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MULTIPLE SOLUTIONS FOR A CLASS OF QUASILINEAR ELLIPTIC PROBLEMS

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Abstract We deal with a class of p-Laplacian Dirichlet boundary-value problems where the combined effects of 'sublinear' and 'superlinear' growths allow us to establish the existence of at least two positive solutions.

Keywords: p-Laplacian; positive solutions; fixed-point index; superlinear problems

AMS 2000 *Mathematics subject classification:* Primary 35J25; 35J60
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1. Introduction

The objective of this paper is to establish the existence of two radial solutions for the quasilinear boundary-value problem

$$\left.\begin{aligned} -\Delta_p u &= f(u) && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a ball of radius b, and where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian with $1 < p < N$. We will assume that the function $f : [0, +\infty) \to [0, +\infty)$ is a given continuous function satisfying $f(0) = 0$ and the following two conditions:

(H_1) $\lim_{t\to 0} f(t)/t^{p-1} = +\infty$, and

(H_2) $\lim_{t\to +\infty} f(t)/t^{p-1} = +\infty$.

It follows from assumptions (H_1) and (H_2) that there exists $R > 0$ such that

$$\frac{f(R)}{R^{p-1}} = \min_{t>0} \frac{f(t)}{t^{p-1}}.$$

Let $\bar{R}$ be a point where f attains its maximum on the interval $(0, R]$. We will assume the following two further conditions:

(H_3) $f(\bar{R})/\bar{R}^{p-1} < \eta = (p/(p-1))^{p-1}N/b^p$, and

(H_4) there exist increasing functions $g_1, g_2 \in C([0, +\infty), [0, +\infty))$ and positive constants δ, η, with $\delta \in (0, 1)$, such that for all $t > 0$,

$$g_2(t) \leqslant \eta g_1(\delta t),$$
$$g_1(t) \leqslant f(t) \leqslant g_2(t).$$

Our main result is Theorem 1.1, which will be proved in § 3 using fixed-point techniques.

Theorem 1.1. *Under the assumptions* (H_1)–(H_4), *problem (1.1) has at least two radial solutions.*

Our study was motivated by some recent work on elliptic problems with concave–convex nonlinearities (see [**1**–**3**, **9**, **11**, **12**]). Ambrosetti *et al.* [**1**] study the second-order elliptic problem

$$\left.\begin{aligned} -\Delta u &= \lambda u^s + u^r && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ (for $N \geqslant 3$) with smooth boundary $\partial\Omega$, Δ is the Laplace operator, λ is a positive real parameter, and $0 < s < 1 < r$. They prove that there exists a positive real constant Λ such that, for all $0 < \lambda < \Lambda$, the problem (1.2) has a solution, which is found using subsolution as well as supersolution methods. Here the essential term is u^s, while the exponent r may be arbitrary. Using variational methods, a second solution of (1.2) is found. In this case, the term u^r plays a fundamental role, where r must satisfy $r \leqslant (N+2)/(N-2)$. Among others, the following question is left open. Suppose that $r > (N+2)/(N-2)$ and that Ω is a ball. Does the problem (1.2) have two positive solutions for λ small enough? In [**12**], it is proved that the assertion is true.

Difficulties arise while extending the study of the problem (1.2) to the p-Laplacian operator. Many known techniques and results for the Laplacian no longer apply for the p-Laplacian due to its nonlinear nature. Using a radial setting, *a priori* estimates and topological arguments, Ambrosetti *et al.* [**2**] obtained a global multiplicity result for elliptic problems of the form

$$\left.\begin{aligned} -\Delta_p u &= \lambda u^s + u^r && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.3}$$

More precisely, they prove that there is $\Lambda > 0$ such that there exist at least two positive solutions of the problem (1.3) in the interval $(0, \Lambda)$, where Ω is a ball and $0 < s < p-1 < r < p^* = Np/(N-p)$, with $p < N$. In [**3**], the critical case considering the following restrictive assumptions on p is studied: $2N/(N+2) < p < 3$ or $p \geqslant 3$ and $p-1 > s > (p^*+1) - 2/(p-1)$. Related results may be found in [**4**, **8**]. For global multiplicity results on a general bounded domain in the subcritical case, see [**9**]. When $1 \leqslant s < p-1 < r \leqslant p^* - 1$, which includes the critical case, see [**11**].

