z].$$

Since $\nu_0 \in L_0$, in $V[G_\kappa * F_\kappa]$ we also have $H_{\nu_0} = \langle h_{\nu_0,\alpha} \mid \alpha < \nu_0^+ \rangle$ which are mutually Cohen-generic over $V[G_{\nu_0} * F_{\nu_0} \restriction Z]$.

By construction, $\forall \tau_1 < \tau_2 \in \phi''\nu_0$, $s_{\tau_1}(\nu_0) < s_{\tau_2}(\nu_0)$, hence $\langle h_{\nu_0, s_\tau(\nu_0)} \mid \tau \in \phi''\nu_0 \rangle$ are Cohen functions over ν_0 which are distinct mutually $V[G_{\nu_0} * F_{\nu_0} \restriction Z]$-generic. Also, $\overline{\mathcal{A}} \subseteq \pi(\text{Cohen}(\kappa, Y)) = \text{Cohen}(\nu_0, \pi(\phi)''\nu_0) = \text{Cohen}(\nu_0, \pi''[\phi''\nu_0])$ is a maximal antichain. Since $|\pi''\phi''\nu_0| = \nu_0 = |\phi''\nu_0|$, we can change the enumeration of the functions $\langle h_{\nu_0, s_\tau(\nu_0)} \mid \tau \in \phi''\nu_0 \rangle$ to $h'_{\pi(\tau)} = h_{\nu_0, s_\tau(\nu_0)}$ so that $\langle h'_\rho \mid \rho \in \pi''\phi''\nu_0 \rangle$ is generic for $\text{Cohen}(\nu_0, \pi''\phi_0)$. Thus pick $d_0 \in \overline{\mathcal{A}}$ such that d_0 is extended by $\langle h'_\rho \mid \rho \in \pi''\phi''\nu_0 \rangle$. It follows that

$$d := \pi^{-1}(d_0) \in \mathcal{A} \cap N_{\nu_0}$$

is a condition with $\text{dom}(d) = \pi^{-1}(\text{dom}(d_0))$. Since the critical point of π is ν_0, for every $\langle \alpha, \beta \rangle \in \text{dom}(d_0)$, $\pi^{-1}(\langle \alpha, \beta \rangle)) = \langle \alpha, \pi^{-1}(\beta) \rangle$, hence

$$d(\langle \alpha, \pi^{-1}(\beta) \rangle) = \pi^{-1}(d_0(\alpha, \beta)) = d_0(\alpha, \beta).$$

In particular for every $\langle \gamma, \alpha \rangle \in \text{dom}(d)$,

$$d(\gamma, \alpha) = d_0(\gamma, \pi(\alpha)) = h'_{\pi(\alpha)}(\gamma) = h_{\nu_0, s_\alpha(\nu_0)}(\gamma).$$

Thus d is extended by $\langle h_{\nu_0, s_\tau(\nu_0)} \mid \tau \in \phi''\nu_0 \rangle$. $\dashv$

It suffices to show that any condition in $\text{Prikry}(W)$ has an extension which forces that G^* meets a member of $\mathcal{A}$.

Let $p = \langle \langle \rangle, B \rangle$ be a condition (we assume for simplicity that its finite sequence is empty) and shrink B to $B \cap C \cap E$. For any $\nu_0 \in B \cap C \cap E$, we split $\phi''\nu_0$ into two sets:

$$X_0^{\nu_0} := \{\tau \in \phi''\nu_0 \mid \nu_0 \in A_\tau\} \text{ and } X_1^{\nu_0} = \phi''\nu_0 \setminus X_0^{\nu_0}.$$

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The condition $p_0 = \langle\langle\nu_0\rangle, B \cap C \cap E \cap X \cap (\bigcap_{\tau\in\phi''\nu_0} A_\tau)\rangle$ forces the following:

(1) The Prikry sequence is included in each A_τ, $\tau \in X_0^{\nu_0}$, i.e., $n_\tau = 0$.
(2) $n_\tau = 1$, for every $\tau \in X_1^{\nu_0}$.

In particular, this condition forces some information about the Cohen functions. Namely that:

(1) For $\tau \in X_0^{\nu_0}$, $f^*_\tau \restriction \nu_0 = h_{\nu_0, s_\tau(\nu_0)}$.
(2) For $\tau \in X_1^{\nu_0}$, $f^*_\tau \restriction \nu_0 = h_{\underset{\sim}{\nu}_1, s_\tau(\underset{\sim}{\nu}_1)} \restriction \nu_0$.

We would like to find a condition in $\mathcal{A}$ which is below these decided parts of the Cohen. By the previous proposition, there is $d \in N_{\nu_0} \cap \mathrm{Cohen}(\kappa, Y) = \mathrm{Cohen}(\nu_0, \phi''\nu_0)$, which is extended by $\langle h_{\nu_0, s_\tau(\nu_0)} \mid \tau \in \phi''\nu_0\rangle$. However, by (1) and (2) we can only ensure that the generic f^*_τ to extend $d \restriction \nu_0 \times X_0^{\nu_0}$ in $X_0^{\nu_0}$. We are left to extend $d \restriction \nu_0 \times X_1^{\nu_0}$. Let us show that for many ν_0, X_0^{ν} is a relatively large subset of $\phi''\nu_0$:

Claim 2.16. *Let*

$$R = \{\nu < \kappa \mid \forall\alpha \in \phi''\pi_{nor}(\nu), \nu \in A_\alpha\}.$$

Then $R \in W$.

Proof. Clearly, for every $\alpha \in j_2^*(\phi)''\kappa$, $\alpha = j_2^*(\phi(\gamma))$, and $f_{\kappa_2,\alpha}(\kappa_1) = 1$, reflecting this, we can find a W-large set of ν's such that for every $\alpha \in \phi''\pi_{nor}(\nu)$, $f_{\kappa,\alpha}(\nu) = 1$. And by definition of A_α, $\nu \in A_\alpha$. ⊣

Denote $B_0 := B \cap C \cap E \cap R$. In order to extend $d \restriction \nu_0 \times X_1$, we will need to pick ν_0 high enough in B_0, but also the next point $\nu_1 \in B_0 \setminus \nu_0 + 1$ in the Prikry sequence such that it will belong to all A_τ with $\tau \in X_1$ and in addition the relevant Cohen functions over ν_1 extend $d \restriction \nu_0 \times X_1$.

Let us look at B_0 more carefully. Let $\underset{\sim}{B_0}$ be its name in V. We fix a condition $m_0 \in G_\kappa * F_\kappa$ which forces that if $\nu_0 \in \underset{\sim}{B_0}$ then the properties of Claims 2.15 and 2.16 hold, namely there is $d \in \mathrm{Cohen}(\nu_0, \underset{\sim}{\phi}''\nu_0) \cap \underset{\sim}{\mathcal{A}}$ which is extended by $\langle \underset{\sim}{h}_{\nu_0, s_\tau(\nu_0)} \mid \nu_0 \in \phi''\nu_0\rangle$, and $\forall\alpha \in \phi''\pi_{nor}(\nu_0)$. $\nu_0 \in \underset{\sim}{A}_\alpha$. Recall that by the construction of G_{κ_2}, we have $m_0 \in G_{\kappa_2} * F_{\kappa_2}$. Let $m_0 \leq t \in G_{\kappa_2} * F_{\kappa_2}$ be a condition such that

$$(1) \quad t \Vdash \kappa_1 \in j_2(\underset{\sim}{B_0}).$$

By the construction of $G_{\kappa_2} * F_{\kappa_2}$, t has the form:

$$t = \langle t_{<\kappa}, t_\kappa, t_{(\kappa,\kappa_1)}, \underbrace{\langle t^0_{\kappa_1}, t^1_{\kappa_1}\rangle}_{t_{\kappa_1}}, t_{(\kappa_1,\kappa_2)}, t_{\kappa_2}\rangle.$$

Since $f_{\kappa_2, j_2(\alpha)}(\kappa_1) = 1$ for every $\alpha < \kappa^+$, this will hold for every $\alpha \in \phi''\kappa$ as well. Also, recall that $Y \in V$, hence $\phi \in V$. Thus $j_2(\phi) \in M_2$ and $j_2(\phi)''\kappa \in M_2$. Also, for $(t_{\kappa_2})_{G_{\kappa_2}} \in M_2[G_{\kappa_2}]$,

$$j_2''\kappa^+ \cap \mathrm{Supp}((t_{\kappa_2})_{G_{\kappa_2}}) \in M_2[G_{\kappa_2}]$$

and $(t_{\kappa_2})_{G_{\kappa_2}} \restriction \kappa \times \{j_2(\alpha)\} \subseteq f_{\kappa,\alpha}$. We also fix $\theta < \kappa^+$ such that $\mathrm{Supp}((t_{\kappa_2})_{G_{\kappa_2}}) \subseteq j_2(\theta)$, there is such θ since $j_2''\kappa^+$ is unbounded in $j_2(\kappa^+)$. Therefore, we can extend if necessary t such that

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$(2a)$ $t_{<\kappa_2} \Vdash (\kappa \cup \{\kappa_1\}) \times j_2(\phi)''\kappa \subseteq \mathrm{dom}(t_{\kappa_2}) \wedge (0, \kappa_1) \in \mathrm{dom}(t_{\kappa_2}) \wedge \mathrm{Supp}(t_{\kappa_2}) \subseteq j_2(\theta)$,

$(2b)$ $t_{<\kappa_2} \Vdash\ t_{\kappa_2}(\kappa_1, \alpha) = 1,$ for every $\alpha \in j_2(\phi)''\kappa$ and $t_{\kappa_2,\kappa_1}(0) = \kappa$,

$(2c)$ $t_{<\kappa_2} \Vdash t_{\kappa_2, j_2(\alpha)} \restriction \kappa = \underset{\sim}{f}_{\kappa,\alpha}$ for every $j_2(\alpha) \in j_2''\kappa^+ \cap \mathrm{Supp}(t_{\kappa_2})$.

Next consider $t_{\kappa_1} = \langle t^0_{\kappa_1}, t^1_{\kappa_1} \rangle$; it is a $\mathcal{P}_{\kappa_1}$-name for a condition in $F_{\kappa_1} \times H_{\kappa_1}$. By the construction of the generic $F_{\kappa_1} \times H_{\kappa_1}$, for every $\alpha < \kappa^+$, we made sure that $h_{\kappa_1, j_1(\alpha)} \restriction \kappa = f_{\kappa,\alpha}$. Also, $j_1(\phi)''\kappa \in M_2$. Let

$$\mu_1 = (j_1 \restriction \phi''\kappa)^{-1} \in M_1.$$

Note that for every $\beta < \kappa^+$, $j_1(s_\beta) = s_{j_1(\beta)} : \kappa_1 \to \kappa_1$ is the canonical function for $j_1(\beta)$ defined in M_U, hence $j_2(s_\beta)(\kappa_1) = k(s_{j_1(\beta)})(\kappa_1) = j_1(\beta)$. Hence

$$\mathrm{dom}(\mu_1) = j_1(\phi)''\kappa = \{s_\gamma(\kappa_1) \mid \gamma \in j_2(\phi)''\kappa\},\ rg(\mu_1) = \phi''\kappa \subseteq \kappa^+.$$

Extend if necessary $t_{<\kappa_1}$, and assume that

$$(3)\ \ t_{<\kappa_1} \Vdash \kappa \times j_1(\phi)''\kappa \subseteq \mathrm{dom}(t^1_{\kappa_1}) \wedge \forall j_1(\alpha) \in j_1(\phi)''\kappa,\ t^1_{\kappa_1, j_1(\alpha)} \restriction \kappa = \underset{\sim}{f}_{\kappa,\alpha}.$$

As for the lower part, due to the Easton support, we have

$$(4)\ \ t_{<\kappa} \in V_\kappa.$$

Fix functions r, Γ_1 which represents t, μ resp. in the ultrapower M_{U^2}, namely $j_2(r)(\kappa, \kappa_1) = t,\ \ j_2(\Gamma_1)(\kappa, \kappa_1) = \mu$. Without loss of generality, suppose that for every (ν', ν), it takes the form

$$r(\nu', \nu) = \langle r_{<\nu'} r_{\nu'}, r_{(\nu',\nu)}, \langle r^0_\nu, r^1_\nu \rangle, r_{(\nu,\kappa)}, r_\kappa \rangle.$$

Reflecting some of the properties of t we obtain a set $B' \in U^2$ such that for every $(\nu', \nu) \in B'$:

$(1)_{(\nu',\nu)}$ $r(\nu', \nu) \Vdash \nu \in B_0$.
$(2a)_{(\nu',\nu)}$ $r_{<\kappa} \Vdash (\nu' \cup \{\hat{\nu}\}) \times \phi''\nu' \subseteq \mathrm{dom}(r_\kappa) \wedge \langle 0, \nu \rangle \in \mathrm{dom}(r_\kappa) \wedge \mathrm{Supp}(r_\kappa) \subseteq \theta$.
$(2b)_{(\nu',\nu)}$ $r_{<\kappa} \Vdash \forall \alpha \in \phi''\nu'. r_{\kappa,\alpha}(\nu) = 1$ and $r_{\kappa,\nu}(0) = \nu'$.
$(3)_{(\nu',\nu)}$ $r_{<\nu} \Vdash \nu' \times \mathrm{dom}(\Gamma_1(\nu', \nu)) \subseteq \mathrm{dom}(r^1_\nu)$ and for every $\beta \in \mathrm{dom}(\Gamma_1(\nu', \nu))$, $r^1_{\nu,\beta} \restriction \nu' = \underset{\sim}{f}_{\nu', \Gamma_1(\nu',\nu)(\beta)}$.
$(4)_{(\nu',\nu)}$ $r_{<\nu'} = t_{<\kappa} \in V_{\nu'}$.

Let

$$B'' = \{\nu \mid \exists (\nu', \nu) \in B'. r(\nu', \nu) \in G_\kappa * F_\kappa\}.$$

Since $B' \in U^2$ we have that $(\kappa, \kappa_1) \in j_2(B')$ and since $j_2(r)(\kappa, \kappa_1) = t \in j_2^*(G_\kappa * F_\kappa) = G_{\kappa_2} * F_{\kappa_2}$, we conclude that $B'' \in W$. Also, $B'' \subseteq B_0$ by clause (1).

We proceed by a density argument, recalling that by the definition of G_2, we have that $\langle t_{<\kappa}, t_\kappa \rangle \in G_\kappa * F_\kappa$.

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CLAIM 2.17. *Let D be the set of all conditions $q \in \mathcal{P}_{\kappa+1}$, such that exists $(\nu_0', \nu_0), (\nu_1', \nu_1) \in B'$, $\nu_1' > \nu_0$ and a $\mathcal{P}_{\nu_0}$-name $\underset{\sim}{d}^{\nu_0}$ such that*:

(a) $r(\nu_0', \nu_0), r(\nu_1', \nu_1) \leq q$.
(b) $q \Vdash \underset{\sim}{d}^{\nu_0} \in \underset{\sim}{A} \cap \mathrm{Cohen}(\nu_0, \phi'' \nu_0)$.
(c) $q \Vdash \forall \tau \in X_1^{\nu_0}.\underset{\sim}{h}_{\nu_1, s_\tau(\nu_1)} \restriction \nu_0 = \underset{\sim}{d}_\tau^{\nu_0}$.

Then D is dense (open) above $\langle t_{<\kappa}, t_\kappa \rangle$ and thus $D \cap G_\kappa * F_\kappa \neq \emptyset$.

PROOF. Work in V, and let $\langle t_{<\kappa}, t_\kappa \rangle \leq p := \langle p_{<\kappa}, p_\kappa \rangle \in \mathcal{P}_{\kappa+1}$. We will define two extensions $p \leq q \leq q^*$ which corresponds to the choice of $(\nu_0', \nu_0), (\nu_1', \nu_1)$ such that $q^* \in D$. By definition of $\mathcal{P}_{\kappa+1}$, $p_{<\kappa} \Vdash p_\kappa \in \mathrm{Cohen}(\kappa, \kappa^+)$, by κ –cc of $\mathcal{P}_\kappa$, for some $Z \subseteq \kappa^+$, $Z \in V$, $|Z| < \kappa$ and some $\gamma < \kappa$, $p_{<\kappa} \Vdash \mathrm{dom}(p_\kappa) \subseteq \gamma \times Z$. Applying j_2, we have that

$$j_2(p_{<\kappa}) = p_{<\kappa} \Vdash \mathrm{dom}(j_2(p_\kappa)) \subseteq j_2(\gamma \times Z) = \gamma \times j_2'' Z \text{ and } j_2(p_\kappa)_{j_2(\alpha)} = p_{\kappa,\alpha} \geq t_{\kappa,\alpha}.$$

Combining with $(2c)$, we have both

$$p_{<\kappa} \Vdash Z \supseteq \mathrm{Supp}(t_\kappa) \wedge \forall \beta \in Z. j_2(p_\kappa)_{j_2(\beta)} \geq t_{\kappa,\beta},$$

$$t_{<\kappa_2} \Vdash \forall j_2(\tau) \in \mathrm{Supp}(t_{\kappa_2}) \cap j_2(Z). t_{\kappa_2, j_2(\tau)} \restriction \gamma = \underset{\sim}{f}_{\kappa,\tau} \restriction \gamma.$$

To reflect this, denote $\mu = (j_2 \restriction (Z \cup \theta))^{-1} \in M_2$, then

$$\mathrm{dom}(\mu) = j_2(Z) \cup j_2'' \theta, \ \mathrm{rng}(\mu) = Z \cup \theta, \ \mu \text{ is 1–1},$$

and we can reformulate

$$p_{<\kappa} \Vdash \mu'' j_2(Z) \supseteq \mathrm{Supp}(t_\kappa) \wedge \forall \beta \in j_2(Z). j_2(p_\kappa)_\beta \geq t_{\kappa,\mu(\beta)},$$

$$t_{<\kappa_2} \Vdash \forall \tau \in \mathrm{Supp}(t_{\kappa_2}) \cap j_2(Z). t_{\kappa_2,\tau} \restriction \gamma = \underset{\sim}{f}_{\kappa,\mu(\tau)} \restriction \gamma.$$

Also, since we can find $\delta < \kappa$ such that $t_{<\kappa} \Vdash \phi''(\delta, \kappa) \cap Z = \emptyset$. There exists such δ since $|Z| < \kappa$, $t_{<\kappa} \Vdash |\mathrm{Supp}(t_\kappa)| < \kappa$ and by κ-cc of $\mathcal{P}_\kappa$. Recall that by the definition of μ_1, $\phi''(\delta, \kappa) = \mu_1''\{s_\gamma(\kappa_1) \mid \gamma \in j_2(\phi)''(\delta, \kappa)\}$ and that $\mu'' \mathrm{Supp}(j_2(p_\kappa)) = Z$. Therefore in M_2 we will have that

$$p_{<\kappa} \Vdash [\mu_1''\{s_\gamma(\kappa_1) \mid \gamma \in j_2(\phi)''(\delta, \kappa)\}] \cap [\mu'' \mathrm{Supp}(j_2(p_\kappa))] = \emptyset.$$

Let Γ be such that $j_2(\Gamma)(\kappa, \kappa_1) = \mu$, and there is a set $\overline{B}_0 \subseteq B'$, $\overline{B}_0 \in U^2$ such that for every $(\nu', \nu) \in \overline{B}_0$,

$$(i) \ \ p_{<\kappa} \Vdash \Gamma(\nu', \nu)'' Z \supseteq \mathrm{Supp}(r_{\nu'}) \wedge \forall \beta \in Z. p_{\kappa,\beta} \geq r_{\nu',\Gamma(\nu',\nu)(\beta)},$$

$$(ii) \ \ r_{<\kappa} \Vdash \forall \tau \in Z \cap \mathrm{Supp}(r_\kappa). r_{\kappa,\tau} \restriction \gamma = \underset{\sim}{f}_{\nu',\Gamma(\nu',\nu)(\tau)} \restriction \gamma,$$

$$(iii) \ \ p_{<\kappa} \Vdash \Gamma_1(\nu', \nu)''\{s_\gamma(\nu) \mid \gamma \in \phi''(\delta, \nu')\} \cap [\Gamma(\nu', \nu)'' \mathrm{Supp}(p_\kappa)] = \emptyset.$$

Let us move to the choice of $(\nu_0', \nu_0), (\nu_1', \nu_1)$. In $V[G_\kappa * F_\kappa]$, there exists $(\nu_0^0, \nu_0), (\nu_1^0, \nu_1) \in \overline{B}_0$ such that $r(\nu_0^0, \nu_0), r(\nu_1^0, \nu_1) \in G_\kappa * F_\kappa$ (hence they are compatible) such that $\nu_0^0 > \delta, \gamma, \sup(\mathrm{Supp}(p_{<\kappa}))$ and $\nu_1^0 > \nu_0, \mathrm{Supp}(r_{<\kappa}(\nu_0^0, \nu_0))$.