Observe that we improve those results for the p-Laplacian operator which involve concave and convex nonlinearities because we impose restrictions neither on $p \in (1, N)$ nor on the growth of the nonlinearities, which may be subcritical, critical or supercritical. Note that the nonlinearities we consider are sublinear at 0 and superlinear at $+\infty$, and hence they contain the concave and convex nonlinearities above. We point out that our result is an improvement even in the case studied in [**12**] because we consider more general nonlinearities. For instance, let $g_1(t) = a_1 t^s + b_1 t^r \leqslant g_2(t) = a_2 t^s + b_2 t^r$, where $0 < s < p-1 < r$, and where a_1, b_1, a_2 and b_2 are positive constants. Assume that $g_1(t) = a_1 t^s + b_1 t^r \leqslant f(t) \leqslant g_2(t) = a_2 t^s + b_2 t^r$. It is easy to see that f satisfies the hypotheses of Theorem 1.1. Finally, note that, in [**6**], De Figueiredo and Lions studied the Laplacian operator with subcritical nonlinearities that satisfy a sublinearity condition at 0 and a superlinearity condition at $+\infty$.

The paper is organized as follows: § 2 contains preliminary results and § 3 is devoted to proving our main result, Theorem 1.1.

2. Preliminary results

We will establish radial solutions of the problem (1.1). In fact, we will obtain solutions $u = u(r)$ of the ordinary equation

$$\left.\begin{aligned} -(r^{N-1}\phi(u'))' = r^{N-1} f(u) \qquad & \text{in } (0, b), \\ u > 0 \qquad & \text{in } (0, b), \\ u(b) = u'(0) = 0, \qquad & \end{aligned}\right\} \tag{2.1}$$

where $\phi(t) = |t|^{p-2} t$. Performing the change of variable $t = a(r)$, define $z(t) = u(r(t))$, where $a : [0, b) \to [0, +\infty)$ is given by

$$a(r) = \frac{p-1}{N-p}[r^{(p-N)/(p-1)} - b^{(p-N)/(N-1)}].$$

Thus (2.1) can be rewritten as

$$\left.\begin{aligned} -(\phi(z'(t)))' = r^{(N-1)p/(p-1)}(t) f(z(t)) \qquad & \text{in } (0, +\infty), \\ z > 0 \qquad & \text{in } (0, +\infty), \\ z(0) = z'(+\infty) = 0. \qquad & \end{aligned}\right\} \tag{2.2}$$

Integrating (2.2) and using the boundary conditions we obtain

$$\phi(z'(t)) = \int_t^{+\infty} r^{(N-1)p/(p-1)}(\tau) f(z(\tau)) \, d\tau,$$

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which is equivalent to

$$z'(t) = \left[\int_t^{+\infty} r^{(N-1)p/(p-1)}(\tau) f(z(\tau))\, \mathrm{d}\tau \right]^{1/(p-1)}.$$

Integrating once again we obtain

$$z(t) = \int_0^t \left[\int_s^{+\infty} G(\tau) f(z(\tau))\, \mathrm{d}\tau \right]^{1/(p-1)} \mathrm{d}s, \tag{2.3}$$

where

$$G(\tau) = \left(b^{(p-N)/(p-1)} + \tau \frac{N-p}{p-1} \right)^{p(1-N)/(N-p)}. \tag{2.4}$$

Consequently, we will solve (2.1) using fixed-point techniques. For this, we state the following well-known abstract result without proof (cf. [**5**, **7**, **10**]).

Lemma 2.1. *Let X be a Banach space with norm $|\cdot|$, and let $K \subset X$ be a cone in X. For $r > 0$, define $K_r = K \cap B[0,r]$, where $B[0,r] = \{x \in X : |x| \leqslant r\}$ is the closed ball of radius r centred at the origin of X. Assume that $F : K_r \to K$ is a compact map such that $Fx \neq x$, for all $x \in \partial K_r = \{x \in K : |x| = r\}$.*

Then the following assertions hold.