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In particular, in V we can find $(\nu_0', \nu_0), (\nu_1', \nu_1) \in \overline{B}_0$ such that $r(\nu_0', \nu_0), r(\nu_1', \nu_1)$ are compatible, $\nu_0' > \delta, \gamma, \sup(\mathrm{Supp}(p_{<\kappa}))$, and $\nu_1' > \nu_0, \sup(\mathrm{Supp}(r_{<\kappa}(\nu_0', \nu_0)))$. Denote

$$r^0 := r(\nu_0', \nu_0) = \langle r^0_{<\nu_0'}, r^0_{\nu_0'}, r^0_{(\nu_0',\kappa)}, r^0_\kappa \rangle,$$

$$r^1 := r(\nu_1', \nu_1) = (r^1_{<\nu_1'}, r_{\nu_1'}, r_{(\nu_1',\nu_1)}, \langle r^{0,1}_{\nu_1}, r^{1,1}_{\nu_1} \rangle, r^1_{(\nu_1,\kappa)}, r^1_\kappa \rangle.$$

Let us define the first extension q, and it has the form:

$$q = p_{<\kappa}{}^\frown q_{\nu_0'}{}^\frown r^0_{(\nu_0',\kappa)}{}^\frown q_\kappa.$$

First, $q_{\nu_0'}$ is a $\mathcal{P}_{\nu_0'}$-name for a condition with $\mathrm{Supp}(q_{\nu_0'}) = \Gamma(\nu_0', \nu_0)''Z$, by (i) $\mathrm{Supp}(q_{\nu_0'}) \supseteq \mathrm{Supp}(r^0_{\nu_0'})$. Set $q_{\nu_0', \Gamma(\nu_0',\nu_0)(\beta)} = p_{\kappa,\beta}$. As for q_κ, we set it to be a $\mathcal{P}_\kappa$-name for $r^0_\kappa \cup p_\kappa$.

Once we will prove that $p_{<\kappa}, r^0_{<\kappa} \leq q_{<\kappa}$, from (i) and (ii) it will follow that $q_{<\kappa}$ forces q_κ to be a partial function. Indeed, for every $\beta \in \mathrm{Supp}(r^0_\kappa) \cap Z$, $q_{<\kappa}$ will force

$$r^0_{\kappa,\beta} \restriction \gamma = \underset{\sim}{f}_{\nu', \Gamma(\nu_0',\nu_0)(\beta)} \restriction \gamma \geq q_{\nu_0', \Gamma(\nu_0',\nu_0)(\beta)} = p_{\kappa,\beta}.$$

Clearly $p \leq q$. To see that $r^0 \leq q$, up to ν_0', we have that by $(4)_{(\nu_0',\nu_0)}$ that

$$q_{<\nu_0'} = p_{<\kappa} \geq t_{<\kappa} = r^0_{<\nu_0'}.$$

At ν_0', if $\alpha = \Gamma(\nu_0', \nu_0)(\beta)$, then (i) insures that $q_{\nu_0',\alpha} = p_{\kappa,\beta} \geq r^0_{\nu',\alpha}$. Since in the interval (ν_0', κ), q and r^0 are the same, it follows that $q_{<\kappa} \geq r^0_{<\kappa}$ and at κ it is clear that $q_{<\kappa} \Vdash r^0_\kappa \leq q_\kappa$.

Next let us move to the choice of $\underset{\sim}{d}^{\nu_0}$. Since $r^0 \leq q$ and $m_0 \leq \langle t_{<\kappa}, t_\kappa \rangle \leq q \Vdash \nu_0 \in \underset{\sim}{B}_0$, use the maximality principal to find a $\mathcal{P}_{\nu_0}$-name, $\underset{\sim}{d}^{\nu_0}$ such that q forces (b).[5]

Define the final condition $q \leq q^*$,

$$q^* = q_{<\kappa}{}^\frown q^*_{\nu_1'}{}^\frown r^1_{(\nu_1',\kappa)}{}^\frown q^*_\kappa.$$

The crucial point here is that by $(2b)_{(\nu_1',\nu_1)}$,

$$r^0_{<\kappa} \Vdash \nu_0^0 = \underset{\sim}{f}_{\kappa,\nu_0}(0) = r^0_{\kappa,\nu_0}(0) = \nu_0'$$

and since $r^0 \Vdash \nu_0 \in \underset{\sim}{R}$ we have that $r^0 \Vdash X_1^{\nu_0} \subseteq \phi''(\nu_0', \nu_0) \subseteq \phi''(\nu_0', \nu_1')$. By (iii) we have that $q_{<\kappa} \Vdash [\Gamma_1(\widetilde{\nu_1'}, \nu_1)''\{s_\gamma(\nu_1) \mid \gamma \in X_1^{\nu_0}\}] \cap [\Gamma(\nu_1', \nu_1)''Z] = \emptyset$. This will permit to code d^{ν_0}, let

$$\mathrm{Supp}(q^*_{\nu_1'}) = [\Gamma_1(\nu_1', \nu_1)''\{s_\gamma(\nu_1) \mid \gamma \in X_1^{\nu_0}\}] \uplus [\Gamma(\nu_1', \nu_1)''Z]$$

and

$$q^*_{\nu_1',\alpha} = \begin{cases} q_{\kappa,\beta}, & \exists \beta \in \Gamma(\nu_1', \nu_1)''Z.\alpha = \Gamma(\nu_1', \nu_1)(\beta), \\ \underset{\sim}{d}^{\nu_0}_\tau, & \exists \tau \in X_1^{\nu_0}.\alpha = \Gamma_1(\nu_1', \nu_1)(s_\tau(\nu_1)), \end{cases}$$

[5] Since the tail forcing $\mathcal{P}_{[\nu_0,\kappa]}$ is ν_0-closed, if there is such $d^{\nu_0} \in V[G_\kappa * F_\kappa]$ then $|d^{\nu_0}| < \nu_0$, hence $d^{\nu_0} \in V[G_{\nu_0}]$.

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and $q^*_\kappa = q_\kappa \cup r^1_\kappa$. Note that if $\tau \in \mathrm{Supp}(q_\kappa) \cap \mathrm{Supp}(r^1_\kappa)$ then either $\tau \in \mathrm{Supp}(r^0_\kappa) \cap \mathrm{Supp}(r^1_\kappa)$, and r^0_κ, r^1_κ are forced to be compatible by $q_{<\kappa}$ and if $\tau \in Z \cap \mathrm{Supp}(r^1_\kappa)$ then the same argument as before works. We conclude that $r^0 \leq q \leq q^*$, $r^1 \leq q^*$, namely (a). Finally, for every $\tau \in X_1^{\nu_0}$, $s_\tau(\nu_1) \in \mathrm{dom}(\Gamma_1(\nu_1', \nu_1))$ and by $(3)_{(\nu_1',\nu_1)}$ we have that q^* forces that

$$\underset{\sim}{h}_{\nu_1, s_\tau(\nu_1)} \restriction \nu_0 = \underset{\sim}{f}_{\nu_1', \Gamma_1(\nu_1',\nu_1)(s_\tau(\nu_1))} \restriction \nu_0 \geq q^*_{\nu_1', \Gamma_1(\nu_1',\nu_1)(s_\tau(\nu_1))} = \underset{\sim}{d}^{\nu_0}_\tau.$$

Then $p \leq q^*$ and $q^* \in D$. $\dashv$

By density, we can find such a condition $p^* \in G_\kappa * F_\kappa \cap D$ and points $(\nu_0', \nu_0), (\nu_1', \nu_1) \in B'$ witnessing $p^* \in D$. It follows that $r(\nu_0', \nu_0), r(\nu_1', \nu_1) \in G_\kappa * F_\kappa$, and by $(1)_{(\nu_0',\nu_0)}, (1)_{(\nu_1',\nu_1)}$, $\nu_0, \nu_1 \in B_0$. Extend $\langle\langle\rangle, B\rangle$ by $p^* = \langle \nu_0, \nu_1, B_0 \cap (\cap_{\tau \in \phi''\nu_0} A_\tau\rangle \setminus \nu_1 + 1\rangle$. By $(2b)_{(\nu_1',\nu_1)}$, for every $\tau \in \phi''\nu_0 \subseteq \phi''\nu_1'$, $f_{\kappa,\tau}(\nu_1) = r_{\kappa,\tau}(\nu_1) = 1$, hence $\nu_1 \in \cap_{\tau \in \phi''\nu_0} A_\tau$ and $p^* \Vdash n_\tau = \begin{cases} 0, & \tau \in X_0, \\ 1, & \tau \in X_1. \end{cases}$ In other words, since $\nu_0 \in B_0$,

$$p^* \Vdash \forall \tau \in X_0 . \underset{\sim}{f}^* \tau \restriction \nu_0 = h_{\nu_0, s_\tau(\nu_0)},$$

$$p^* \Vdash \forall \tau \in X_1 . \underset{\sim}{f}^* \tau \restriction \nu_1 = h_{\nu_1, s_\tau(\nu_1)}.$$

Let $d = (\underset{\sim}{d}^{\nu_0})_{G_{\nu_0}} \in \mathrm{Cohen}(\nu_0, \phi''\nu_0) \cap \mathcal{A}$, it follows that $p^* \Vdash \forall \tau \in X_0 . \underset{\sim}{f}^*_\tau$ extends d_τ, and by (c) of the definition of D, $p^* \Vdash \forall \tau \in X_1 \underset{\sim}{f}^*_\tau$ extends d_τ. Thus $p^* \Vdash d \in \underset{\sim}{G}^* \cap \mathcal{A}$. This concludes the genericity proof. $\dashv$

$\dashv$

§3. The results where $2^\kappa = \kappa^{++}$.

3.1. Strong non-Galvin witnesses of length $2^\kappa = \kappa^{++}$. In this section we produce a model with a non-Galvin ultrafilter with a strong witnessing sequence of length $2^\kappa = \kappa^{++}$. This will of course require to violate GCH on a measurable cardinal and in turn to start with a stronger large cardinal assumption (see [15, 32]). We will follow a similar construction to the one given in the case of κ^+ addressed in previous sections. Indeed, instead of iterating $\mathrm{Cohen}(\alpha, \alpha^+)$ we will iterate $\mathrm{Cohen}(\alpha, \alpha^{++})$ aiming to force $\mathrm{Cohen}(\kappa, \kappa^{++})$, from which we will be able to define a non-Galvin ultrafilter and a strong witness of length κ^{++} in a similar fashion to the one we have on κ^+, distinguishing between α's which are in the image of the second iteration and those which are in the image of the factor map. The difficulty is, as always, to extend a ground model embedding. By the large cardinal lower bound, we can no longer work with an ultrapower by an ultrafilter. The usual embedding to lift in the context of violation of GCH at measurables is a (κ, κ^{++})-extender ultrapower embedding, which we will use here. This makes the lifting argument more involved and the existence of generic filters for the iteration requires variations of *Woodin's surgery method* (see [12, Section 25]).

THEOREM 3.1. *Assume GCH and that there is a (κ, κ^{++})-extender over κ in V. Then there is a cofinality preserving forcing extension V^* such that $V^* \models 2^\kappa = \kappa^{++}$, and in V^* there is a κ-complete ultrafilter W over κ which concentrates on regulars, extends Cub_κ, and has a strong witness of length κ^{++} for the failure of Galvin's property.*

Proof. Let E be a (κ,κ^{++})-extender. Let $j_1 = j_E : V \to M_E =: M_1$ be its ultrapower embedding with $crit(j_E) = \kappa$ and ${}^{\kappa}M_E \subseteq M_E$. Denote by E_α the ultrafilter

$$E_\alpha := \{X \subseteq \kappa \mid \alpha \in j_E(X)\}.$$

Denote $U := E_\kappa$ the normal ultrafilter and let $k : M_U \to M_E$ be the factor map defined by setting $k(j_U(f)(\kappa)) = j_E(f)(\kappa)$ such that $j_E = k \circ j_U$. Define an Easton support iteration $\langle \mathcal{P}_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa+1, \beta \leq \kappa\rangle$ as follows:

$\underset{\sim}{Q}_\beta$ is trivial unless β is inaccessible, in which case $Q_\beta = \mathrm{Cohen}(\beta,\beta^{++})$.

Let $G_{\kappa+1} := G_\kappa * g_\kappa$ be a V-generic subset of $\mathcal{P}_{\kappa+1} = \mathcal{P}_\kappa * \underset{\sim}{Q}_\kappa$. Keeping similar notations to those from previous sections, let $\langle f_{\kappa,\alpha} \mid \alpha < \kappa^{++}\rangle$ be the Cohen generic functions from κ to 2 introduced by g_κ.

Now we apply Woodin's argument (see [12, Section 25], and [10] for constructing generics without additional forcing) to see that there will be $G_{j_E(\kappa)+1} * H^* \subseteq j_E(\mathcal{P}_{\kappa+1}) * \mathbb{S}_0$ in $V_1^* := V[G_{\kappa+1}][H]$, where $H \subseteq \mathbb{S}_0$ is a $V[G_{\kappa+1}]$-generic filter, where $\mathbb{S}_0$ is some κ^+-distributive in $V[G_{\kappa+1}]$ (in the case of Ben-Shalom, there is no need for H^* and we can work directly in $V[G_{\kappa+1}]$) generic over M_E and an elementary embedding

$$j_1^* : V_1^* \to M_E[G_{j_E(\kappa)+1} * f^*]$$

which extends j_1. Recall that the generic filter constructed for $j_1(Q_\kappa)$ is obtained by a surgery argument, making small changes on an $M_1[G_{j_1(\kappa)}]$-generic filter f to be compatible with $j_1''g_\kappa$. For our purposes, we need some additional changes to be made; for every $p \in f$ we change p to p^* such that $\mathrm{dom}(p^*) = \mathrm{dom}(p)$ and

$$p^*(\langle\gamma,\alpha\rangle) = \begin{cases} f_\beta(\gamma), & \gamma < \kappa \wedge \alpha = j_1(\beta),\\ \beta, & \gamma = \kappa \wedge \alpha = j_1(\beta),\\ p(\langle\gamma,\alpha\rangle), & \text{else.}\end{cases}$$

To see that p was only changed at κ-many places, find $a \in [\kappa^{++}]^{<\omega}$ such that $j_E(P)(a) = p$, where $P : \kappa^{|a|} \to Q_\kappa$. By elementarity, for every $\langle\alpha, j_1(\beta)\rangle \in \kappa \times j_1''\kappa^{++} \cap \mathrm{dom}(p)$, there is $x \in [\kappa]^{|a|}$ such that $\langle\alpha,\beta\rangle \in \mathrm{dom}(P(x))$. It follows that $|\kappa \times j_1''\kappa^{++} \cap \mathrm{dom}(p)| \leq \kappa$. Moreover, $|\{\kappa\} \times j_1''\kappa^{++} \cap \mathrm{dom}(p)| \leq \kappa$, since otherwise there would be some $\alpha < \kappa^{++}$ such that

$$\mathrm{cf}(\alpha) = \kappa^+ \text{ and } \sup\{j_E(\beta) \mid \langle\kappa, j_E(\beta)\rangle \in \mathrm{dom}(p)\} = j_E(\alpha).$$

But $|\mathrm{dom}(p)|^{M_1} < j_1(\kappa)$ and $\mathrm{cf}^{M_1}(j_1(\alpha)) = j_1(\kappa)^+$ which is a contradiction. Hence $p^* \in M_1[G_{j_1(\kappa)}]$ since we have only changed p at κ-many values and ${}^{\kappa}M_1[G_{j_1(\kappa)}] \subseteq M_1[G_{j_1(\kappa)}]$.

The argument that such changes do not affect the genericity is the same as in [12]. So we additionally obtain that $f_{\kappa_1, j_1(\beta)}(\kappa) = \beta$, for every $\beta < \kappa^{++}$.

We also claim that j_1^* is actually the ultrapower embedding by the normal ultrafilter

$$U^* = \{X \subseteq \kappa \mid \kappa \in j_1^*(X)\}$$

extending U. To see this, consider $k^* : M_{U^*} \to M_1[G_{j_1(\kappa)+1} * H^*]$ defined by $k^*([f]_{U^*}) = j_1^*(f)(\kappa)$, which is clearly elementary. To see that $k^* = id$, let us prove that k^* is onto. Fix $A = (\underset{\sim}{A})_{G_{j_1(\kappa)+1} * H^*} \in M_1[G_{j_1(\kappa)+1}]$ and let $f \in V$, $a = \{\alpha_1, ..., \alpha_r\} \in [\kappa^{++}]^{<\omega}$ be such that $j_1(f)(a) = \underset{\sim}{A}$. Define in $V[G_{\kappa+1}]$ the function $f^*(x) = (f(\{f_{\alpha_1}(x), ..., f_{\alpha_r}(x)\}))_{G_{\kappa+1}*H}$. Then

$$k^*(j_{U^*}(f^*)(\kappa)) = j_1^*(f^*)(\kappa) = (j_1(f)(\{j_1^*(f_{\alpha_1})(\kappa), ..., j_1^*(f_{\alpha_2})(\kappa)\}))_{G_{j_1(\kappa)+1}*H^*}$$

$$= (j_1(f)(a))_{G_{j_1(\kappa)+1}} = (\underset{\sim}{A})_{G_{j_1(\kappa)+1}*H^*} = A.$$

We would like now to construct a κ-complete ultrafilter $W \in V[G_{\kappa+1}]$ over κ which includes Cub_κ and the family $\langle A_\alpha \mid \alpha < \kappa^{++}\rangle$ which is a strong witness that W fails to satisfy the Galvin Property. Set

$$A_\alpha := \{\nu < \kappa \mid f_\alpha(\nu) \text{ is odd}\},$$

for every $\alpha < \kappa^{++}$.