(1) *If $|x| \leqslant |Fx|$ for all $x \in \partial K_r$, then $\imath(F, K_r, K) = 0$.*

(2) *If $|x| \geqslant |Fx|$ for all $x \in \partial K_r$, then $\imath(F, K_r, K) = 1$.*

Now we consider the space

$$X = \{z : [0, +\infty) \to \mathbb{R} : z \text{ is a bounded, continuous function}\}$$

endowed with the norm $|z|_\infty = \sup\{|z(t)| : t \in [0, +\infty)\}$. Let $A : K_1 \to X$ be the operator defined by

$$(Az)(t) = \int_0^t \left[\int_s^{+\infty} G(\tau) f(z(\tau))\, \mathrm{d}\tau \right]^{1/(p-1)} \mathrm{d}s, \tag{2.5}$$

where K_1 is the cone defined by

$$K_1 = \{z \in X : z \text{ is non-negative, concave and } z(0) = 0\}.$$

Note that the elements of K_1 are increasing functions.

Lemma 2.2. *A is well defined, $A(K_1) \subset K_1$, and A is a completely continuous operator.*

Proof. For all $s \geqslant 0$, note that

$$\int_s^{+\infty} G(\tau)\, \mathrm{d}\tau = \frac{1}{N} G(s)^{N(p-1)/p(N-1)}$$

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and that

$$\int_0^{+\infty}\left[\int_s^{+\infty}G(\tau)\,\mathrm{d}\tau\right]^{1/(p-1)}\mathrm{d}s=\eta^{1/(1-p)}.$$

Hence A is well defined.

Also, note that the function $(Az)(t)$ is of class C^2, whose derivatives are given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(Az)(t)&=\left[\int_t^{+\infty}G(\tau)f(z(\tau))\,\mathrm{d}\tau\right]^{1/(p-1)},\\ \frac{\mathrm{d}^2}{\mathrm{d}t^2}(Az)(t)&=\frac{1}{1-p}G(t)\left[\frac{\mathrm{d}}{\mathrm{d}t}(Az)(t)\right]^{p-2}f(z(t)).\end{aligned}$$

Thus $(Az)(t)$ is increasing and concave. Therefore, $A(K_1)\subset K_1$.

It remains to prove that A is a completely continuous operator. Let $|z_n|_\infty\leqslant C_0$, and let $M_1=\max\{f(t):t\in[0,C_0]\}$. It follows that

$$\begin{aligned}|(Az_n)(t)|&\leqslant M_1^{1/(p-1)}\int_0^{+\infty}\left[\int_s^{+\infty}G(\tau)\,\mathrm{d}\tau\right]^{1/(p-1)}\mathrm{d}s,\\ \left|\frac{\mathrm{d}}{\mathrm{d}t}(Az_n)(t)\right|&\leqslant\left[M_1\int_0^{+\infty}G(\tau)\,\mathrm{d}\tau\right]^{1/(p-1)}.\end{aligned}$$

By the Arzelá–Ascoli compactness criterion for uniform convergence, there exists a uniformly convergent subsequence $(Az_{n_k})\subset(Az_n)$ on compact subsets of $[0,+\infty)$. To prove that there exists a uniformly convergent subsequence of (Az_n) it suffices to recall that given $\epsilon>0$, there is $T=T(\epsilon)$ such that

$$\int_T^{+\infty}\left[\int_s^{+\infty}G(\tau)\,\mathrm{d}\tau\right]^{1/(p-1)}\mathrm{d}s<\epsilon.$$

We now verify that A is continuous. Let $(z_n)\in X$ such that $|z_n-z_0|_\infty\to 0$ as $n\to\infty$. Thus

$$|(Az_n)(t)-(Az_0)(t)|\leqslant\int_0^{+\infty}|\Gamma_n(s)-\Gamma_0(s)|\,\mathrm{d}s,$$

where

$$\Gamma_n(s)=\int_s^{+\infty}G(\tau)f(z_n(\tau))\,\mathrm{d}\tau\quad\text{and}\quad\Gamma_0(s)=\int_s^{+\infty}G(\tau)f(z_0(\tau))\,\mathrm{d}\tau.$$