Consider the second ultrapower (of V) by E, i.e., $\mathrm{Ult}(M_E, j_E(E))$. In order to simplify the notation let us denote M_E by M_1 and $\mathrm{Ult}(M_1, j_1(E))$ by M_2 and $j_{2,1} := j_{j_1(E)} : M_1 \to M_2$. Also, let $\kappa_1 = j_1(\kappa), E_1 = j_1(E)$, and $\kappa_2 = j_{2,1}(\kappa_1)$. Let $j_2 : V \to M_2$ be the composition of j_1 with $j_{2,1}$.

Work in $M_1[G_{\kappa_1+1} * H^*]$, and apply there the Woodin argument to E_1. There will be $G_{\kappa_2+1} * H^{**} \subseteq j_2(P_\kappa * Q_\kappa * \mathbb{S}_0)$ (in $M_1[G_{\kappa_1+1} * H^*]$) generic over M_2 and an elementary embedding

$$j_{2,1}^* : M_1[G_{\kappa_1+1} * H^*] \to M_2[G_{\kappa_2+1} * H^{**}]$$

which extends j_{E_1}. Additionally, for every $\alpha < (\kappa_1^{++})^{M_1}$ let us arrange the following:

(1) $f_{\kappa_2, j_{2,1}(\alpha)}(\kappa_1)$ is odd, if $\alpha \in j_E''\kappa^{++}$.
(2) $f_{\kappa_2, j_{2,1}(\alpha)}(\kappa_1)$ is an even, if $\alpha \in (\kappa_1^{++})^{M_1} \setminus j_E''\kappa^{++}$.
(3) $f_{\kappa_2, \kappa_1}(\kappa_1) = \kappa$.

The point being that this requires only small changes of conditions in $(\mathrm{Cohen}(\kappa_2, \kappa_2^{++}))^{M_2}$, and so preserves the genericity.

Namely, given $p \in (\mathrm{Cohen}(\kappa_2, (\kappa_2)^{++}))^{M_2}$, define p^* such that $\mathrm{dom}(p^*) = \mathrm{dom}(p)$ and

$$p^*(\langle \gamma, \alpha\rangle) = \begin{cases} f_{\kappa_1, \beta}(\gamma), & \gamma < \kappa_1 \wedge \exists \beta < \kappa_1^{++} \alpha = j_{2,1}(\beta), \\ \beta \cdot 2 + 1, & \gamma = \kappa_1 \wedge \exists \beta \in j_1''\kappa^{++}. \alpha = j_{2,1}(\beta), \\ \beta \cdot 2, & \gamma = \kappa_1 \wedge \exists \beta \in \kappa_1^{++} \setminus j_1''\kappa^{++}. j_{2,1}(\beta) = \alpha, \\ \kappa, & \gamma = \alpha = \kappa_1, \\ p(\langle \gamma, \alpha \rangle), & \text{otherwise}. \end{cases}$$

In $V[G_{\kappa+1} * H]$, $|\mathrm{Supp}(p) \cap j_2^{*\prime\prime}\kappa^{++}| \leq \kappa$ and $M_1[G_{\kappa_1+1} * H^*]$ is closed under κ-sequences, hence $p^* \in M_1$. The argument we have seen before applied in $M_1[G_{\kappa_1+1} * H^*]$ shows that

$$M_1[G^*_{\kappa_1+1}] \models |\mathrm{dom}(p) \cap (\kappa_1 + 1) \times j_{2,1}''(\kappa_1^{++})^{M_1[G_{\kappa_1+1}]}| \leq \kappa_1.$$

This implies that $p^* \in M_2[G_{\kappa_2+1} * H^{**}]$ since $M_2[G_{\kappa_2+1} * H^{**}]$ is closed under κ_1-sequences from $M_1[G_{\kappa_1+1} * H^*]$. Then the embedding $j_2 : V \to M_2$ extends to

$$j_2^* : V[G_{\kappa+1} * H^*] \to M_2[G_{\kappa_2+1} * H^{**}].$$

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Define now

$$W = \{X \subseteq \kappa \mid \kappa_1 \in j_2^*(X)\}.$$

CLAIM 3.2. (1) $j_W = j_2^*$, $[id]_W = \kappa_1$, $U^* \leq_{R\text{-}K} W$.
(2) $Cub_\kappa \subseteq W$, $\{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \alpha)\} \in W$.
(3) *The sequence* $\langle A_\alpha \mid \alpha < \kappa^{++}\rangle$ *is a strong witness for* $\neg Gal(W, \kappa, \kappa^{++})$, *where*

$$A_\alpha := \{\nu < \kappa \mid f_{\kappa,\alpha}(\nu) \text{ is odd}\}.$$

PROOF. Indeed $Cub_\kappa \subseteq W$ and $\{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \alpha\} \in W$, is the same as in Claim 2.8 from the last section. To see (1), we let $k_W : M_W \to M_2[j_2^*(G)]$ be the usual factor map $k_W([f]_W) = j_2^*(f)(\kappa_1)$ and we prove that $k_W = id$ by proving that k_W is onto. Let $A \in M_2[G_{\kappa_2+1} * H^{**}]$, then $A = (\underset{\sim}{A})_{G_{\kappa_2+1}*H^{**}}$ where $\underset{\sim}{A} \in M_2$ is a $\mathcal{P}_{\kappa_2+1} * j_2(\mathbb{S}_0)$-name. Since $j_{2,1}$ is a $(\kappa_1, \kappa_1^{++})$-extender ultrapower, there is $f \in M_1$ and $a \in [\kappa_1^{++}]^{<\omega}$ such that $\underset{\sim}{A} = j_{2,1}(f)(a)$. Suppose that $a = \{\alpha_1, ..., \alpha_n\}$ is an increasing enumeration. Then by construction, $f_{\kappa_2, j_{2,1}(\alpha_i)}(\kappa_1) \in \{\alpha_i \cdot 2, \alpha_i \cdot 2 + 1\}$. In particular we derive α_i from $f_{\kappa_2, j_{2,1}(\alpha_i)}(\kappa_1)$ [6]. Define $g_{\alpha_i} : \kappa_1 \to \kappa_1 \in M_1[G_{\kappa_1+1} * H^*]$ by $g_{\alpha_i}(\alpha) = \lfloor \frac{f_{\kappa_1,\alpha_i}(\alpha)}{2} \rfloor$, then $j_{2,1}^*(g_{\alpha_i})(\kappa_1) = \lfloor \frac{f_{\kappa_2, j_{2,1}(\alpha_i)}(\kappa_1)}{2} \rfloor = \alpha_i$. Finally, let $g(\alpha) = f(g_{\alpha_1}(\alpha), ..., g_{\alpha_n}(\alpha))$. Then,

$$j_{2,1}^*(g)(\kappa_1) = j_{2,1}(f)(j_{2,1}^*(g_{\alpha_1})(\kappa_1), ..., j_{2,1}^*(g_{\alpha_n})(\kappa_1)) = j_{2,1}(f)(a) = \underset{\sim}{A}.$$

We already know that $M_1[G_{\kappa_1+1} * H^*]$ is the ultrapower by U^*, hence $g = j_1^*(h)(\kappa)$ for some $h \in V[G_{\kappa+1} * H]$ and in turn $\underset{\sim}{A} = j_2^*(h)(\kappa, \kappa_1)$. Finally, we made sure that κ is expressible by κ_1, so we define in $V[G_{\kappa+1} * H]$ $f^* : \kappa \to \kappa$ by

$$f^*(\alpha) = (h(f_{\kappa,\alpha}(\alpha), \alpha))_G.$$

It follows that

$$k_W([f^*]_W) = j_2(f^*)(\kappa_1) = (j_2^*(h)(f_{\kappa_2,\kappa_1}(\kappa_1), \kappa_1))_{G_{\kappa_2+1}*H^{**}}$$

$$= (j_2^*(h)(\kappa, \kappa_1))_{G_{\kappa_2+1}*H^{**}} = (\underset{\sim}{A})_{G_{\kappa_2+1}*H^{**}} = A;$$

this concludes (1). (2) and (3) are completely analogous to Claim 2.8. ⊣

⊣

3.2. Adding κ^{++}-Cohens using Prikry forcing. The construction of the previous section can be modified to obtain a model in which there is a κ-complete ultrafilter U^* over κ such that Prikry(U^*) adds a generic filter for Cohen(κ, κ^{++}). This will require the violation of SCH and in turn larger cardinals [16, 33].

THEOREM 3.3. *Assume GCH and that E is a* (κ, κ^{++})*-extender in V. Then there is a cofinality preserving forcing extension* V^* *in which* $2^\kappa = \kappa^{++}$ *and a non-Galvin ultrafilter* $W \in V^*$ *such that forcing with* Prikry(W) *introduces a* V^**-generic filter for* Cohen$^{V^*}(\kappa, \kappa^{++})$*-generic filter.*

[6] An easy transfinite induction proves that if an ordinal $\gamma = \beta \cdot 2$ or $\gamma = \beta \cdot 2 + 1$, then β is unique, and we denote $\beta = \lfloor \frac{\gamma}{2} \rfloor$.

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PROOF. Let $j_1 : V \to M_E =: M_1$ be the ultrapower embedding of E with $crit(j_1) = \kappa$ and ${}^{\kappa}M_1 \subseteq M_1$ and $\kappa_1 = j_1(\kappa)$. Denote by E_α the ultrafilter $\{X \subseteq \kappa \mid \alpha \in j_E(X)\}$. As before, denote E_κ by U and let $k : M_U \to M_E$ be defined by setting $k(j_U(f)(\kappa)) = j_E(f)(\kappa)$. Define an Easton support iteration $\langle \mathcal{P}_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa+1, \beta < \kappa\rangle$ as follows:

$\underset{\sim}{Q}_\beta$ is trivial unless β is inaccessible. If $\beta < \kappa$ is inaccessible, then

$$\underset{\sim}{Q}_\beta = \mathrm{LOTT}(\mathrm{Cohen}(\beta, \beta^{++}), \mathrm{Cohen}(\beta, \beta^{++}) \times \mathrm{Cohen}(\beta, \beta^{++})).$$

Over κ, we let $Q_\kappa = \mathrm{Cohen}(\kappa, \kappa^{++})$.

Let $G_{\kappa+1} = \widetilde{G}_\kappa * F_\kappa$ be a V-generic filter of $\mathcal{P}_{\kappa+1}$. We denote by $F_\alpha := \langle f_{\alpha,\gamma} \mid \gamma < \alpha^{++}\rangle$ the generic Cohen function if $\mathrm{Cohen}(\alpha, \alpha^{++})$ was forced in G_κ and by

$$F_\alpha := \langle f_{\alpha,\gamma} \mid \gamma < \alpha^{++}\rangle,\ H_\alpha := \langle h_{\alpha,\gamma} \mid \gamma < \alpha^{++}\rangle$$

if $\mathrm{Cohen}(\alpha, \alpha^{++}) \times \mathrm{Cohen}(\alpha, \alpha^{++})$ was. The elementary embedding j_1 extends to $j_1^* : V[G_{\kappa+1}] \to M_1[G_{\kappa_1+1}]$ such that at κ we forced one block of Cohen's, $\mathrm{Cohen}(\kappa, \kappa^{++})$, and for every $\alpha < \kappa^{++}$,

$$f_{\kappa_1, j_1(\alpha)}(\kappa) = \alpha.$$

Indeed, in the Woodin and Ben-Shalom argument we first build the generic G_{κ_1} up to κ_1 not including κ_1 in the same standard fashion as in [12]. The original construction of Woodin or Ben-Shalom of the Cohen generic F_{κ_1} which is $M_1[G_{\kappa_1}]$-generic for $\mathrm{Cohen}(\kappa_1, \kappa_1^{++})^{M_1[G_{\kappa_1}]}$ applies in our case, as it only uses the fact that $M_1[G_{\kappa_1}]$ is closed under κ-sequences and properties of $\mathrm{Cohen}(\kappa_1, \kappa_1^{++})$. Since

$$\mathrm{Cohen}(\kappa_1, \kappa_1^{++}) \simeq \mathrm{Cohen}(\kappa_1, \kappa_1^{++}) \times \mathrm{Cohen}(\kappa_1, \kappa_1^{++}),$$

we can split the generic F_{κ_1} and assume it is of the form $F_{\kappa_1} \times H_{\kappa_1}$, which is $M_1[G_{\kappa_1}]$-generic for $\mathrm{Cohen}(\kappa_1, \kappa_1^{++}) \times \mathrm{Cohen}(\kappa_1, \kappa_1^{++})$. Work inside $V[G_\kappa * F_\kappa]$, and modify the values of F_{κ_1} and H_{κ_1}, as in the previous section so that for every $\alpha < \kappa^{++}$,

$$f_{\kappa_1, j_1(\alpha)} \restriction \kappa = h_{\kappa_1, j_1(\alpha)\cdot 2+1} \restriction \kappa = f_{\kappa,\alpha}$$

and for every $\alpha < \kappa^{++}$, $f_{\kappa_1, j_1(\alpha)}(\kappa) = \alpha$.

Lift j_1 to the embedding $j_1 \subseteq j_1^* : V[G_{\kappa+1}] \to M_E[G_{\kappa_1} * F_{\kappa_1}]$. Note that H_{κ_1} will be used only later. Set

$$U^* = \{X \subseteq \kappa \mid \kappa \in j_1^*(X)\},$$

then $U \subseteq U^*$ and j_1^* is actually the ultrapower embedding by U^*. Continuing as before, consider the second ultrapower (of V) by E. Denote M_E by M_1 and $\mathrm{Ult}(M_E, j_E(E))$ by M_2, $j_{2,1} = j_{j_1(E)} : M_1 \to M_2$ the ultrapower embedding. Also, let $E_1 = j_1(E)$ and $\kappa_2 = j_{2,1}(\kappa_1)$. Let $j_2 : V \to M_2$ be the composition of j_1 with $j_{2,1}$. The extension of $j_{2,1}$ will be such that at κ_1 we force with $\mathrm{Cohen}(\kappa_1, \kappa_1^{++}) \times \mathrm{Cohen}(\kappa_1, \kappa_1^{++})$ part of the Lottery sum. To realize this, we define in $M_1[G_{\kappa_1} * (F_{\kappa_1} \times H_{\kappa_1})]$ we take the generic G_{κ_1} up to κ_1. At κ_1 we take $F_{\kappa_1} \times H_{\kappa_1}$, then in $M_1[G_{\kappa_1} * (F_{\kappa_1} \times H_{\kappa_1})]$ we construct as in Woodin and Ben-shalom argument in $V[G_\kappa * F_\kappa]$ an $M_2[G_{\kappa_1} * (F_{\kappa_1} \times H_{\kappa_1})]$-generic $G_{(\kappa_1,\kappa_2)} * F_{\kappa_2}$ such that

$j''_{2,1}G_{\kappa_1} * F_{\kappa_1} \subseteq G_{\kappa_2} * F_{\kappa_2}$. Denote by $\langle f_{\kappa_2,\alpha} \mid \alpha < (\kappa_2^{++})^{M_2}\rangle$ the Cohen function induced by F_{κ_2}. We also secure that for every $\alpha < (\kappa_1^{++})^{M_1}$:

(1) $f_{\kappa_2 k(\alpha)}(\kappa_1) = \alpha \cdot 2 + 1$, if $\alpha \in j''_E\kappa^{++}$.
(2) $f_{\kappa_2 k(\alpha)}(\kappa_1) = \alpha \cdot 2$, if $\alpha \in (\kappa_1^{++})^{M_1} \setminus j''_E\kappa^{++}$.
(3) $f_{\kappa_2,\kappa_1}(\kappa_1) = \kappa$.

Formally, given $p \in (\mathrm{Cohen}(\kappa_2, (\kappa_2)^{++}))^{M_2[G_{\kappa_2}]}$, define p^* such that $\mathrm{dom}(p^*) = \mathrm{dom}(p)$ and

$$p^*(\langle\gamma,\beta\rangle) = \begin{cases} f_{\kappa_1,\alpha}(\gamma), & \gamma < \kappa_1 \wedge \beta = k(\alpha), \\ \alpha \cdot 2 + 1, & \gamma = \kappa_1 \wedge \beta = k(\alpha) \wedge \alpha \in j''_E\kappa^{++}, \\ \alpha \cdot 2, & \gamma = \kappa_1 \wedge \beta = k(\alpha) \wedge \alpha \in (\kappa_1^{++})^{M_1} \setminus j''_E\kappa^{++}, \\ \kappa, & \alpha = \gamma = \kappa_1, \\ p(\langle\gamma,\alpha\rangle), & \text{otherwise.} \end{cases}$$

In $V[G_{\kappa+1}]$, $|\,\mathrm{dom}(p) \cap j''_{E^2}\kappa^{++}| \leq \kappa$ and $M_1[G_{\kappa_1+1}]$ is closed under κ-sequences, hence $p^* \in M_1[G_{\kappa_1+1}]$. The argument we have seen before applied in $M_1[G_{\kappa_1+1}]$, thus

$$M_1[G_{\kappa_1+1}] \models |\,\mathrm{dom}(p) \cap (\kappa_1+1) \times j''_{12}(\kappa_1^{++})^{M_1[G_{\kappa_1+1}]}| \leq \kappa_1.$$

This implies that $p^* \in M_2[G_{\kappa_2+1}]$ since $M_2[G_{\kappa_2+1}]$ is closed under κ_1-sequences from $M_1[G_{\kappa_1+1}]$.

Extend in $V[G_\kappa * F_\kappa]$, $j_{2,1} \subseteq j_2^* : M_1[G_{\kappa_1} * F_{\kappa_1} \to M_2[G_{\kappa_2} * F_{\kappa_2}]$ and let $j_2^* : V[G_\kappa * F_\kappa] \to M_2[G_{\kappa_2} * F_{\kappa_2}]$ be the composition $j_{2,1}^* \circ j_1^*$. Note that $j_{2,1}^*$ is definable only in $V[G_\kappa * (F_\kappa]$. Denote by $V[G_\kappa * F_\kappa] = V^*$, define

$$W = \{X \subseteq \kappa \mid \kappa_1 \in j_2^*(X)\} \in V^* \text{ and } A_\alpha = \{\beta < \kappa \mid f_\alpha(\beta) \text{ is odd}\}.$$

Claim 3.4. *W is a κ-complete ultrafilter over κ such that*:

(1) $j_W = j_2^*$, $[id]_W = \kappa_1$, $U^* \leq_{R\text{–}K} W$.
(2) $Cub_\kappa \subseteq W$, $\{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \alpha)\} \in W$.
(3) $L_0 := \{\beta < \kappa \mid \mathrm{Cohen}(\beta, \beta^{++}) \times \mathrm{Cohen}(\beta, \beta^{++})$ *was forced in* $G_{\kappa+1}\} \in W$.
(4) *For every* $\alpha < \kappa^{++}$, $L_{1,\alpha} := \{\nu < \kappa \mid f_{\kappa,\alpha}(\nu) < \nu^{++}\} \in W$.
(5) *The sequence* $\langle A_\alpha \mid \alpha < \kappa^{++}\rangle$ *is a strong witness for* $\neg Gal(W, \kappa, \kappa^{++})$. *Moreover, the sequence* $\langle A_\alpha \cap L_{1,\alpha} \mid \alpha < \kappa^{++}\rangle$ *is a witness for* $\neg Gal(W, \kappa, \kappa^{++})$.

Proof. (1), (2), and the first part of (5) are the same argument as in Claim 3.2. As for (3), note that we have constructed the generic $G_{\kappa_2+1} = j_2^*(G_{\kappa+1})$ so that on κ_1 we have forced $\mathrm{Cohen}(\kappa_1, \kappa_1^{++}) \times \mathrm{Cohen}(\kappa_1, \kappa_1^{++})$. To see (4), for every $\alpha < \kappa^{++}$,

$$j_2^*(f_{\kappa,\alpha})(\kappa_1) = f_{\kappa_2, j_{2,1}(j_1(\alpha))}(\kappa_1) = j_1(\alpha) \cdot 2 + 1 < \kappa_1^{++}.$$

Hence by elementarity, $\kappa_1 \in j_2^*(L_{1,\alpha})$. Finally, the moreover part of (5), toward a contradiction if there would be a set $I \in [\kappa^{++}]^\kappa$ such that $\cap_{i\in I} A_\alpha \cap L_{1,\alpha} \in W$ then clearly $\cap_{i\in I} A_\alpha \in W$, contradicting the first part of (5) that A_α's form a witness for $\neg\mathrm{Gal}(W, \kappa, \kappa^{++})$. ⊣

Denoted by $\nu \mapsto \pi_{nor}(\nu)$ the Rudin–Keisler projection from W to U^*, and let us prove that W witnesses the theorem:

PROPOSITION 3.5. *Let $H \subseteq$ Prikry(W) be a V^*-generic filter. There is $G^* \in V^*[H]$ which is V^*-generic for Cohen$(\kappa, \kappa^{++})^{V^*}$.*

PROOF OF PROPOSITION 3.5. Let $\langle c_n \mid n < \omega \rangle$ be the W-Prikry sequence corresponding to H. Suppose without loss of generality that for every $n < \omega$, $c_n \in L_0$.

Define, for every $n < \omega$, the set

$$Z_n = \{\alpha < \kappa^{++} \mid \{c_m \mid n \leq m < \omega\} \subseteq A_\alpha \cap L_{1,\alpha} \text{ and } n \text{ is least possible}\}.$$

For every $\alpha < \kappa^{++}$, let n_α be the unique n such that $\alpha \in Z_n$. Let $\alpha < \kappa^+$, and define $f^*_\alpha : \kappa \to \kappa$ as follows:

Denote by

$$\langle f_{c_n,\alpha} \mid \alpha < c_n^{++} \rangle,\ \langle h_{c_n,\alpha} \mid \alpha < c_n^{++} \rangle$$

the generic c_n-Cohen functions forced by G and define the function $f^*_\alpha : \kappa \to \kappa$ by

$$f^*_\alpha = h_{c_{n_\alpha}, f_{\kappa,\alpha}(c_{n_\alpha})} \cup \left(\bigcup_{n_\alpha < n < \omega} h_{c_n, f_{\kappa,\alpha}(c_n)} \restriction [c_{n-1}, c_n) \right).$$

Note that the Cohen functions on κ play the role of the canonical functions from the previous section. Let us prove that $F = \langle f^*_\alpha \mid \alpha < \kappa^{++} \rangle$ are Cohen generic functions over V^*.

CLAIM 3.6. *Let $G^* = \{p \in$ Cohen$(\kappa, \kappa^{++})^{V^*} \mid p \subseteq F\}$, then G^* is a V^*-generic filter.*

Let $\mathcal{A} \in V^*$ be a maximal antichain in the forcing Cohen$(\kappa, \kappa^{++})^{V^*}$. Note that since Cohen$(\kappa, \kappa^{++})^{V^*}$ is κ-closed then

$$Cohen(\kappa, \kappa^{++})^{V[G_\kappa]} = Cohen(\kappa, \kappa^{++})^{V^*}.$$

By κ^+-cc of the forcing, there is $Y' \subseteq \kappa^{++}$, $Y' \in V$ such that $|Y'| = \kappa$ and $\mathcal{A} \subseteq$ Cohen$(\kappa, Y')^{V^*}$. Also, since $|\mathcal{A}| = \kappa$, $\mathcal{A} \in V[G_\kappa * F_\kappa]$, there is $Z \subseteq \kappa^{++}$ such that $|Z| = \kappa$ such that $\mathcal{A} \in V[G_\kappa * F_\kappa \restriction Z]$. Without loss of generality assume that $Z = Y \in V$. Let $V \ni \phi : \kappa \to Y$ be a bijection.

As in Claim 2.13, we can construct an $\in$-increasing continuous chain $\langle N_\beta \mid \beta < \kappa \rangle \in V^*$ of elementary submodels of H_χ such that:

(1) $|N_\beta| < \kappa$.
(2) $G_{\kappa+1}, \mathcal{A}, \phi, Y \in N_0$.
(3) $N_\beta \cap \kappa = \gamma_\beta$ is a cardinal $< \kappa$, and $\gamma_{\beta+1}$ is regular.
(4) If γ_β is regular, then Cohen$(\gamma_\beta, \phi''\gamma_\beta) =$ Cohen$(\kappa, Y) \cap N_\beta$.

Set

$$C = \{\beta < \kappa \mid \gamma_\beta = \beta\}.$$

This is club in κ since the sequence γ_β is continuous and since the set $\{\beta \mid \gamma_\beta = \beta\}$ is a club.

Recall that by construction $j^*_2(\langle f_{\kappa,\alpha} \mid \alpha < \kappa^{++} \rangle) = \langle f_{\kappa_2,\alpha} \mid \alpha < \kappa_2^{++} \rangle$. Also, for every $\nu \in j_2(\phi)''\kappa_1$ there is $\gamma < \kappa_1$ such that $\nu = j_2(\phi)(\gamma)$, and since $crit(j_{2,1}) = \kappa_1$,

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$\nu = j_{2,1}(j_1(\phi)(\gamma))$. Since $j_1(\phi) : \kappa_1 \to \kappa_1^{++}$ we conclude that $\nu = j_{2,1}(\alpha)$ for some $\alpha < (\kappa_1^{++})^{M_1}$ which implies that

$$f_{\kappa_2,\nu}(\kappa_1) \in \{\alpha \cdot 2, \alpha \cdot 2 + 1\}.$$

Since ϕ is a bijection, for every distinct $\nu_1, \nu_2 \in j_2(\phi)''\kappa_1$, $f_{\kappa_2,\nu_1}(\kappa_1) \neq f_{\kappa_2,\nu_2}(\kappa_1)$. Reflecting this, we obtain that the set

$$E := \{\nu < \kappa \mid \forall \nu_1, \nu_2 \in \phi''\nu.\nu_1 \neq \nu_2 \to f_{\kappa,\nu_1}(\nu) \neq f_{\kappa,\nu_2}(\nu)\} \in W.$$

Also, by construction, for every $\alpha < \kappa_1^{++}$, $f_{\kappa_2,j_{2,1}(\alpha)} \restriction \kappa_1 = f_{\kappa_1,\alpha}$ and therefore for every $\alpha \in j_2(\phi)''\kappa_1$, there is $\nu < \kappa_1^{++}$ such that

$$\alpha = j_{2,1}(j_1(\phi))(\nu) = j_{2,1}(j_1(\phi)(\nu))$$

and $j_1(\phi)(\nu) < \kappa_1^{++}$. Hence $f_{\kappa_2,\alpha} \restriction \kappa_1 = f_{\kappa_1,\beta}$ for some $\beta < \kappa_1^{++}$. Reflecting this we obtain that the set

$$F := \{\beta < \kappa \mid \forall \gamma \in \phi''\beta.\exists \delta < \beta^{++}.f_{\kappa,\gamma} \restriction \beta = f_{\beta,\delta}\} \in W.$$

Now the argument of Claim 2.15 applies since for every $\nu_0 \in C \cap E \cap F$, $\forall \tau_1 < \tau_2 \in \phi''\nu_0$, $f_{\kappa,\tau_1}(\nu_0) \neq f_{\kappa,\tau_2}(\nu_0)$, hence $\langle h_{\nu_0,f_{\kappa,\tau}(\nu_0)} \mid \tau \in \phi''\nu_0\rangle$ are distinct mutually $V[G_{\nu_0} * F_{\nu_0}]$-generic Cohen functions over ν_0. Thus, we can find $d \in \mathcal{A} \cap \text{Cohen}(\nu_0, \nu_0^{++})$ such that d is extended by $\langle h_{\nu_0,f_{\kappa,\alpha}(\nu_0)} \mid \alpha \in \phi''\nu_0\rangle$. Finally we note that

$$R := \{\nu < \kappa \mid \forall \alpha \in \phi''\pi_{nor}(\nu).f_{\kappa,\alpha}(\nu) \text{ is odd}\} \in W.$$

Let $p = \langle\langle\rangle, B\rangle$ be a condition, shrink B to $B_0 := B \cap C \cap E \cap F \cap R \in W$, and pick now any $\nu_0 \in B_0$. Split $\phi''\nu_0$ into two sets:

$$X_0^{\nu_0} := \{\tau \in \phi''\nu_0 \mid \nu_0 \in A_\tau\} \text{ and } X_1^{\nu_0} = \phi''\nu_0 \setminus X_0^{\nu_0}.$$

Since $\nu_0 \in R$ we have that $X_1 \subseteq \phi''(\pi_{nor}(\nu_0), \nu_0)$. The condition $p_0 = \langle\langle\nu_0\rangle, B_0 \cap (\bigcap_{\tau\in\phi''\nu_0} A_\tau)\rangle$ forces the following:

(1) The Prikry sequence is included in each A_τ, $\tau \in X_0^{\nu_0}$, i.e., $n_\tau = 0$.
(2) $n_\tau = 1$, for every $\tau \in X_1^{\nu_0}$.

In particular, this condition forces some information about the Cohen functions. Namely that:

(1) For $\tau \in X_0^{\nu_0}$, $f_\tau^* \restriction \nu_0 = h_{\nu_0,f_{\kappa,\tau}(\nu_0)}$.
(2) For $\tau \in X_1^{\nu_0}$, $f_\tau^* \restriction \nu_0 = h_{\underset{\sim}{C}_1,f_{\kappa,\tau}(\underset{\sim}{C}_1)} \restriction \nu_0$.

We would like to find a condition in $\mathcal{A}$ which is below these decided parts of the Cohen. By the previous paragraph, there is $d \in N_{\nu_0} \cap \text{Cohen}(\kappa, Y) = \text{Cohen}(\nu_0, \phi''\nu_0)$, which is extended by $\langle h_{\nu_0,f_{\kappa,\tau}(\nu_0)} \mid \tau \in \phi''\nu_0\rangle$. As before we will need to pick ν_0, ν_1 so that $d^{\nu_0} \in G^*$.

Let $\underset{\sim}{B_0}$ be a name in V for B_0. We fix a condition $m_0 \in G_\kappa * F_\kappa$ which forces that if $\nu_0 \in \widetilde{B}_0$ then there is $d \in \text{Cohen}(\nu_0, \phi''\nu_0) \cap \underset{\sim}{\mathcal{A}}$ which is extended by $\langle \underset{\sim}{h}_{\nu_0,f_{\kappa,\tau}(\nu_0)} \mid \nu_0 \in \phi''\widetilde{\nu_0}\rangle$, and $\forall \alpha \in \phi''\pi_{nor}(\nu_0)$. $\nu_0 \in \underset{\sim}{A}_\alpha$. Recall that by the construction of G_{κ_2}, we have $m_0 \in G_{\kappa_2} * F_{\kappa_2}$. Let $\widetilde{m}_0 \leq t \in \widetilde{G}_{\kappa_2} * F_{\kappa_2}$ be a condition such that

$$(1) \quad t \Vdash \kappa_1 \in j_2(\underset{\sim}{B_0}).$$

By the construction of $G_{\kappa_2} * F_{\kappa_2}$, t has the form:

$$t = \langle t_{<\kappa}, t_\kappa, t_{(\kappa,\kappa_1)}, \underbrace{\langle t^0_{\kappa_1}, t^1_{\kappa_1}\rangle}_{t_{\kappa_1}}, t_{(\kappa_1,\kappa_2)}, t_{\kappa_2}\rangle.$$

Distinguishing from the case of κ^+, we now have that $f_{\kappa_2, j_2(\alpha)}(\kappa_1) = j_1(\alpha)\cdot 2+1$ for every $\alpha < \kappa^+$; this will hold for every $\alpha \in \phi''\kappa$ as well. Also, recall that $Y \in V$, hence $\phi \in V$. Thus $j_2(\phi) \in M_2$ and $j_2(\phi)''\kappa \in M_2$. Also, for $(t_{\kappa_2})_{G_{\kappa_2}} \in M_2[G_{\kappa_2}]$,

$$j_2''\kappa^{++} \cap \mathrm{Supp}((t_{\kappa_2})_{G_{\kappa_2}}) \in M_2[G_{\kappa_2}]$$

and $(t_{\kappa_2})_{G_{\kappa_2}} \restriction \kappa \times \{j_2(\alpha)\} \subseteq f_{\kappa,\alpha}$. We also fix $X \in V$, $X \subseteq \kappa^{++}$, $|N_0| \le \kappa$ such that $\mathrm{Supp}((t_{\kappa_2})_{G_{\kappa_2}}) \subseteq j_2(N_0)$.

Therefore, we can extend if necessary t such that:

$$(2a)\ \ t_{<\kappa_2} \Vdash (\kappa \cup \{\kappa_1\}) \times j_2(\phi)''\kappa \subseteq \mathrm{dom}(t_{\kappa_2}) \wedge (0,\kappa_1) \in \mathrm{dom}(t_{\kappa_2}) \wedge \mathrm{Supp}(t_{\kappa_2}) \subseteq j_2(N_0),$$

$$(2b)\ \ t_{<\kappa_2} \Vdash\ t_{\kappa_2}(\kappa_1, j_2(\alpha)) = j_1(\alpha)\cdot 2+1, \text{ for every } j_2(\alpha) \in j_2(\phi)''\kappa \text{ and } t_{\kappa_2,\kappa_1}(0) = \kappa,$$

$$(2c)\ \ t_{<\kappa_2} \Vdash t_{\kappa_2, j_2(\alpha)} \restriction \kappa = \underset{\sim}{f}_{\kappa,\alpha} \text{ for every } j_2(\alpha) \in j_2''\kappa^+ \cap \mathrm{Supp}(t_{\kappa_2}).$$

Next consider $t_{\kappa_1} = \langle t^0_{\kappa_1}, t^1_{\kappa_1}\rangle$; it is a $\mathcal{P}_{\kappa_1}$-name for a condition in $F_{\kappa_1} \times H_{\kappa_1}$. By the construction of the generic $F_{\kappa_1} \times H_{\kappa_1}$, for every $\alpha < \kappa^{++}$, we made sure that, $h_{\kappa_1, j_1(\alpha)2+1} \restriction \kappa = f_{\kappa,\alpha}$. Also, $(j_1(\phi)''\kappa)\cdot 2+1 \in M_2$[7]. Let

$$\mu_1 = \{\langle j_1(\alpha)\cdot 2+1, \alpha\rangle \mid \alpha \in \phi''\kappa\} \in M_1.$$

The fact that for every $\beta < \kappa^{++}$, $f_{\kappa_2, j_2(\beta)}(\kappa_1) = j_1(\beta)\cdot 2+1$ implies

$$\mathrm{dom}(\mu_1) = (j_1(\phi)''\kappa)\cdot 2+1 = \{f_{\kappa_2,\gamma}(\kappa_1) \mid \gamma \in j_2(\phi)''\kappa\},\ \mathrm{rng}(\mu_1) = \phi''\kappa \subseteq \kappa^{++}.$$

Extend if necessary $t_{<\kappa_1}$, and assume that

$$(3)\ \ t_{<\kappa_1} \Vdash \kappa \times (j_1(\phi)''\kappa)\cdot 2+1 \subseteq \mathrm{dom}(t^1_{\kappa_1}) \wedge \forall j_1(\alpha) \in j_1(\phi)''\kappa,\ t^1_{\kappa_1, j_1(\alpha)\cdot 2+1} \restriction \kappa = \underset{\sim}{f}_{\kappa,\alpha}.$$

As for the lower part, due to the Easton support, we have

$$(4)\ \ t_{<\kappa} \in V_\kappa.$$

Fix functions r, Γ_1 which represent t, μ resp. in the ultrapower M_{E^2}, namely for some $\vec{\xi} \in [\kappa_1^{++}]^{<\omega}$, $j_2(r)(\vec{\xi}) = t$, $j_2(\Gamma_1)(\vec{\xi}) = \mu$. Without loss of generality, suppose that both κ and κ_1 appear in $\vec{\xi}$, $\kappa = \min(\vec{\xi}) = \vec{\xi}(0)$ and $\kappa_1 = \vec{\xi}(i_0)$. Then the functions $\vec{\nu} \in [\kappa]^{|\vec{\xi}|} \mapsto (\vec{\nu}(0), \vec{\nu}(i_0))$ represent (κ,κ_1). Without loss of generality, suppose that for every $\vec{\nu}$, it takes the form

$$r(\vec{\nu}) = \langle r_{<\vec{\nu}(0)}, r_{\vec{\nu}(0)}, r_{(\vec{\nu}(0),\vec{\nu}(i_0))}, \langle r^0_{\vec{\nu}(i_0)}, r^1_{\vec{\nu}(i_0)}\rangle, r_{(\vec{\nu}(i_0),\kappa)}, r_\kappa\rangle.$$

Reflecting some of the properties of t we obtain a set $B' \in E(\vec{\xi})$ such that for every $\vec{\nu} \in B'$:

[7]For a set of ordinals A, let $A\cdot 2+1 = \{\alpha\cdot 2+1 \mid \alpha \in A\}$.

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$(1)_{\vec{v}}$ $r(\vec{v}) \Vdash \vec{v}(i_0) \in B_0$.
$(2a)_{\vec{v}}$ $r_{<\kappa} \Vdash (\vec{v}(\widetilde{0}) \cup \{\vec{v}(i_0)\}) \times \phi''\vec{v}(0) \subseteq \mathrm{dom}(r_\kappa) \wedge \langle 0, \vec{v}(i_0)\rangle \in \mathrm{dom}(r_\kappa) \wedge \mathrm{Supp}(r_\kappa) \subseteq N_0$.
$(2b)_{\vec{v}}$ $r_{<\kappa} \Vdash \forall \alpha \in \phi''\vec{v}(0).r_{\kappa,\alpha}(\vec{v}(i_0))$ is odd and $r_{\kappa,\vec{v}(i_0)}(0) = \vec{v}(0)$.
$(3)_{\vec{v}}$ $r_{<\vec{v}(i_0)} \Vdash \vec{v}(0) \times \mathrm{dom}(\Gamma_1(\vec{v})) \subseteq \mathrm{dom}(r^1_{\vec{v}(i_0)})$ and for every $\beta \in \mathrm{dom}(\Gamma_1(\vec{v}))$, $r^1_{\vec{v}(i_0),\beta} \restriction \vec{v}(0) = \underset{\sim}{f}_{\vec{v}(0),\Gamma_1(\vec{v})(\beta)}$.
$(4)_{\vec{v}}$ $r_{<\vec{v}(0)} = t_{<\kappa} \in V_{\vec{v}(0)}$.

Let

$$B'' = \{v(i_0) \mid \exists \vec{v} \in B'.r(\vec{v}) \in G_\kappa * F_\kappa\}.$$

Since $B' \in E(\vec{\xi})$ we have that $\vec{\xi} \in j_2(B')$ and since $j_2(r)(\vec{\xi}) = t \in j_2^*(G_\kappa * F_\kappa) = G_{\kappa_2} * F_{\kappa_2}$, we conclude that $B'' \in W$. Also, $B'' \subseteq B_0$ by clause (1).