It follows from $|z_n-z_0|_\infty\to 0$ that $\Gamma_n(s)\to\Gamma_0(s)$ and that

$$\Gamma_n(s)\leqslant\frac{C}{NG(s)^{N(p-1)/p(N-1)}}$$

for all $s\in[0,+\infty)$. By the Lebesgue-dominated convergence theorem,

$$|Az_n-Az_0|_\infty\to 0,$$

which implies that A is continuous. □

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Given $\omega \in K_1$, there clearly exists a unique $\tau = \tau(\omega)$ such that $2\omega(\tau) = |\omega|_\infty$. Define

$$\tau^* = \sup\{\tau(A(z)) : z \in K_1\}$$

and

$$K = \{z \in K_1 : 2 \inf_{t \geqslant \tau^*} z(t) \geqslant |z|_\infty\}.$$

Lemma 2.3. *τ^* is a positive real number and K is an invariant cone by A.*

The proof is based on the following assertion.

Assertion 2.4. $\{\omega/|\omega|_\infty : \omega \in A(K_1) \setminus \{0\}\}$ is a relatively compact subset of X.

Proof. Since $\{Az/|Az|_\infty : z \in K_1 \text{ and } Az \neq 0\}$ is a bounded subset of X, it suffices to prove that

$$\{[Az]'/|Az|_\infty : z \in K_1 \text{ and } Az \neq 0\}$$

is also a bounded subset of X.

Integrating by parts we have

$$\begin{aligned}\left[\frac{[Az]'(t)}{|Az|_\infty}\right]^{p-1} &= \frac{\int_t^{+\infty} G(\tau) f(z(\tau)\,\mathrm{d}\tau}{[\int_0^{+\infty}[\int_s^{+\infty} G(\tau)f(z(\tau))\,\mathrm{d}\tau]^{1/(p-1)}\,\mathrm{d}s]^{p-1}} \\ &= \frac{(p-1)^{p-1}\int_t^{+\infty} G(\tau)f(z(\tau))\,\mathrm{d}\tau}{[\int_0^{+\infty}[\int_s^{+\infty} G(\tau)f(z(\tau))\,\mathrm{d}\tau]^{(2-p)/(p-1)} sG(s)f(z(s))\,\mathrm{d}s]^{p-1}}. \qquad (2.6)\end{aligned}$$

We consider two cases.

Case 1 ($1 < p < 2$). In this case, it follows from condition (H_4) that

$$\begin{aligned}\left[\frac{[Az]'(t)}{|Az|_\infty}\right]^{p-1} &\leqslant \frac{(p-1)^{p-1}\int_0^{+\infty} G(\tau)g_2(z(\tau))\,\mathrm{d}\tau}{[\int_0^{+\infty}[\int_s^{+\infty} G(\tau)g_1(z(\tau))\,\mathrm{d}\tau]^{(2-p)/(p-1)} sG(s)g_1(z(s))\,\mathrm{d}s]^{p-1}} \\ &\leqslant \frac{(p-1)^{p-1}\int_0^{+\infty} G(\tau)g_2(z(\tau))\,\mathrm{d}\tau}{[\int_0^{+\infty}[\int_s^{+\infty} G(\tau)\,\mathrm{d}\tau]^{(2-p)/(p-1)} sG(s)g_1(z(s))^{1/(p-1)}\,\mathrm{d}s]^{p-1}} \\ &\leqslant I_1 + I_2,\end{aligned}$$

where I_1 and I_2 are given by

$$I_1 = \frac{(p-1)^{p-1}\int_0^1 G(\tau)g_2(z(\tau))\,\mathrm{d}\tau}{[\int_0^1[\int_s^{+\infty} G(\tau)\,\mathrm{d}\tau]^{(2-p)/(p-1)} sG(s)g_1(z(s))^{1/(p-1)}\,\mathrm{d}s]^{p-1}}$$

and

$$I_2 = \frac{(p-1)^{p-1}\int_1^{+\infty} G(s)g_2(z(s))\,\mathrm{d}s}{[\int_1^{+\infty}[\int_s^{+\infty} G(\tau)\,\mathrm{d}\tau]^{(2-p)/(p-1)} sG(s)g_1(z(s))^{1/(p-1)}\,\mathrm{d}s]^{p-1}}.$$

We estimate each integral separately.

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To estimate I_1, we use condition (H_4) to obtain

$$\begin{aligned}
I_1 &= \frac{(p-1)^{p-1}\int_0^1 G(\tau)g_2(z(\tau))\,\mathrm{d}\tau}{[\int_0^1[\int_s^{+\infty} G(\tau)\,\mathrm{d}\tau]^{(2-p)/(p-1)} sG(s)g_1(z(s))^{1/(p-1)}\,\mathrm{d}s]^{p-1}} \\
&\leqslant \frac{(p-1)^{p-1}\int_0^1 G(\tau)g_2(z(\tau))\,\mathrm{d}\tau}{[\int_\delta^1[\int_s^{+\infty} G(\tau)\,\mathrm{d}\tau]^{(2-p)/(p-1)} sG(s)g_1(z(\delta))^{1/(p-1)}\,\mathrm{d}s]^{p-1}} \\
&\leqslant \frac{(p-1)^{p-1}\int_0^1 G(\tau)g_2(z(1))\,\mathrm{d}\tau}{[\int_\delta^1[\int_s^{+\infty} G(\tau)\,\mathrm{d}\tau]^{(2-p)/(p-1)} sG(s)g_1(\delta z(1))^{1/(p-1)}\,\mathrm{d}s]^{p-1}} \\
&\leqslant \frac{\eta(p-1)^{p-1}\int_0^1 G(\tau)\,\mathrm{d}\tau}{[\int_\delta^1[\int_s^{+\infty} G(\tau)\,\mathrm{d}\tau]^{(2-p)/(p-1)} sG(s)\,\mathrm{d}s]^{p-1}}.
\end{aligned}$$

To estimate I_2, we use again condition (H_4) to get

$$\begin{aligned}
&\left[\int_s^{+\infty} G(\tau)\,\mathrm{d}\tau\right]^{(2-p)/(p-1)} sG(s)(g_1(z(s)))^{1/(p-1)} \\
&\qquad\geqslant N^{(p-2)/(p-1)}[sG(s)g_1(z(s))]^{1/(p-1)} \\
&\qquad\geqslant N^{(p-2)/(p-1)}[sG(s)g_1(\delta z(s))]^{1/(p-1)} \\
&\qquad\geqslant \frac{N^{(p-2)/(p-1)}}{\eta^{1/(p-1)}}[sG(s)g_2(z(s))]^{1/(p-1)},
\end{aligned}$$

which implies

$$\begin{aligned}
I_2 &= \frac{(p-1)^{p-1}\int_1^{+\infty} G(s)g_2(z(s))\,\mathrm{d}s}{[\int_1^{+\infty}[\int_s^{+\infty} G(\tau)\,\mathrm{d}\tau]^{(2-p)/(p-1)} sG(s)g_1(z(s))^{1/(p-1)}\,\mathrm{d}s]^{p-1}} \\
&\leqslant \frac{\eta(p-1)^{p-1}\int_1^{+\infty} G(s)g_2(z(s))\,\mathrm{d}s}{N^{p-2}[\int_1^{+\infty}[sG(s)g_2(z(s))]^{1/(p-1)}\,\mathrm{d}s]^{p-1}} \\
&= \frac{\eta(p-1)^{p-1}\int_1^{+\infty} s(1/s)G(s)g_2(z(s))\,\mathrm{d}s}{N^{p-2}[\int_1^{+\infty}[sG(s)g_2(z(s))]^{1/(p-1)}\,\mathrm{d}s]^{p-1}} \\
&\leqslant \frac{\eta(p-1)^{p-1}\|1/s\|_{L^{1/(2-p)}[1,+\infty)}}{N^{p-2}}.
\end{aligned}$$