We proceed by a density argument, and recall that by the definition of G_2, we have that $\langle t_{<\kappa}, t_\kappa\rangle \in G_\kappa * F_\kappa$.

Claim 3.7. *Let D be the set of all conditions $q \in \mathcal{P}_{\kappa+1}$, such that there exist $\vec{v}_0, \vec{v}_1 \in B'$, $\vec{v}_1(0) > \vec{v}_0(i_0)$, and a $\mathcal{P}_{\vec{v}_0(i_0)}$-name $\underset{\sim}{d}^{\vec{v}_0(i_0)}$ such that*:

(a) $r(\vec{v}_0), r(\vec{v}_1) \leq q$.
(b) $q \Vdash \underset{\sim}{d}^{\vec{v}_0(i_0)} \in \underset{\sim}{A} \cap \mathrm{Cohen}(\vec{v}_0(i_0), \phi''\vec{v}_0(i_0))$.
(c) $q \Vdash \forall \tau \in X_1^{\vec{v}_0(i_0)}.\underset{\sim}{h}_{v_1, f_{\kappa,\tau}(\vec{v}_1(i_0))} \restriction \vec{v}_0(i_0) = \underset{\sim}{d}_\tau^{\vec{v}_0(i_0)}$.

Then D is dense (open) above $\langle t_{<\kappa}, t_\kappa\rangle$ and thus $D \cap G_\kappa * F_\kappa \neq \emptyset$.

Proof. Work in V, and let $\langle t_{<\kappa}, t_\kappa\rangle \leq p := \langle p_{<\kappa}, p_\kappa\rangle \in \mathcal{P}_{\kappa+1}$. We will define two extensions $p \leq q \leq q^*$ as before such that $q^* \in D$. By definition of $\mathcal{P}_{\kappa+1}$, $p_{<\kappa} \Vdash p_\kappa \in \mathrm{Cohen}(\kappa, \kappa^{++})$, by κ–cc of $\mathcal{P}_\kappa$, for some $Z \subseteq \kappa^{++}$, $Z \in V$, $|Z| < \kappa$ and some $\gamma < \kappa$, $p_{<\kappa} \Vdash \mathrm{dom}(p_\kappa) \subseteq \gamma \times Z$. The same argument as before indicates that

$$p_{<\kappa} \Vdash Z \supseteq \mathrm{Supp}(t_\kappa) \wedge \forall \beta \in Z.j_2(p_\kappa)_{j_2(\beta)} \geq t_{\kappa,\beta},$$

$$t_{<\kappa_2} \Vdash \forall j_2(\tau) \in \mathrm{Supp}(t_{\kappa_2}) \cap j_2(Z).t_{\kappa_2, j_2(\tau)} \restriction \gamma = \underset{\sim}{f}_{\kappa,\tau} \restriction \gamma.$$

Denote $\mu = (j_2 \restriction (Z \cup N_0))^{-1} \in M_2$, then

$$\mathrm{dom}(\mu) = j_2(Z) \cup j_2'' N_0,\ \mathrm{rng}(\mu) = Z \cup \theta,\ \mu \text{ is 1–1},$$

and we can reformulate

$$p_{<\kappa} \Vdash \mu'' j_2(Z) \supseteq \mathrm{Supp}(t_\kappa) \wedge \forall \beta \in j_2(Z).j_2(p_\kappa)_\beta \geq t_{\kappa,\mu(\beta)},$$

$$t_{<\kappa_2} \Vdash \forall \tau \in \mathrm{Supp}(t_{\kappa_2}) \cap j_2(Z).t_{\kappa_2,\tau} \restriction \gamma = \underset{\sim}{f}_{\kappa,\mu(\tau)} \restriction \gamma.$$

Also, find $\delta < \kappa$ such that $t_{<\kappa} \Vdash \phi''(\delta, \kappa) \cap Z = \emptyset$. We have that

$$\phi''(\delta, \kappa) = \mu_1''\{f_{\kappa_2,\gamma}(\kappa_1) \mid \gamma \in j_2(\phi)''(\delta, \kappa)\},\ \text{and } \mu'' \mathrm{Supp}(j_2(p_\kappa)) = Z.$$

Therefore in M_2 we will have that

$$p_{<\kappa} \Vdash [\mu_1''\{\underset{\sim}{f}_{\kappa_2,\gamma}(\kappa_1) \mid \gamma \in j_2(\phi)''(\delta, \kappa)\}] \cap [\mu'' \mathrm{Supp}(j_2(p_\kappa))] = \emptyset.$$

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Let Γ be such that $j_2(\Gamma)(\vec{\xi}) = \mu$, and there is a set $\overline{B}_0 \subseteq B'$, $\overline{B}_0 \in E(\vec{\xi})$ such that for every $\vec{v} \in \overline{B}_0$:

$$(i)\ \ p_{<\kappa} \Vdash \Gamma(\vec{v})''Z \supseteq \mathrm{Supp}(r_{\vec{v}(0)}) \wedge \forall\beta \in Z.p_{\kappa,\beta} \geq r_{\vec{v}(0),\Gamma(\vec{v})(\beta)},$$

$$(ii)\ \ r_{<\kappa} \Vdash \forall\tau \in Z \cap \mathrm{Supp}(r_\kappa).r_{\kappa,\tau} \restriction \gamma = \underset{\sim}{f}_{\vec{v}(0),\Gamma(\vec{v})(\tau)} \restriction \gamma,$$

$$(iii)\ \ p_{<\kappa} \Vdash \Gamma_1(\vec{v})''\{\underset{\sim}{f}_{\kappa,\gamma}(\vec{v}(i_0)) \mid \gamma \in \phi''(\delta, \vec{v}(0))\} \cap [\Gamma(\vec{v})''\,\mathrm{Supp}(p_\kappa)] = \emptyset.$$

Find $\vec{v}_0, \vec{v}_1 \in \overline{B}_0$ such that $r(\vec{v}_0), r(\vec{v}_1)$ are compatible, $\vec{v}_0(0) > \delta, \gamma, \sup(\mathrm{Supp}(p_{<\kappa}))$, and $\vec{v}_1(0) > \vec{v}_0(i_0), \sup(\mathrm{Supp}(r_{<\kappa}(\vec{v}))$. Denote

$$r^0 := r(\vec{v}_0) = \langle r^0_{<\vec{v}_0(0)}, r^0_{\vec{v}_0(0)}, r^0_{(\vec{v}_0(0),\kappa)}, r^0_\kappa\rangle,$$

$$r^1 := r(\vec{v}_1) = (r^1_{<\vec{v}_1(0)}, r_{\vec{v}_1(0)}, r_{(\vec{v}_1(0),\vec{v}_1(i_0))}, \langle r^{0,1}_{\vec{v}_1(i_0)}, r^{1,1}_{\vec{v}_1(i_0)}\rangle, r^1_{(\vec{v}_1(i_0),\kappa)}, r^1_\kappa\rangle.$$

As before, q has the form: $q = p_{<\kappa}{}^\frown q_{\vec{v}_0(0)}{}^\frown r^0_{(\vec{v}_0(0),\kappa)}{}^\frown q_\kappa$. We have $q_{\vec{v}_0(0)}$ is a $\mathcal{P}_{\vec{v}_0(0)}$-name for a condition with $\mathrm{Supp}(q_{\vec{v}_0(0)}) = \Gamma(\vec{v}_0)''Z$ and $q_{v_0',\Gamma(v_0',v_0)(\beta)} = p_{\kappa,\beta}$. As for q_κ, we set it to be a $\mathcal{P}_\kappa$-name for $r^0_\kappa \cup p_\kappa$.

The argument that $r^0 \leq q$ is the same as in the case of κ^+.

The choice of $\underset{\sim}{d}^{\vec{v}_0(i_0)}$ is possible since $r^0 \leq q$ and $m_0 \leq \langle t_{<\kappa}, t_\kappa\rangle \leq q \Vdash \vec{v}_0(i_0) \in \underset{\sim}{B}_0$.

Define the final condition $q \leq q^*$,

$$q^* = q_{<\kappa}{}^\frown q^*_{\vec{v}_1(0)}{}^\frown r^1_{(\vec{v}_1(0),\kappa)}{}^\frown q^*_\kappa.$$

Again we have that $r^0 \Vdash X_1^{\vec{v}_0(i_0)} \subseteq \phi''(\vec{v}_0(0), \vec{v}_0(i_0)) \subseteq \phi''(\vec{v}_0(0), \vec{v}_1(0))$ and by (iii)

$$q_{<\kappa} \Vdash [\Gamma_1(\vec{v}_1)''\{\underset{\sim}{f}_{\kappa,\gamma}(v_1) \mid \gamma \in X_1^{v_0}\}] \cap [\Gamma(\vec{v}_1)''Z] = \emptyset.$$

Now for the code of $\underset{\sim}{d}^{\vec{v}_0(i_0)}$, let

$$\mathrm{Supp}(q^*_{\vec{v}_1(0)}) = [\Gamma_1(\vec{v}_1)''\{\underset{\sim}{f}_{\kappa,\gamma}(\vec{v}_1(i_0)) \mid \gamma \in X_1^{\vec{v}_0(i_0)}\}] \uplus [\Gamma(\vec{v}_1)''Z]$$

and

$$q^*_{\vec{v}_1(0),\alpha} = \begin{cases} q_{\kappa,\beta}, & \exists\beta \in \Gamma(\vec{v}_1)''Z.\alpha = \Gamma(\vec{v}_1)(\beta), \\ \underset{\sim}{d}_\tau^{\vec{v}_0(i_0)}, & \exists\tau \in X_1^{\vec{v}_0(0)}.\alpha = \Gamma_1(\vec{v}_1)(\underset{\sim}{f}_{\kappa,\tau}(\vec{v}_1(i_0))), \end{cases}$$

and $q^*_\kappa = q_\kappa \cup r^1_\kappa$. We conclude that $r^0 \leq q \leq q^*$, $r^1 \leq q^*$, namely (a). Finally, for every $\tau \in X_1^{\vec{v}_0(i_0)}$, $\underset{\sim}{f}_{\kappa,\tau}(\vec{v}_1(i_0)) \in \mathrm{dom}(\Gamma_1(\vec{v}))$ and by $(3)_{(\vec{v}_1)}$ we have that q^* forces that

$$\underset{\sim}{h}_{\vec{v}_1(i_0),\underset{\sim}{f}_{\kappa,\tau}(\vec{v}_1(i_0))} \restriction \vec{v}_0(i_0) = \underset{\sim}{f}_{\vec{v}_1(0),\Gamma_1(\vec{v}_1)(\underset{\sim}{f}_\tau(\vec{v}_1(i_0)))} \restriction \vec{v}_0(i_0) \geq$$

$$\geq q^*_{\vec{v}_1(0),\Gamma_1(\vec{v}_1)(\underset{\sim}{f}_{\kappa,\tau}(\vec{v}_1(i_0)))} = \underset{\sim}{d}_\tau^{\vec{v}_0(i_0)}.$$

Then $p \leq q^*$ and $q^* \in D$. $\dashv$

The rest of the argument remains unchanged. $\dashv$

$\dashv$

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§4. On the Extender-based Prikry forcings and adding subsets to κ. H. Woodin asked in the early 90s whether, assuming that there is no inner model with a strong cardinal, it is possible to have a model M in which $2^{\aleph_\omega} \geq \aleph_{\omega+3}$, GCH holds below $\aleph_\omega$, there is an inner model N such that $\kappa = (\aleph_\omega)^M$ is a measurable and $2^\kappa \geq (\aleph_{\omega+3})^M$. His question was natural given the results known back then: Magidor [26] proved that it is consistent relative to a supercompact cardinal and a huge cardinal above it to have $2^{\aleph_\omega} \geq \aleph_{\omega+m}$ and $GCH_{<\aleph_\omega}$ using the supercompact Prikry forcing with collapses. Woodin, in an unpublished work which can be found in [11] reduced Magidor's large cardinal assumption to get $2^{\aleph_\omega} = \aleph_{\omega+2} + GCH_{<}\aleph_\omega$ to a strong cardinal (actually to a $p_2\kappa$-hypermeasurable). Later, Gitik and Magidor [21] proved using the Extender-based Prikry forcing with collapses that starting from the optimal large cardinal assumption, it is possible to obtain $\aleph_{\omega+m} = 2^{\aleph_\omega}$ and $GCH_{<\aleph_\omega}$. However, Woodin's question remained unanswered.

A natural approach to answer Woodin's question is to force with the Extender-based Prikry forcing over κ and then argue that in some intermediate where κ is measurable we added $\lambda \geq \kappa^{++}$ many subsets to κ.

Our purpose will be to show that this direction is doomed. More precisely, we will prove that in any intermediate model of the Extender-based Prikry forcing where κ^{++}-many subsets of κ were introduced, κ is singularized (and in particular not measurable). We will analyze the situation in both the original version of Gitik and Magidor from [21] and Merimovich version of the Extender-based Prikry forcing from [29–31]. We will rely on the following theorem from [6, Theorem 6.7]:

THEOREM 4.1. *Suppose that $\mathbb{U} = \langle U_a \mid a \in [\kappa]^{<\omega}\rangle$ is a tree of P-point ultrafilters. Let $G \subseteq P(\mathbb{U})$ be V-generic, then for every set of ordinals $A \in V[G] \setminus V$, $cf^{V[A]}(\kappa) = \omega$.*

Note that if U is any κ-complete ultrafilter, then the forcing Prikry(U) which we use in this paper is forcing equivalent to $P(\mathbb{U})$ where $\mathbb{U} = \langle U_a \mid a \in [\kappa]^{<\omega}\rangle$ is such that $U_a = U$ for every a.

Assume $2^\kappa = \kappa^+$. Let E be an extender over κ. We consider two sorts of Extender-based Prikry forcings—the original one (see [21] or [17]) and a more elegant version of Merimovich [29–31].

Let us start with the Merimovich version, but in which the measures of E are P-points as in [21].

4.1. The Merimovich version with *P*-points. Suppose that there is $h : \kappa \to \kappa$ such that all the generators of E are below $j_E(h)(\kappa)$.

For example, if E is a (κ, κ^{++})-extender, this holds with $h(\nu) = \nu^{++}$, $\nu < \kappa$. This is sufficient to ensure that for every $\alpha < \lambda$, U_α is a P-point ultrafilter.

Denote by $\mathbb{P}_E$ the Merimovich Extender-based Prikry forcing with E, as defined in [31] (or see Definition 1.5).

THEOREM 4.2. *Let $G \subseteq \mathbb{P}_E$ be a generic. Suppose that $A \in V[G] \setminus V$ is a subset of κ. Then κ changes its cofinality to ω in $V[A]$.*

PROOF. Work in V. Suppose that $\underset{\sim}{A}$ is a name of a subset of κ and some $p \in \mathbb{P}_E$ forces that it is a new subset.

Let us use κ^+-properness of the forcing $\mathbb{P}_E$ (see [31, Claim 2.7] or [29, Claim 3.29]). Pick now $N \preceq H_\chi$, for some χ large enough such that:

(1) $|N| = \kappa$,
(2) $N \supseteq {}^{\kappa>}N$,
(3) $E, \mathbb{P}_E, p, \underset{\sim}{A} \in N$.

The properness implies that there is $p^* \geq^* p$ which is $\langle N, \mathbb{P}_E\rangle$-generic, i.e.,

$$p^* \Vdash (\forall D \in N(\text{if } D \text{ is a dense open, then } D \cap N \cap \underset{\sim}{G} \neq \emptyset)).$$

In particular, for every $\nu < \kappa$, the dense open set

$$D_\nu := \{q \mid \exists \alpha. q \Vdash otp(\underset{\sim}{A}) > \nu \rightarrow \text{ the } \nu\text{-th element of } \underset{\sim}{A} \text{ is } \alpha\}$$

is definable from $\underset{\sim}{A}$ and ν, hence in N and it is dense open by elementarity.

Consider $X = \bigcup_{p \in \mathbb{P}_E \cap N} \operatorname{Supp}(p)$, since $\operatorname{Supp}(p)$, N are of size κ, and we have that $|X| \leq \kappa$. There exists $\alpha^* < \lambda$ such that for some $f \in V$, $j_E(f)(\alpha^*) = (j \restriction X)^{-1}$(see, for example, [17, Lemma 3.3]).

Denote $Y = X \cup \{\alpha^*\}$ and fix a set $R \in E_Y$ such that if $\mu \in R$, then $f(\mu(\alpha^*)) = \mu \restriction X$. Such a set exists since $j_E(f)(j^{-1}(j(\alpha^*))) = (j \restriction X)^{-1}$, hence

$$(j \restriction Y)^{-1} \in j_E(\{\mu \in ob(Y) \mid f(\mu(\alpha^*)) = \mu \restriction X\}).$$

Find a condition $p_* \in G$ such that $Y \subseteq \operatorname{Supp}(p)$ and $A^{p_*} \restriction Y \subseteq R$. Define $G \restriction Y = \{p \restriction Y \mid p \in G/p_*\}$. Then by genericity of p^* and definition of Y, for every $\alpha < \kappa$ there is $p_\alpha \in G \cap D_\nu \cap N$, hence $\operatorname{Supp}(p_\alpha) \subseteq Y$ and we can find $p_\alpha \leq p^*_\alpha \in G \restriction Y \cap D_\nu$. It follows that $A \in V[G \restriction Y]$. Let $G_{\alpha^*} = \{p \restriction \{\alpha^*\} \mid p \in G/p_*\}$, in particular, $p_0 := p_* \restriction \{\alpha^*\} \in G_{\alpha^*}$. Note that G_{α^*} is essentially a Prikry generic filter for Prikry(U_{α^*}).

Claim 4.3. $V[G \restriction Y] = V[G_{\alpha^*}]$.

Proof. Inclusion from right to left is clear as $\alpha^* \in Y$. For the other direction, let $p_0 = \langle t_0, B_0\rangle \leq q = \langle t, B\rangle \in G_{\alpha^*}$. For every $|t_0| < i \leq |t|$ $t(i) \in B \subseteq B_0$, by the property of R, we have that $\mu_i := f(t(i)) \frown t(i) \in A^{p^*}$ such that $\mu_i(\alpha^*) = t(i)$. Now define $q' = \langle f, B'\rangle$ as follows: $\operatorname{dom}(f) = Y$ and

$$f = f^{p^*} {}^\frown \mu_{|t_0|+1} {}^\frown \ldots {}^\frown \mu_{|t|}.$$

In particular $f(\alpha^*) = t \geq f^{p_*}(\alpha^*)$. Also, let $B' = \{\mu \mid \mu(\alpha^*) \in B', \ f(\mu(\alpha^*)) = \mu \restriction X\}$. We claim that $G \restriction Y = \{q' \mid q \in G_{\alpha^*}/p_0\}$. Indeed if $p \in G/p_*$ then $q = p \restriction \{\alpha^*\} \in G_{\alpha^*}$ and it is straightforward to check that $q' = p \restriction Y$. It follows that $G \restriction Y$ is definable in $V[G_{\alpha^*}]$. ⊣

By our assumption U_{α^*} is a P-point ultrafilter. Now, Theorem 4.1 applies, so ⊣

$$V[A] \models \operatorname{cof}(\kappa) = \omega.$$

4.2. The original version. The difference here from the forcing of the previous section is that the order $\leq^*$ is not κ^+-closed. However, we will show that the forcing is still κ^+-proper.

Assume for simplicity that E is a (κ, κ^{++})-extender and the function $\nu \mapsto \nu^{++}$ represents κ^{++} in the ultrapower.

Let $\mathcal{P}_E$ be the forcing of [21] with E.

LEMMA 4.4. *Assume $p \in \mathcal{P}_E$. Let $N \preceq H_\chi$, for some χ large enough such that*:

(1) $|N| = \kappa$,
(2) $N \supseteq {}^{\kappa>}N$,
(3) $E, \mathcal{P}_E, p \in N$.

Then there is $p^* \geq p$ which is $\langle N, \mathcal{P}_E\rangle$-generic.

PROOF. Let $\langle D_\nu \mid \nu < \kappa\rangle$ be an enumeration of all dense open subsets of $\mathcal{P}_E$ which are in N. Proceed by induction and define a $\leq^*$-increasing sequence $\langle p_\nu \mid \nu < \kappa\rangle$ of extensions of p such that, for every $\nu < \kappa$:

(a) $p_\nu \in N$.
(b) $\min(A_\nu^0) > \nu$, where $A_\nu^0 = \{\rho^0 \mid \rho \in A_\nu\}$ is the projection of A_ν to the normal measure.
(c) There is $k < \omega$ such that for every $\langle \rho_1, ..., \rho_k\rangle \in [A_\nu]^k$, $p_\nu{}^\frown\langle \rho_1, ..., \rho_k\rangle \in D_\nu$.

It is natural now to move now to a coordinate η which is above everything in N and to take the diagonal intersection Δ^* of the pre-images of A_ν's according to the normal measure. However, in order to have the property (c) above, something more is needed. Namely, we would like to have the following:

(d) for every $\langle \xi_1, ..., \xi_m\rangle \in [\min(A_\nu^0)]^{<\omega}$, if $p_\nu{}^\frown\langle \xi_1, ..., \xi_m\rangle \in \mathcal{P}_E$ then there is $k < \omega$ such that:

$$\text{for every } \langle \rho_1, ..., \rho_k\rangle \in [A_\nu]^k,\ \ p_\nu{}^\frown\langle \xi_1, ..., \xi_m\rangle^\frown\langle \rho_1, ..., \rho_k\rangle \in D_\nu.$$

Given (d), as we will see, the idea above works fine. Let us construct a sequence which satisfies the conditions (a)–(d).

Pick $p_0 \in N$ such that $p_0 \geq^* p$ and (d) is satisfied. To define p_1, use the strong Prikry property to pick a condition $p_1' \in N$, $p_1' \geq^* p_0$ and

$$\text{there is } k < \omega \text{ such that for every } \langle \rho_1, ..., \rho_k\rangle \in [A_1']^k,\ p_1'{}^\frown\langle \rho_1, ..., \rho_k\rangle \in D_1.$$

Let $\eta_0 = \min((A_1')^0)$, by definition of $\pi_{\alpha,\kappa}$ it follows that η_0 is an inaccessible cardinal.

Let $\langle \vec{\xi}_i \mid i < \eta_0\rangle$ be an enumeration of $[\eta_0]^{<\omega}$.

Define $\leq^*$-increasing sequence $\langle q_i \mid i < \eta_0\rangle$.

Consider $p_1'{}^\frown\vec{\xi}_0$. If it does not extend p_0, then set $q_0 = p_1'$. Otherwise, pick (inside N) $r_0 \geq^* p_1'{}^\frown\vec{\xi}_0$ such that

$$\text{there is } k < \omega \text{ such that for every } \langle \rho_1, ..., \rho_k\rangle \in [A(r_0)]^k, r_0{}^\frown\langle \rho_1, ..., \rho_k\rangle \in D_1.$$

Let $q_0 = \langle f^{q_0}, A^{q_0}\rangle$ be obtained from r_0 by removing $\vec{\xi}_0$ from all coordinates which appear in p_1' (and leaving at new ones), and then, adding a larger maximal coordinate. Namely, $\mathrm{dom}(f^{q_0}) = \mathrm{dom}(f^{r_0}) \cup \{\alpha_0\}$ where α_0 is $\leq_E$ strictly above all the ordinals in $\mathrm{dom}(f^{r_0})$. Let t be such that $\pi''_{\alpha,\kappa} t = f^{p_1'}(\kappa)$ and for every $\gamma \in \mathrm{dom}(f^{q_0})$,

$$f^{q_0}(\gamma) = \begin{cases} f^{p_1'}(\gamma), & \gamma \in Supp(p_1'), \\ f^{r_0}(\gamma), & \gamma \in Supp(r_0) \setminus Supp(p_1'), \\ t, & \gamma = \alpha_0. \end{cases}$$

Let $A^{q_0} = \pi^{-1}_{\alpha_0, mc(r_0)}[A^{r_0}]$. Then $q_0 \in N$ and also $q_0 \in \mathcal{P}_E$. By shrinking A^{q_0} a bit more (as in [17, Lemma 3.10]) we secure condition (6), and $p_1' \leq^* q_0$.

Define q_1 in the exact same fashion only replacing p'_1 by q_0 and $\vec{\xi}_0$ by $\vec{\xi}_1$.

Continue similarly for every $i < \eta_0$, and finally, let q_{η_0} be a $\leq^*$-extension of all q_i's.

If $\eta_0 = \min((A(q_{\eta_0}))^0)$, then set $p_1 = q_{\eta_0}$. Otherwise, let $\eta_1 = \min((A(q_{\eta_0}))^0)$. Repeat the process above with η_1 replacing η_0 and q_{η_0} replacing p'_1. Continuing in a similar fashion, we hope to reach some η which is a fixed point, i.e., $\eta = \min((A(q_\eta))^0)$. However, we need to do this a bit more carefully at limit stages. Let us pick an elementary substructure $N' \prec V_\mu$ for sufficiently large μ of cardinality κ^+, closed under κ-sequences, including $p'_1, p_0, \mathcal{P}_E, E, \ldots$. We can find some $\alpha < \kappa^{++}$ such that for every $p \in N' \cap \mathcal{P}_E$ and every $\gamma \in \mathrm{Supp}(p)$, $\gamma <_E \alpha$. Define a sequence of condition $\langle q_{\eta_i} \mid i < \eta \rangle$ of conditions of N'.

We start with q_{η_0} which is already defined. Let $Y_0 \in U_\alpha$ such that the commutativity requirement from Definition 1.6(6) holds with respect to $\mathrm{Supp}(q_{\eta_0})$. If $\eta_0 = \min(Y_0^0)$ we are done. Otherwise, let $\eta_1 = \min(Y_0^0)$ and construct q_{η_1} in a similar fashion going over all possible $\vec{\xi} \in [\eta_1]^{<\omega}$, and construct $Y_1 \in U_\alpha$ to satisfy (6) with respect to $\mathrm{Supp}(q_{\eta_1})$. At a general successor step, we are given η_i, q_{η_i}, and Y_i. Check if $\eta_i = \min(Y_i^0)$, if yes, stop the construction, set $p_1 = q_{\eta_i}$, and we are done. Otherwise, let $\eta_{i+1} = \min(Y_i^0)$, construct $q_{\eta_{i+1}}$ above q_{η_i} as we did with q_{η_0}, going over all possible $\vec{\xi} \in [\eta_{i+1}]^{<\omega}$, then find $Y_{i+1} \in U_\alpha$ satisfying (6) with respect to $\mathrm{Supp}(q_{\eta_{i+1}})$. At limit stages δ take $\eta_\delta = \sup_{i<\delta} \eta_i$, check if $\eta_\delta = \min((\cap_{i<\delta} Y_i)^0)$, if yes, stop the construction and consider the condition $p_1 = q_{\eta_\delta}$ with maximal coordinate α, putting $\cap_{i<\delta} Y_i$ as his measure one set. Then q_{η_δ} will be as desired. Otherwise, we find any $q_{\eta_\delta} \in N'$ above all the previous q_{η_i}, and construct $Y_\delta \in U_\alpha$ with respect to $\mathrm{Supp}(q_{\eta_\delta})$. We can further require that $\pi''_{\alpha, mc(q_{\eta_i})} Y_i \subseteq A(q_{\eta_i})$ and that $\min(A(q_{\eta_i})^0) > i$.

Assume toward a contradiction that no suitable q_{η_δ} was found and that the process goes all the way up to κ. Consider $Y^* = \Delta^*_{i<\kappa} Y_i \in U_\alpha$ and let μ be any limit point of Y^*. Consider step μ^0 of the construction, and we have $\eta_{\mu^0} = \sup_{i<\mu^0} \eta_i$. For every $i < \mu^0$, we have that $\mu \in Y_i$, hence $\mu \in \cap_{i<\mu^0} Y_i$ and $\mu^0 \in (\cap_{i<\mu^0} Y_i)^0$, and it follows that $\eta_{\mu^0} \geq \mu^0 \geq \min((\cap_{i<\mu^0} Y_i)^0) \geq \eta_{\mu^0}$. This means that $\eta_{\mu^0} = \mu^0 = \min((\cap_{i<\mu^0} Y_i)^0)$ which indicates that the construction should have terminated at step μ_0, contradiction.

We conclude that p_1 is defined. The further construction of p_ν's is similar, exploiting the κ-closure of $\leq^*$.

Pick now some $\alpha \geq_E \beta$, for every $\beta \in N \cap \mathrm{dom}(E)$ which exists since $|N| = \kappa$. Set

$$A = \Delta^*_{\nu<\kappa} \tilde{A}(p_\nu) = \{\rho < \kappa \mid \forall \nu < \rho^0 (\rho \in \tilde{A}(p(\nu)))\},$$

where $\tilde{A}(p_\nu)$ is the pre-image of $A(p_\nu)$ under the projection from α to $mc(p_\nu)$. Define a condition $p^* = \langle f^*, A^* \rangle$ from the sequence $\langle p_\nu \mid \nu < \kappa \rangle$ as follows: $\mathrm{Supp}(p^*) = \cup_{\nu<\kappa} \mathrm{Supp}(p_\nu) \cup \{\alpha\}$, from the way we defined p_ν there is no problem defining $f^* = \cup_{\nu<\kappa} f^{p_\nu} \cup \{\langle \alpha, t \rangle\}$ where t is any sequence such that $\pi''_{\alpha,\kappa} t = f^*(\kappa)$. Then we take $A^* = A$. It follows that $p^* \in \mathcal{P}_E$, and it has the property that for every $\nu < \kappa$ and any sequence

$$\xi_1 < \ldots, \xi_k < \min(A^0_\nu) \leq \xi_{k+1} < \ldots < \xi_n$$

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of ordinals from A, $p_\nu^\frown\langle\xi_1,...,\xi_n\rangle \leq p^{*\frown}\langle\xi_1,...,\xi_n\rangle$.[8] Let us argue that it is $\langle N, \mathcal{P}_E\rangle$-generic. Let G be generic with $p^* \in G$. We need to prove that $G \cap N \cap D_\nu \neq \emptyset$ for every $\nu < \kappa$. By density, pick any $p^\frown\langle\xi_1,...,\xi_{k_1}\rangle \leq^* q \in D_\nu \cap G$, and let m be such that $\xi_1,...,\xi_m < \min(A(p_\nu)) \leq \xi_{m+1} < \cdots < \xi_{k_1}$. By condition (d), there is k_2 such that any $\langle\nu_1,...,\nu_{k_2}\rangle \in [A_\nu]^{k_2}$ extension $p_\nu^\frown\langle\xi_1,...,\xi_m\rangle^\frown\langle\nu_1,...,\nu_{k_2}\rangle \in D_\nu$. If necessary, extend q to

$$q^\frown\langle\xi_{k_1+1},...,\xi_{k_1+k_2}\rangle \in G \cap D_\nu,$$

and suppose without loss of generality that $k_1 \geq m + k_2$. Since $\nu < \min(A(p_\nu)^0) \leq \xi_{m+1}$, by definition of $\pi_{\alpha,\kappa}$, it follows that $\nu < \xi^0_{m+1}$, and by diagonal intersection, $\xi_{m+1},...,\xi_{k_1} \in A_\nu$. It follows that

$$p_\nu^\frown\langle\xi_1,...,\xi_m\rangle^\frown\langle\xi_{m+1},...,\xi_{m+k}\rangle \in D_\nu.$$

Also, $p_\nu^\frown\langle\xi_1,...,\xi_m\rangle^\frown\langle\xi_{m+1},...,\xi_{m+k}\rangle \leq q$ hence in G. Hence

$$p_\nu^\frown\langle\xi_1,...,\xi_m\rangle^\frown\langle\xi_{m+1},...,\xi_{m+k}\rangle \in G \cap D_\nu \cap N$$

as wanted. $\dashv$

Now, as in the previous section, the following holds.

THEOREM 4.5. *Let $G \subseteq \mathcal{P}_E$ be a generic. Suppose that $A \in V[G] \setminus V$ is a subset of κ. Then κ changes its cofinality to ω in $V[A]$.*

4.3. The Merimovich version. The previous subsection implies in particular that $\mathcal{P}_E$ and $\mathbb{P}_E$ with P-points cannot add κ^{++}-many mutually generic Cohen functions. In this subsection, we will provide the general argument that the Extender-based Prikry forcing $\mathbb{P}_E$ cannot add κ^{++}-many distinct subsets of κ which preserves even the regularity of κ.

THEOREM 4.6. *Assume GCH[9] and let E be an extender over κ. Let G be a generic subset of $\mathbb{P}_E$ and let $\langle A_\alpha \mid \alpha < \kappa^{++}\rangle$ be different subsets of κ in $V[G]$. Then there is $I \subseteq \kappa^{++}, I \in V, |I| = \kappa$ such that κ is a singular cardinal of cofinality ω in $V[\langle A_\alpha \mid \alpha \in I\rangle]$. In particular, there is no intermediate model of $V[G]$ where κ is measurable and $2^\kappa > \kappa^+$.*

PROOF. Let $\langle \underset{\sim}{A}_\alpha \mid \alpha < \kappa^{++}\rangle$ be $\mathbb{P}_E$-names of subsets of κ. We will confuse them sometimes with their characteristic functions. Work in V, and for every $\alpha < \kappa^{++}$, let N_α be an elementary submodel of H_θ of cardinality κ such that ${}^{\kappa>}N_\alpha \subseteq N_\alpha$, $E, P_E, \alpha, \langle \underset{\sim}{A}_\alpha \mid \alpha < \kappa^{++}\rangle \in N_\alpha$.

Let $f_\alpha \in \mathbb{P}^*_E$ be N_α-completely generic, i.e., $f_\alpha^\frown\langle\vec{\nu}_1,...,\vec{\nu}_n\rangle \in \mathbb{P}^*_E$ is N_α-generic.

Using Δ-system-like arguments, we can assume that $\langle f_\alpha \mid \alpha < \kappa^{++}\rangle$ form a Δ-system such that for every $\alpha, \beta < \kappa^{++}$,

(1) $otp(\mathrm{dom}(f_\alpha)) = otp(\mathrm{dom}(f_\beta))$, and the order isomorphism between $\mathrm{dom}(f_\alpha)$ and $\mathrm{dom}(f_\beta)$, $\sigma_{\alpha,\beta}$ is constant on the intersection $\mathrm{dom}(f_\alpha) \cap \mathrm{dom}(f_\beta)$.
(2) for every $\rho \in \mathrm{dom}(f_\alpha)$, $f_\alpha(\rho) = f_\beta(\sigma_{\alpha\beta}(\rho))$.

[8]Although $\xi_1,...,\xi_k \notin A_\nu$, the condition $p_\nu^\frown\langle\xi_1,...,\xi_n\rangle$ is a legitimate condition which is simply not above p_ν.

[9]$2^\kappa = \kappa^+$ is enough, since κ is a measurable, and so $2^\nu = \nu^+$ on relevant sets.

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Attach to each $\alpha < \kappa^+$ an $E(\mathrm{dom}(f_\alpha))$-large tree T_α. Define T_α level by level as follows. Set $Lev_1(T_\alpha) = S^0_\alpha \cup S^1_\alpha$, where:

(1) for every $\vec{\nu} \in S^0_\alpha$, $\mathrm{dom}(\vec{\nu})$ contains elements in $\mathrm{dom}(f_\alpha) \setminus \mathrm{dom}(f_0)$, if $\alpha > 0$,
(2) if $\alpha = 0$, then $S^0_\alpha = S^1_\alpha$,
(3) $S^1_\alpha = \{\vec{\nu} \mid \vec{\nu}$ is an increasing partial function from $\mathrm{dom}(f_0) \cap \mathrm{dom}(f_\alpha)$ to $\kappa\}$, if $\alpha > 0$,
(4) for every $\vec{\nu} \in S^0_\alpha$, the following holds:
$\langle f_\alpha {}^\frown \vec{\nu}, B_{\vec{\nu}} \rangle$ decides $\underset{\sim}{A}_\alpha \cap \vec{\nu}(\kappa)$ for some $E(\mathrm{dom}(f_\alpha))$-tree $B_{\vec{\nu}}$ and such that the decision depends only on $\vec{\nu}(\kappa)$.