Case 2 ($p \geqslant 2$). In this case, in accordance with conditions (2.6) and (H_4):

$$\begin{aligned}
\frac{[Az]'(t)}{|Az|_\infty} &\leqslant \frac{(p-1)\int_0^{+\infty} G(s)f(z(s))\,\mathrm{d}s}{\int_0^{+\infty} G(s)sf(z(s))\,\mathrm{d}s} \\
&\leqslant (p-1)\left[\frac{\int_0^1 G(s)f(z(s))\,\mathrm{d}s}{\int_0^1 G(s)sf(z(s))\,\mathrm{d}s} + 1\right]
\end{aligned}$$

$$\leqslant (p-1)\left[\frac{\int_0^1 G(s)g_2(z(s))\,\mathrm{d}s}{\int_0^1 G(s)sg_1(z(s))\,\mathrm{d}s}+1\right]$$

$$\leqslant (p-1)\left[\frac{\int_0^1 G(s)g_2(z(s_M))\,\mathrm{d}s}{\int_\delta^1 sG(s)g_1(z(s_m))\,\mathrm{d}s}+1\right],$$

where $z(s_M)=\max\{z(s): s\in[0,1]\}$ and $z(s_m)=\min\{z(s): s\in[\delta,1]\}$. It now follows from the fact that $z(s_m)\geqslant \delta z(s_M)$ and condition (H_4) that

$$\frac{[Az]'(t)}{|Az|_\infty}\leqslant (p-1)\left[\eta\frac{\int_0^1 G(s)\,\mathrm{d}s}{\int_\delta^1 sG(s)\,\mathrm{d}s}+1\right].$$

The result follows by the Arzelá–Ascoli compactness criterion. □

Proof of Lemma 2.3. We first show that τ^* is a positive real number. Suppose on the contrary that $\tau^*=+\infty$. Then there must exist a sequence $(z_n)\subset K_1\setminus\{0\}$ such that $(\tau(z_n/|z_n|_\infty))$ is a strictly increasing sequence of positive real numbers converging to $+\infty$. By Assertion 2.4, there exists a subsequence of $(z_n/|z_n|_\infty)$, which is denoted by the same index, such that $(z_n/|z_n|_\infty)$ converges to some ω_0 in X. Hence $|\omega_0|_\infty=1$ and, for large n, we must have

$$\tau(z_n/|z_n|)>\tau(\omega_0).$$

Note that $\omega_0(t)\leqslant\frac{1}{2}$, for all $t\in[0,\tau(\omega_0)]$. On the other hand, given $t>\tau(\omega_0)$, we have $t<\tau(z_n/|z_n|)$ for large n. It follows that

$$\omega_0(t)=\lim_{n\to+\infty} z_n(t)/|z_n|_\infty\leqslant\tfrac{1}{2},\quad \text{for } t>\tau(\omega_0)$$

We conclude that $\omega_0(t)\leqslant\frac{1}{2}$, for all $t\geqslant 0$. But this is impossible, since $|\omega_0|_\infty=1$.

That K is an invariant cone by A is clear. The proof of the lemma is now complete. □

Lemma 2.5. *We have $\imath(A,K_R,K)=1$.*

Proof. According to condition (H_3), for $z\in\partial K_R$,

$$\begin{aligned}|Az|_\infty &= \max_{t\geqslant 0}\int_0^t\left[\int_s^{+\infty}G(\tau)f(z(\tau))\,\mathrm{d}\tau\right]^{1/(p-1)}\mathrm{d}s\\ &\leqslant \int_0^{+\infty}\left[\int_s^{+\infty}G(\tau)f(\bar R)\,\mathrm{d}\tau\right]^{1/(p-1)}\mathrm{d}s\\ &= \frac{f(\bar R)^{1/(p-1)}(p-1)b^{p/(p-1)}}{pN^{1/(p-1)}}\\ &< \bar R.\end{aligned}$$

Since $\bar R\leqslant R$, we have $|Az|_\infty<R=|z|_\infty$. The result now follows from Lemma 2.1 (2). □

Lemma 2.6. *There is $r_1\in(0,R)$ such that $\imath(A,K_{r_1},K)=0$.*

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Proof. According to condition (H_1), given $M > 0$ there exists $r_1 \in (0, R)$ such that

$$f(t) \geqslant M t^{p-1}, \quad \text{for all } t \in [0, r_1].$$

Thus for $z \in \partial K_{r_1}$,

$$\begin{aligned}(Az)(\tau^*) &= \int_0^{\tau^*} \left[\int_s^{+\infty} G(\tau) f(z(\tau))\, \mathrm{d}\tau \right]^{1/(p-1)} \mathrm{d}s \\ &\geqslant \int_0^{\tau^*} \left[\int_s^{+\infty} G(\tau) M z(\tau)^{p-1}\, \mathrm{d}\tau \right]^{1/(p-1)} \mathrm{d}s \\ &\geqslant \int_0^{\tau^*} \left[\int_{\tau^*}^{+\infty} G(\tau) M z(\tau)^{p-1}\, \mathrm{d}\tau \right]^{1/(p-1)} \mathrm{d}s \\ &\geqslant \left[\int_{\tau^*}^{+\infty} G(\tau)\, \mathrm{d}\tau \right]^{1/(p-1)} \frac{\tau^* M^{1/(p-1)}}{2} |z|_\infty.\end{aligned}$$

Choosing $M > 0$ such that

$$\tau^* G(\tau^*)^{N/p(N-1)} \left[\frac{M}{N} \right]^{1/(p-1)} > 2, \tag{2.7}$$

we have that $|Az|_\infty > |z|_\infty$, for all $z \in \partial K_{r_1}$. The result now follows from Lemma 2.1 (1). □

Lemma 2.7. *There is $r_2 > R$ such that $\imath(A, K_{r_2}, K) = 0$.*

Proof. It follows from condition (H_2) that there exists $r_3 > R$ such that

$$f(t) \geqslant M t^{p-1}, \quad \text{for all } t \geqslant r_3.$$

Note that for $z \in \partial K_{2r_3}$ we have

$$2 \min_{t \geqslant \tau^*} z(t) \geqslant |z|_\infty = 2r_3,$$

which implies

$$f(z(t)) \geqslant M z(t)^{p-1}, \quad \text{for all } t \geqslant \tau^*.$$

Thus

$$\begin{aligned}(Az)(\tau^*) &= \int_0^{\tau^*} \left[\int_s^{+\infty} G(\tau) f(z(\tau))\, \mathrm{d}\tau \right]^{1/(p-1)} \mathrm{d}s \\ &\geqslant \int_0^{\tau^*} \left[\int_{\tau^*}^{+\infty} G(\tau) f(z(\tau))\, \mathrm{d}\tau \right]^{1/(p-1)} \mathrm{d}s \\ &\geqslant \int_0^{\tau^*} \left[\int_{\tau^*}^{+\infty} G(\tau) M z(\tau)^{p-1}\, \mathrm{d}\tau \right]^{1/(p-1)} \mathrm{d}s \\ &\geqslant \tau^* G(\tau^*)^{N/p(N-1)} \left[\frac{M}{N} \right]^{1/(p-1)} \frac{|z|_\infty}{2}.\end{aligned}$$

Define the number $r_2 = 2r_3$. By (2.7), we have $|Az|_\infty > |z|_\infty$, for $z \in \partial K_{r_2}$, and the result now follows from Lemma 2.1 (1). □

3. Proof of the main result

Proof of Theorem 1.1. It follows from Lemmas 2.5–2.7 and the additivity of the fixed-point index that

$$\imath(A, K_R \setminus K_{r_1}, K_{r_1}) = 1$$

and that

$$\imath(A, K_{r_2} \setminus K_R, K_R) = -1.$$

Consequently, the operator A has two fixed points, namely z_1 in $K_R \setminus K_{r_1}$ and z_2 in $K_{r_2} \setminus K_R$. □

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