In order to find such a tree, we will use the fact that $f_\alpha \in \mathbb{P}^*_E$ is N_α-generic, and the set

$$E = \{f \mid \exists B. \langle f, B \rangle \text{ decides } \underset{\sim}{A}_\alpha \cap \vec{\nu}(\kappa)\}$$

being dense open in $\mathbb{P}^*_E$. This implies the existence of an $E(\mathrm{dom}(f_\alpha))$-tree $B_{\vec{\nu}}$ such that

$$\langle f_\alpha {}^\frown \vec{\nu}, B_{\vec{\nu}} \rangle \text{ decides } \underset{\sim}{A}_\alpha \cap \vec{\nu}(\kappa).$$

Next, in order to make the decision to depend only on $\vec{\nu}(\kappa)$, we use ineffability: Suppose that $\langle f_\alpha {}^\frown \vec{\nu}, B_{\vec{\nu}} \rangle$ forces that $\underset{\sim}{A}_\alpha \cap \vec{\nu}(\kappa) = A_\alpha(\vec{\nu})$. Let g be the function $g(\vec{\nu}) = A_\alpha(\vec{\nu})$. It follows that

$$X_\alpha(\langle\rangle) := j(g)((j \restriction \mathrm{dom}(f_\alpha))^{-1}) \subseteq \kappa.$$

Also, since $crit(j) = \kappa$, it follows that $j(X_\alpha(\langle\rangle)) \cap \kappa = X_\alpha(\langle\rangle)$. Combine this together with the fact that

$$j'' \,\mathrm{dom}(f_\alpha) \text{ contains elements not in } j(\mathrm{dom}(f_0))$$

to find an $E(\mathrm{dom}(f_\alpha))$-large set S^0_α, such that (1) holds and for all $\vec{\nu} \in S^0_\alpha$,

$$A_\alpha(\vec{\nu}) = X_\alpha(\langle\rangle) \cap \vec{\nu}(\kappa).$$

Finally, we let $Lev_1(T_\alpha) = S^0_\alpha \cup S^1_\alpha$. Note that if $\alpha > 0$, then S^0_α and S^1_α are disjoint and therefore $S^1_\alpha \notin E(\mathrm{dom}(f_\alpha))$. In general, we define by induction on n, then n^{th} level of T_α. So let $\langle \vec{\rho}_1, ..., \vec{\rho}_n \rangle \in Lev_n(T_\alpha)$ and let us define $\mathrm{Succ}_{T_\alpha}(\langle \vec{\rho}_1, ..., \vec{\rho}_n \rangle) = S^0_{\alpha,\langle \vec{\rho}_1,...,\vec{\rho}_n \rangle} \cup S^1_{\alpha,\langle \vec{\rho}_1,...,\vec{\rho}_n \rangle}$, where:

(1) For every $\vec{\nu} \in S^0_{\alpha,\langle \vec{\rho}_1,...,\vec{\rho}_n \rangle} \cup S^1_{\alpha,\langle \vec{\rho}_1,...,\vec{\rho}_n \rangle}$, $\vec{\nu}(\kappa) > \sup(\mathrm{rng}(\rho_n))$.
(2) $S^0_{\alpha,\langle \vec{\rho}_1,...,\vec{\rho}_n \rangle} \subseteq Suc_{B_{\langle \vec{\rho}_1,...,\vec{\rho}_n \rangle}}(\langle \vec{\rho}_1, ..., \vec{\rho}_n \rangle)$.
(3) If $\alpha > 0$, then for every $\vec{\nu} \in S^0_{\alpha,\langle \vec{\rho}_1,...,\vec{\rho}_n \rangle}$, $\mathrm{dom}(\vec{\nu})$ contains elements in $\mathrm{dom}(f_\alpha) \setminus \mathrm{dom}(f_0)$.
(4) If $\alpha = 0$, then $S^0_{\alpha,\langle \vec{\rho}_1,...,\vec{\rho}_n \rangle} = S^1_{\alpha,\langle \vec{\rho}_1,...,\vec{\rho}_n \rangle}$.
(5) If $\vec{\rho}_n \in S^0_{\alpha,\langle \vec{\rho}_1,...,\vec{\rho}_{n-1} \rangle}$ and $\alpha > 0$, then $S^1_{\alpha,\langle \vec{\rho}_1,...,\vec{\rho}_n \rangle} = \emptyset$.
(6) If $\vec{\rho}_n \in S^1_{\alpha,\langle \vec{\rho}_1,...,\vec{\rho}_{n-1} \rangle}$ and $\alpha > 0$, then

$$S^1_{\alpha,\langle \vec{\rho}_1,...,\vec{\rho}_n \rangle} = \{\vec{\nu} \mid \vec{\nu} \text{ is an increasing partial function from}$$

$$\mathrm{dom}(f_0) \cap \mathrm{dom}(f_\alpha) \text{ to } \kappa, \vec{\nu}(\kappa) > \sup(\mathrm{rng}(\vec{\rho}_n))\}.$$

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(7) For every $\vec{\nu} \in S^0_{\alpha,\langle\vec{\rho}_1,\dots,\vec{\rho}_n\rangle}$, the following holds:
$\langle f_\alpha{}^\frown\langle\vec{\rho}_1,\dots,\vec{\rho}_n\rangle^\frown\langle\vec{\nu}\rangle, B_{\langle\vec{\rho}_1,\dots,\vec{\rho}_n\rangle,\vec{\nu}}\rangle$ decides $\underset{\sim}{A}_\alpha \cap \vec{\nu}(\kappa)$ and the decision depends only on $\langle\vec{\rho}_1,\dots,\vec{\rho}_n\rangle^\frown\vec{\nu}(\kappa)$,
for some $E(\mathrm{dom}(f_\alpha))$-tree $B_{\langle\vec{\rho}_1,\dots,\vec{\rho}_n\rangle,\vec{\nu}}$, which is a subtree of $B_{\langle\vec{\rho}_1,\dots,\vec{\rho}_n\rangle}$.

Denote by T^0_α the tree T_α with $S^1_{\alpha\vec{\nu}_1,\dots,\vec{\nu}_n}$ removed from $\mathrm{Succ}_{T_\alpha}(\langle\vec{\nu}_1,\dots,\vec{\nu}_n\rangle)$[10]. Clearly, T^0_α is still $E(\mathrm{dom}(f_\alpha))$-tree.

The tree T^0_α has the property that for every $\langle\vec{\nu}_1,\dots,\vec{\nu}_n\rangle \in T_\alpha$ (!), and every $\vec{\nu} \in \mathrm{Succ}_{T^0_\alpha}(\langle\vec{\nu}_1,\dots,\vec{\nu}_n\rangle)$, item (2) above ensures that $(T^0_\alpha)_{\langle\vec{\nu}_1,\dots,\vec{\nu}_n,\vec{\nu}\rangle} \subseteq B_{\langle\vec{\nu}_1,\dots,\vec{\nu}_n,\vec{\nu}\rangle}$ and by item (7) we obtain

$$(*)\ \langle f_\alpha^\frown\langle\vec{\nu}_1,\dots,\vec{\nu}_n,\vec{\nu}\rangle, (T^0_\alpha)_{\langle\vec{\nu}_1,\dots,\vec{\nu}_n,\vec{\nu}\rangle}\rangle \Vdash X_\alpha(\langle\vec{\nu}_1,\dots,\vec{\nu}_n\rangle) \cap \vec{\nu}(\kappa) = \underset{\sim}{A}_\alpha \cap \vec{\nu}(\kappa).$$

By shrinking if necessary, we can assume that the trees are isomorphic under the obvious isomorphism induced by the Δ-system. Moreover, by GCH, there are only κ^+-many possible decisions on a fixed isomorphism-type of trees, and therefore we can stabilize the decisions, so they do not depend on a particular choice of α. Let us now take κ elements and combine them into a single condition. Namely, we consider $\langle\langle f_\alpha, T_\alpha\rangle \mid 0 < \alpha < \kappa\rangle$ and define a condition $\langle f^*, T^*\rangle$ as follows:

Let $f^* = \bigcup_{0<\alpha<\kappa} f_\alpha$. Define an $E(\mathrm{dom}(f^*))$-tree T^*. It will be a sort of a diagonal intersection of $T_\alpha, 0 < \alpha < \kappa$. Set

$$X = \{\vec{\nu} \mid \vec{\nu} \text{ is an increasing partial function from } \mathrm{dom}(f^*) \text{ to } \kappa,$$

$$\mathrm{dom}(\vec{\nu}) \subseteq \bigcup_{\xi<\vec{\nu}(\kappa)} \mathrm{dom}(f_\xi), (\forall\xi < \vec{\nu}(\kappa))|\,\mathrm{dom}(\vec{\nu}) \cap \mathrm{dom}(f_\xi)| = \vec{\nu}(\kappa)\}.$$

To see that $X \in E(\mathrm{dom}(f^*))$, note that

$$\mathrm{dom}((j \restriction \mathrm{dom}(f^*))^{-1}) = j''\,\mathrm{dom}(f^*) \subseteq \cup_{\xi<\kappa} \mathrm{dom}(j(f_\xi)).$$

Also, for every $\xi < \kappa$, $|j''\,\mathrm{dom}(f^*) \cap \mathrm{dom}(j(f_\xi))| = |j''\,\mathrm{dom}(f_\xi)| = |\,\mathrm{dom}(f_\xi)|$ and since f_ξ is completely generic we conclude that this cardinality must be κ. Hence $(j \restriction \mathrm{dom}(f^*))^{-1} \in j(X)$. Define the first level of the tree[11]

$$Lev_1(T^*) = \mathrm{Succ}_{T^*}(\langle\rangle) := X \cap \Delta^*_{\xi<\kappa}\pi^{-1}_{\mathrm{dom}(f^*)\,\mathrm{dom}(f_\xi)}\,\mathrm{Succ}_{T^0_\xi}(\langle\rangle).$$

Then $Lev_1(T^*) \in E(\mathrm{dom}(f^*))$. To see this, it suffices to prove that the $E(\mathrm{dom}(f^*))$ is closed under the diagonal intersection Δ^*, so if $\langle X_\alpha \mid \alpha < \kappa\rangle \subseteq E(\mathrm{dom}(f^*))$, we claim that $(j \restriction \mathrm{dom}(f^*))^{-1} \in j(\Delta^*_{\alpha<\kappa}X_\alpha)$. Indeed, for every $\alpha < \kappa = (j \restriction \mathrm{dom}(f^*))^{-1}(j(\kappa))$, $j(\alpha) = \alpha$ and the α^{th} element in the sequence $j(\langle X_\alpha \mid \alpha < \kappa\rangle)$ is $j(X_\alpha)$. Since X_α is assumed to be in $E(\mathrm{dom}(f^*))$ we conclude that $(j \restriction \mathrm{dom}(f^*))^{-1} \in j(X_\alpha)$. By the definition of Δ^*, and elementarity of j, we conclude that $(j \restriction \mathrm{dom}(f^*))^{-1} \in j(\Delta^*_{\alpha<\kappa}X_\alpha)$.

[10] Even if $\langle\vec{\nu}_1,\dots,\vec{\nu}_n\rangle \in T_\alpha \setminus T^0_\alpha$ the set $\mathrm{Succ}_{T^0_\alpha}(\langle\vec{\nu}_1,\dots,\vec{\nu}_n\rangle)$ is still defined.

[11] We define the diagonal intersection for the ultrafilter $E(d)$ as follows: for $\langle X_\alpha \mid \alpha < \kappa\rangle \subseteq E(d)$, $\Delta^*_{\alpha<\kappa}X_\alpha = \{\vec{\nu} \in Ob(d) \mid \forall\xi < \vec{\nu}(\kappa).\vec{\nu} \in X_\xi\}$.

We continue to define inductively the level of T^*. Let now $\langle \vec{\rho}_1, ..., \vec{\rho}_n \rangle \in Lev_n(T^*)$, and define $\mathrm{Succ}_{T^*}(\langle \vec{\rho}_1, ..., \vec{\rho}_n \rangle)$. As above, we consider first the set

$$X_{\langle \vec{\rho}_1,...,\vec{\rho}_n \rangle} = \{\vec{v} \mid \vec{v} \text{ is an increasing partial function from } \mathrm{dom}(f^*) \text{ to } \kappa, \vec{v}(\kappa) > \sup(\mathrm{rng}(\vec{\rho}_n)),$$

$$\mathrm{dom}(\vec{v}) \subseteq \bigcup_{\xi < \vec{v}(\kappa)} \mathrm{dom}(f_\xi), (\forall \xi < \vec{v}(\kappa)) |\mathrm{dom}(\vec{v}) \cap \mathrm{dom}(f_\xi)| = \vec{v}(\kappa)\}.$$

Clearly, $X_{\langle \vec{\rho}_1,...,\vec{\rho}_n \rangle} \in E(\mathrm{dom}(f^*))$. Let $\mathrm{Succ}_{T^*}(\langle \vec{\rho}_1, ..., \vec{\rho}_n \rangle)$ be the set

$$X_{\langle \vec{\rho}_1,...,\vec{\rho}_n \rangle} \cap \Delta^*_{\xi<\kappa} \pi^{-1}_{\mathrm{dom}(f^*)\,\mathrm{dom}(f_\xi)} \mathrm{Succ}_{T^0_\xi}(\langle \vec{\rho}_1 \restriction \mathrm{dom}(f_\xi), ..., \vec{\rho}_n \restriction \mathrm{dom}(f_\xi) \rangle).$$

Once we ensure that for every $\xi < \kappa$, $\mathrm{Succ}_{T^0_\xi}(\langle \vec{\rho}_1 \restriction \mathrm{dom}(f_\xi), ..., \vec{\rho}_n \restriction \mathrm{dom}(f_\xi) \rangle)$ is well defined, then T^* will form an $E(\mathrm{dom}(f^*))$-fat tree. Namely, we need to prove that:

Claim 4.7. *For every $\xi < \kappa$, $\langle \vec{\rho}_1 \restriction \mathrm{dom}(f_\xi), ..., \vec{\rho}_n \restriction \mathrm{dom}(f_\xi) \rangle \in Lev_n(T_\xi)$. Moreover, $\xi < \vec{\rho}_1(\kappa)$ iff $\langle \vec{\rho}_1 \restriction \mathrm{dom}(f_\xi), ..., \vec{\rho}_n \restriction \mathrm{dom}(f_\xi) \rangle \in Lev_n(T^0_\xi)$.*

Proof of Claim 4.7. For every $\xi < \vec{\rho}_1(\kappa)$, we have

$$\vec{\rho}_1 \in \pi^{-1}_{\mathrm{dom}(f^*)\,\mathrm{dom}(f_\xi)}(Lev_1(T^0_\xi))$$

and therefore $\vec{\rho}_1 \restriction \mathrm{dom}(f_\xi) \in Lev_1(T^0_\xi)$. If $\xi \geq \vec{\rho}_1(\kappa)$, then since $\vec{\rho}_1 \in X$, the Δ-system ensures that $\mathrm{dom}(\vec{\rho}_1) \cap \mathrm{dom}(f_\xi) = \mathrm{dom}(\vec{\rho}_1) \cap \mathrm{dom}(f_0) \subseteq \mathrm{dom}(f_0) \cap \mathrm{dom}(f_\xi)$. It follows from the definition that $\vec{\rho}_1 \restriction \mathrm{dom}(f_\xi) = \vec{\rho}_1 \restriction \mathrm{dom}(f_0) \in S^1_\alpha \subseteq Lev_1(T_\alpha)$. Suppose that $\langle \vec{\rho}_1 \restriction \mathrm{dom}(f_\xi), ..., \vec{\rho}_n \restriction \mathrm{dom}(f_\xi) \rangle \in Lev_n(T_\xi)$, and let $\vec{\rho}_{n+1} \in Succ_{T^*}(\langle \vec{\rho}_1, ..., \vec{\rho}_n \rangle)$. Then for every $\xi < \vec{\rho}_{n+1}(\kappa)$, $\vec{\rho}_{n+1} \restriction \mathrm{dom}(f_\xi) \in \mathrm{Succ}_{T^0_\xi}(\langle \vec{\rho}_1 \restriction \mathrm{dom}(f_\xi), ..., \vec{\rho}_n \restriction \mathrm{dom}(f_\xi) \rangle)$ by the definition of the diagonal intersection. If $\xi \geq \vec{\rho}_{n+1}(\kappa)$, then, as before, $\vec{\rho}_{n+1} \restriction \mathrm{dom}(f_\xi) \in S^1_{\xi, \langle \vec{\rho}_1,...,\vec{\rho}_n \rangle}$. ⊣

$Lev_1(T^*)$ has the property that for all $\vec{\rho} \in Lev_1(T^*)$ and $\alpha < \vec{\rho}(\kappa)$,

$$\langle f^{*\frown}\vec{\rho}, (T^*)_{\vec{\rho}} \rangle \geq^* \langle f_\alpha{}^\frown \vec{\rho} \restriction \mathrm{dom}(f_\alpha), (T^0_\alpha)_{\vec{\rho} \restriction \mathrm{dom}(f_\alpha)} \rangle.$$

Hence, by $(*)$, $\langle f^{*\frown}\vec{\rho}, (T^*)_{\vec{\rho}} \rangle$ also forces $X_\alpha(\langle\rangle) \cap \vec{\rho}(\kappa) = \underset{\sim}{A}_\alpha \cap \vec{\rho}(\kappa)$. In addition, if we have $\alpha, \beta < \vec{\rho}(\kappa)$, then $\underset{\sim}{A}_\alpha \cap \vec{\rho}(\kappa)$, $\underset{\sim}{A}_\beta \cap \vec{\rho}(\kappa)$ depends only on $(\vec{\rho} \restriction \mathrm{dom}(f_\alpha))(\kappa) = \vec{\rho}(\kappa) = (\vec{\rho} \restriction \mathrm{dom}(f_\beta))(\kappa)$, and since the isomorphism $\sigma_{\alpha,\beta}$ fixes κ (as $\kappa \in \mathrm{dom}(f_\alpha) \cap \mathrm{dom}(f_\beta)$) it follows that $\underset{\sim}{A}_\beta \cap \vec{\rho}(\kappa)$, $\underset{\sim}{A}_\alpha \cap \vec{\rho}(\kappa)$ are decided to be the same set.

Next consider $\langle \vec{\rho}_1, ..., \vec{\rho}_n \rangle \in T^*$, by Claim 4.7, and we have that for all $\alpha < \vec{\rho}_n(\kappa)$,

$$(**) \qquad \langle f^{*\frown}\langle \vec{\rho}_1, ..., \vec{\rho}_n \rangle, (T^*)_{\langle \vec{\rho}_1,...,\vec{\rho}_n \rangle} \rangle \geq$$

$$\geq \langle f_\alpha^\frown \langle \vec{\rho}_1 \restriction \mathrm{dom}(f_\alpha), ..., \vec{\rho}_n \restriction \mathrm{dom}(f_\alpha) \rangle, (T^0_\alpha)_{\langle \vec{\rho}_1 \restriction \mathrm{dom}(f_\alpha),...,\vec{\rho}_n \restriction \mathrm{dom}(f_\alpha) \rangle} \rangle.$$

However, since the decision about $\underset{\sim}{A}_\alpha \cap \vec{\rho}_n(\kappa)$ depends now on $\langle \vec{\rho}_1, ..., \vec{\rho}_{n-1} \rangle^\frown \vec{\rho}_n(\kappa)$, then if α or β are below $\vec{\rho}_{n-1}(\kappa)$, then $\vec{\rho}_{n-1} \restriction \mathrm{dom}(f_\alpha)$ (or $\vec{\rho}_i \restriction \mathrm{dom}(f_\alpha)$ for $i < n$) might include in its domain ordinals which are moved under the isomorphism $\sigma_{\alpha,\beta}$ and therefore we are not guaranteed that the decision about $\underset{\sim}{A}_\alpha \cap \vec{v}(\kappa)$, $\underset{\sim}{A}_\beta \cap \vec{v}(\kappa)$

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is the same (up to $\vec{\rho}_1(\kappa)$ it is still the same decision). However, if both $\alpha, \beta \in [\vec{\rho}_{n-1}(\kappa), \vec{\rho}_n(\kappa))$, we have the following claim:

CLAIM 4.8. *If $\alpha, \beta \in [\vec{\rho}_{n-1}(\kappa), \vec{\rho}_n(\kappa))$ then $\langle f^{*\frown}\langle\vec{\rho}_1, ..., \vec{\rho}_n\rangle, (T^*)_{\langle\vec{\rho}_1,...,\vec{\rho}_n\rangle}\rangle$ decides the values of $\underset{\sim}{A}_\alpha \cap \vec{\rho}_n(\kappa)$ and $\underset{\sim}{A}_\beta \cap \vec{\rho}_n(\kappa)$ to be the same.*

PROOF OF CLAIM 4.8. By definition, for every $1 \leq i < n$, $\vec{\rho}_i \in X_{\langle\vec{\rho}_1,...,\vec{\rho}_{i-1}\rangle}$. Since $\alpha, \beta \geq \vec{\rho}_{n-1}(\kappa) \geq \vec{\rho}_i(\kappa)$,

$$\vec{\rho}_i \restriction \mathrm{dom}(f_\alpha) = \vec{\rho}_i \restriction \mathrm{dom}(f_\beta) \text{ and } \mathrm{dom}(\vec{\rho}_i \restriction \mathrm{dom}(f_\alpha)) \subseteq \mathrm{dom}(f_\alpha) \cap \mathrm{dom}(f_0).$$

Since the isomorphism $\sigma_{\alpha,\beta}$ fixes the kernel of the Δ-system, we have that the decision of

$$\langle f_\alpha^\frown\langle\vec{\rho}_1 \restriction \mathrm{dom}(f_\alpha), ..., \vec{\rho}_n \restriction \mathrm{dom}(f_\alpha)\rangle, (T^0_\alpha)_{\langle\vec{\rho}_1\restriction\mathrm{dom}(f_\alpha),...,\vec{\rho}_n\restriction\mathrm{dom}(f_\alpha)\rangle}\rangle$$

about $\underset{\sim}{A}_\alpha \cap \vec{\rho}_n(\kappa)$ and the decision of

$$\langle f_\beta^\frown\langle\vec{\rho}_1 \restriction \mathrm{dom}(f_\beta), ..., \vec{\rho}_n \restriction \mathrm{dom}(f_\beta)\rangle, (T^0_\beta)_{\langle\vec{\rho}_1\restriction\mathrm{dom}(f_\beta),...,\vec{\rho}_n\restriction\mathrm{dom}(f_\beta)\rangle}\rangle$$

about $\underset{\sim}{A}_\beta \cap \vec{\rho}_n(\kappa)$ is the same. By $(**)$, the condition $\langle f^{*\frown}\langle\vec{\rho}_1, ..., \vec{\rho}_n\rangle, (T^*)_{\vec{\rho}_1,...,\vec{\rho}_n}\rangle$ decides the values the same way. ⊣

Using density arguments we can assume that such defined condition $\langle f^*, T^*\rangle$ is in the generic subset G of $\mathbb{P}_E$. Denote by $\langle\kappa_n \mid n < \omega\rangle$ the Prikry sequence for the normal measure E_κ.

It follows that the sets $\langle A_\alpha \mid \alpha < \kappa\rangle$ have the following property in $V[G]$:

$$(***)\ \forall n < \omega . \forall\alpha, \beta \in [\kappa_{n-1}, \kappa_n). A_\alpha \cap \kappa_n = A_\beta \cap \kappa_n.$$

Now, let us turn to the model $M^* = V[\langle A_\alpha \mid \alpha < \kappa\rangle]$ and prove that $cf^{M^*}(\kappa) = \omega$. Let us define in M^* an ω-sequence $\langle\zeta_n \mid n < \omega\rangle$ as follows:

First, let ζ'_0 be the least such that for some for some $\alpha, \beta < \kappa$, $A_\alpha \cap \zeta'_0 \neq A_\beta \cap \zeta'_0$. There exists such ζ'_0 since the sets in the sequence $\langle A_\alpha \mid \alpha < \kappa\rangle$ are distinct. Let ζ''_0 be the least such that for some $\alpha < \zeta''_0$, $A_\alpha \cap \zeta'_0 \neq A_{\zeta''_0} \cap \zeta'_0$. Define $\zeta_0 = \max(\zeta'_0, \zeta''_0)$

CLAIM 4.9. $\zeta_0 \geq \kappa_0$.

PROOF OF CLAIM 4.9. If $\zeta'_0 \geq \kappa_0$ then we are done. Otherwise, suppose $\zeta'_0 \leq \kappa_0$, then by $(***)$ for every $\alpha < \beta < \kappa_0$, we have $A_\alpha \cap \zeta'_0 = A_\beta \cap \zeta'_0$. Hence by the definition of ζ''_0, we have $\zeta''_0 \geq \kappa_0$ and also $\zeta_0 \geq \kappa_0$. ⊣

Suppose that $\zeta_n < \kappa$ was defined. Then the sequence $\langle A_\alpha \mid \zeta_n < \alpha < \kappa\rangle$ consists of κ-many distinct subsets of κ. Since κ is strong limit in $V[G]$, $2^{\zeta_n} < \kappa$, hence there must be $\zeta_n < \alpha < \beta < \kappa$ such that $A_\alpha \setminus \zeta_n + 1 \neq A_\beta \setminus \zeta_n + 1$. Let ζ'_{n+1} be the minimal such that for some $\zeta_n < \alpha < \beta < \kappa$, $A_\alpha \cap \zeta'_{n+1} = A_\beta \cap \zeta'_{n+1}$. Finally, let $\zeta_n < \zeta''_{n+1}$ be the minimal such that for some $\alpha < \zeta''_{n+1}$, $A_\alpha \cap \zeta'_{n+1} \neq A_{\zeta''_{n+1}} \cap \zeta'_{n+1}$ and $\zeta_{n+1} = \max(\zeta'_{n+1}, \zeta''_{n+1})$. To conclude that $cf^{M^*}(\kappa) = \omega$ is suffices to prove the following lemma:

CLAIM 4.10. *For every $n < \omega$, $\zeta_n \geq \kappa_n$.*

PROOF OF CLAIM 4.10. By induction, for $n = 0$ this is just the previous claim. Suppose that $\zeta_n \geq \kappa_n$, and toward a contradiction suppose that $\zeta_{n+1} < \kappa_{n+1}$. Then

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by definition, there is α, such that $\kappa_n \leq \zeta_n < \alpha < \zeta''_{n+1} < \kappa_{n+1}$ such that $A_\alpha \cap \zeta'_{n+1} \neq A_{\zeta''_{n+1}} \cap \zeta'_{n+1}$. However, since $\zeta'_{n+1} < \kappa_{n+1}$ we reached a contradiction to $(***)$, since we found two indices $\alpha, \beta \in [\kappa_n, \kappa_{n+1})$ such that $A_\alpha \cap \kappa_{n+1} \neq A_\beta \cap \kappa_{n+1}$. $\dashv$

The sequence $\langle \zeta_n \mid n < \omega \rangle$ will be a cofinal sequence in κ which belongs to $V[\langle A_\alpha \mid \alpha < \kappa \rangle]$. $\dashv$

It turns out that $\mathbb{P}_E$ can add κ^+-many mutually generic over V Cohen functions, for specially chosen extender E.

THEOREM 4.11. *Assume GCH and suppose that E is a (κ, κ^{++})-extender. Then after the preparation of Theorem 2.10, there exists an extender E' such that $\mathbb{P}_{E'}$ adds κ^+ mutually generic over V Cohen functions.*

PROOF. Let $j = j_E : V \to M$ be the natural ultrapower by the (κ, κ^{++}) – extender E, then $j(\kappa) > \kappa^{++}$, $crit(j) = \kappa$, and ${}^\kappa M \subseteq M$. Recall that the preparation forcing in Theorem 2.10 is an Easton support iteration

$$\langle \mathcal{P}_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa + 1, \beta \leq \kappa \rangle$$

such that Q_β is trivial unless β is inaccessible in which case if $\beta < \kappa$ then Q_β is a $\mathcal{P}_\beta$-name for LOTT$(\text{Cohen}(\beta, \beta^+), \text{Cohen}(\beta, \beta^+)^2)$. At κ, Q_κ is a name for $\text{Cohen}(\kappa, \kappa^+)$. Let $G_\kappa * g_\kappa$ be V-generic for $P_\kappa * Q_\kappa$. In $V[G_\kappa * g_\kappa]$ we can construct an M-generic filter for $j(\mathcal{P}_\kappa * Q_\kappa)$ by taking $G_\kappa * g_\kappa$ to be the generic up to κ, including κ and choosing that the lottery sum forces $\text{Cohen}(\kappa, \kappa^+)$ (this forcing is the same in $V[G_\kappa]$ and $M[G_\kappa]$ since $(\kappa^+)^{M[G_\kappa]} = \kappa^+$ and $M[G_\kappa]$ is closed under κ-sequences of $V[G_\kappa]$). Above κ we have sufficient closure, from the point of view of $V[G_\kappa * g_\kappa]$, and by GCH there are not too many dense open subsets of the tail forcing $\mathcal{P}_{(\kappa, j(\kappa)]}$ to meet, hence the embedding j lifts to

$$j \subseteq j^* : V[G_\kappa * g_\kappa] \to M[j(G_\kappa) * j(g_\kappa)].$$

Since the cardinals in all the models are preserved, it follows that [12, Proposition 8.4]

$$(\kappa^{++})^{M[j(G_\kappa)*j(g_\kappa)]} = \kappa^{++} < j(\kappa) \text{ and } {}^\kappa M[j(G_\kappa) * j(g_\kappa)] \subseteq M[j(G_\kappa) * j(g_\kappa)].$$

So in $V[G_\kappa * g_\kappa]$ the extender E extends to an extender $E' = \langle E'_a \mid a \in [\kappa]^{<\omega} \rangle$ defined by $E'_a = \{X \subseteq \kappa^{|a|} \mid a \in j^*(X)\}$.

Let W be the non-Galvin, κ-complete ultrafilter over κ with preparation for adding κ^+-many Cohens (See Theorem 2.11).

Combine E', W together as follows. First take an ultrapower with E'. Let $j_{E'} : V \to M_{E'}$ be the corresponding embedding. Denote $j_{E'}(\kappa)$ by κ_1 and let $W' = j_{E'}(W)$. Then take an ultrapower of $M_{E'}$ with W'. Let $j_{W'} : M_{E'} \to M$ be the corresponding embedding.

Consider $j_* = j_{W'} \circ j_{E'} : V \to M$. Let E^* be the derived (κ, λ)-extender for some $\kappa_1 < \lambda \leq j_*(\kappa)$.

Note that $E^*(\kappa_1) = W$, since for any $X \subseteq \kappa$,

$$X \in E^*(\kappa_1) \Leftrightarrow \kappa_1 \in j_*(X) \Leftrightarrow \kappa_1 \in j_{W'}(j_{E'}(X)) \Leftrightarrow j_{E'}(X) \in W'$$

$$\Leftrightarrow j_{E'}(X) \in j_{E'}(W) \Leftrightarrow X \in W.$$

The Prikry forcing with W adds κ^+-many Cohens over V. This forcing is a part of $\mathbb{P}_{E^*}$, since W appears as one of the measures of E^*, which implies the theorem. ⊣

4.4. Cohen subsets of κ^+. Let us argue here that both versions add κ^{++}-many (or λ-many if the extender has λ generators for a regular $\lambda > \kappa$) Cohen subsets of κ^+ mutually generic over V.

Start with $\mathcal{P}_E$ of [21].

THEOREM 4.12. *Let $G \subseteq \mathcal{P}_E$ be a generic. Then in $V[G]$ there is a sequence $\langle Z_\xi \mid \xi < \kappa^{++}\rangle$ of mutually generic over V Cohen subsets of κ^+.*

PROOF. Let $\langle t_\alpha \mid \alpha < \kappa^{++}\rangle$ be the Prikry sequences added by G.

Split, in V, κ^{++} into disjoint intervals $\langle I_\xi \mid \xi < \kappa^{++}\rangle$ order type of each κ^+. Denote by σ_ξ the order isomorphism between I_ξ and κ^+.

Now, in $V[G]$, set

$$Z_\xi = \{\sigma_\xi(\alpha) \in I_\xi \mid t_\alpha(0) \text{ is even }\}.$$

Let us argue that such a sequence is as desired.

Work in V. Let $p \in \mathcal{P}$ and let D be a dense open subset of Cohen(κ^+, κ^{++}).

Let us find $q \geq p$ such that

$$q \Vdash \langle \underset{\sim}{Z}_\xi \mid \xi < \kappa^{++}\rangle \text{ extends an element of } D.$$

Extend first p to some r such that for every $\gamma \in \mathrm{Supp}(r)$, r^γ is not equal to the empty sequence. Now, using I_ξ, σ_ξ's turn $\langle r^\gamma(0) \mid \gamma \in \mathrm{Supp}(r)\rangle$ into a condition in Cohen(κ^+, κ^{++}). Extend it to one in D and move back to $\mathcal{P}$ using I_ξ, σ_ξ^{-1}'s. Finally, turn the result into a condition q in $\mathcal{P}$ stronger than r. It will be as desired. ⊣

The situation in the case of the Merimovich version is very similar:

THEOREM 4.13. *Let $G \subseteq \mathbb{P}_E$ be a generic. Then in $V[G]$ there is a sequence $\langle Z_\xi \mid \xi < \kappa^{++}\rangle$ of mutually generic over V Cohen subsets of κ^+.*

PROOF. Proceed as in Theorem 4.12 and define $\langle Z_\xi \mid \xi < \kappa^{++}\rangle$.

Work in V. Let $p \in \mathcal{P}$ and let D be a dense open subset of Cohen(κ^+, κ^{++}).

Let us find $q \geq p$ such that

$$q \Vdash \langle \underset{\sim}{Z}_\xi \mid \xi < \kappa^{++}\rangle \text{ extends an element of } D.$$

A slight difference here is that the support of $p = \langle f, T\rangle$, i.e., dom(f) may have κ many places γ with $f(\gamma) = \langle\rangle$.

As a result, for such γ, $t_\gamma(0)$ will be determined only after an element of the corresponding set of measure one is picked, and there are κ-many such γ's.

However, we do not need the exact value of $t_\gamma(0)$, but rather to know whether it is even or odd. This is determined (on a set of measure one) by γ itself. Namely, in this situation, $t_\gamma(0)$ will be even iff γ is even.

The rest of the argument is as in Theorem 4.12. ⊣

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REFERENCES

[1] U. ABRAHAM and S. SHELAH, *Forcing closed unbounded sets*, this JOURNAL, vol. 48 (1983), no. 3, pp. 643–657.

[2] J. E. BAUMGARTNER, A. HAJNAL, and A. MATE, *Weak saturation properties of ideals*, ***Infinite and Finite Sets (Colloq., Keszthely, 1973; Dedicated to P. Erdős on His 60th Birthday), vol. I***, Colloquia Mathematica Societatis János Bolyai, 10, North-Holland, Amsterdam, 1975, pp. 137–158.

[3] T. BENHAMOU, *Prikry forcing and tree Prikry forcing of various filters*. ***Archive for Mathematical Logic***, vol. 58 (2019), pp. 787–817.

[4] T. BENHAMOU, S. GARTI, and A. POVEDA, *Negating the Galvin property*. ***Journal of the London Mathematical Society***, vol. 108 (2023), pp. 190–237.

[5] T. BENHAMOU, S. GARTI, and S. SHELAH, *Kurepa trees and the failure of the Galvin property*. ***Proceedings of the American Mathematical Society***, vol. 151 (2023), pp. 1301–1309.

[6] T. BENHAMOU and M. GITIK, *Sets in Prikry and Magidor generic extensions*. ***Annals of Pure and Applied Logic***, vol. 172 (2021), no. 4, p. 102926.

[7] ———, *Intermediate models of Magidor–Radin forcing—part II*. ***Annals of Pure and Applied Logic***, vol. 173 (2022), p. 103107.

[8] ———, *Intermediate models of Magidor–Radin foring—part I*. ***Israel Journal of Mathematics***, vol. 252 (2022), pp. 47–94.

[9] T. BENHAMOU, M. GITIK, and Y. HAYUT, *The variety of projections of a tree-Prikry forcing*. ***Journal of Mathematical Logic***, 2021, to appear. Preprint available from arXiv:2109.09069.

[10] Y. BEN-SHALOM, *On the Woodin construction of failure of GCH at a measurable cardinal, preprint*, 2017, arXiv:1706.08143.

[11] J. CUMMINGS, *A model in which GCH holds at successors but fails at limits*. ***Transactions of the American Mathematical Society***, vol. 329 (1992), no. 1, pp. 1–39.

[12] ———, *Iterated forcing and elementary embeddings*, ***Handbook of Set Theory*** (M. Foreman and A. Kanamori, editors), Springer, Dordrecht, 2010, pp. 775–883.

[13] S. GARTI, *Weak diamond and Galvin's property*. ***Periodica Mathematica Hungarica***, vol. 74 (2017), no. 1, pp. 128–136.

[14] ———,*Tiltan*. ***Comptes Rendus de l'Académie des Sciences***, vol. 356 (2018), no. 4, pp. 351–359.

[15] M. GITIK,*The negation of the singular cardinal hypothesis from $o(\kappa) = \kappa^{++}$*. ***Annals of Pure and Applied Logic***, vol. 43 (1989), pp. 209–234.

[16] ———, *The strength of the failure of the singular cardinal hypothesis*. ***Annals of Pure and Applied Logic***, vol. 51 (1991), no. 3, pp. 215–240.

[17] ———, *Prikry-type forcings*, ***Handbook of Set Theory*** (M. Foreman and A. Kanamori, editors), Springer, Dordrecht, 2010, pp. 1351–1447.

[18] ———, *On density of old sets in Prikry type extensions*. ***Proceedings of the American Mathematical Society***, vol. 145 (2017), no. 2, pp. 881–887.

[19] ———, *On κ-compact cardinals*. ***Israel Journal of Mathematics***, vol. 237 (2020), pp. 457–483.

[20] M. GITIK, V. KANOVEI, and P. KOEPKE, *Intermediate models of Prikry generic extensions*, 2010. Available at http://www.math.tau.ac.il/gitik/spr-kn.pdf.

[21] M. GITIK and M. MAGIDOR, *Extender based forcings*, this JOURNAL, vol. 59 (1994), no. 2, pp. 445–460.

[22] J. D. HAMKINS, *The lottery preparation*. ***Annals of Pure and Applied Logic***, vol. 101 (2000), nos. 2–3, pp. 103–146.

[23] T. JECH, ***Set Theory***, the third millennium edition, revised and expanded, Springer Monographs in Mathematics, Springer, Berlin, 2003.

[24] A. KANAMORI, ***The Higher Infinite***, Springer, Berlin, 1994.

[25] P. KOEPKE, K. RASCH, and P. SCHLICHT, *Minimal Prikry-type forcing for singularizing a measurable cardinal*, this JOURNAL, vol. 78 (2013), pp. 85–100.

[26] M. MAGIDOR, *On the singular cardinals problem II*. ***Annals of Mathematics***, vol. 106 (1977), no. 3, pp. 517–547.

[27] D. MAHARAM, *An algebraic characterization of measure algebras*. ***Annals of Mathematics***, vol. 48 (1947), pp. 154–167.

[28] A. R. D. MATHIAS, *On sequences generic in the sense of Prikry*. ***Journal of Australian Mathematical Society***, vol. 15 (1973), pp. 409–414.

[29] C. MERIMOVICH, *Prikry on extenders, revisited*. ***Israel Journal of Mathematics***, vol. 160 (2003), pp. 253–280.

[30] ———, *Supercompact extender based Prikry forcing*. ***Archive for Mathematical Logic***, vol. 50 (2011), pp. 591–602.

[31] ———, *Mathias like criterion for the extender based Prikry forcing*. ***Annals of Pure and Applied Logic***, vol. 172 (2021), p. 102994.

[32] W. J. MITCHELL, *The core model for sequences of measures, I*. ***Mathematical Proceedings of the Cambridge Philosophical Society***, vol. 95 (1984), no. 2, pp. 229–260.

[33] ———, *On the singular cardinal hypothesis*. ***Transactions of the American Mathematical Society***, vol. 329 (1992), no. 2, pp. 507–530.

[34] G. E. SACKS, *Forcing with perfect closed sets, axiomatic set theory*. ***Proceedings of Symposia in Pure Mathematics***, vol. 13 (1971), no. 1, pp. 331–355.

SCHOOL OF MATHEMATICAL SCIENCES
RAYMOND AND BEVERLY SACKLER FACULTY OF EXACT SCIENCE, TEL-AVIV UNIVERSITY
RAMAT AVIV 69978
ISRAEL

E-mail: tombenhamou@tauex.tau.ac.il
E-mail: gitik@tauex.tau.ac.il