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PROPAGATION OF SINGULARITIES FOR SEMILINEAR HYPERBOLIC EQUATIONS

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ABSTRACT In this paper we use a particular kind of weighted Sobolev space and pseudo-differential operators to study H^{3s} propagation of singularities for the solution $u \in H^s$ of the equations with second order

1. **Introduction.** From 1978, B. Lascar [7], J. Rauch [11], M. Bony [5], M. Beals and M. Reed [4] and others obtained H^{2s} propagation of singularities for solutions $u \in H^s$ of nonlinear partial differential equations. From 1983, M. Beals [3], Lingi Liu [9], [10] obtained H^{3s} propagation of singularities for $\Box u = f(u)$, $\Box = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2}$. M. Beals [2] and J. Y. Chemin [6] obtained H^{3s} propagation for second order nonlinear equations. In this paper, we mainly study H^{3s} propagation of singularities for $\Box u = f(t,x,u,Du)$. We use a particular kind of weighted Sobolev space [9], [10] and the idea from [12], [4]. All previous main results of propagation of singularities theorem, as described in Corollary 1.7, used in the wave equations $\Box u = f(t,x,u,Du)$, are shown by the following table.

1981	M. Bony [5]	$r < 2s - \frac{n}{2} - 1$
1982	M. Beals and M. Reed [12]	$r < 2s - \frac{n}{2} - 1$
1985	M. Beals [2]	$r < 3s - n - 2$
1986	J.Y. Chemin [6]	$r \leq 3s - n - 2$
1991	In this paper	$r < 3s - n - 1$

In this paper, we normally only consider $s, r, \alpha, \beta \geq 0$ and write $\langle(\tau,\xi)\rangle = \left(1 + |(\tau,\xi)|^2\right)^{1/2}$.

DEFINITION 1.1. Let $(t_0, x_0, \tau_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$. If there exists a smooth function $\varphi(t,x) \in C_0^\infty(\mathbb{R}^n)$, supported near (t_0, x_0), $\varphi(t_0,x_0) = 1$ and a conic neighbourhood $\Gamma \subset \mathbb{R}^n \setminus 0$ of (τ_0, ξ_0) such that for $\alpha, \beta \geq 0$

(i) $\langle(\tau,\xi)\rangle^s \langle\tau - |\xi|\rangle^\alpha \widehat{\varphi u}(\tau,\xi) \in L^2(\mathbb{R}^n)$

(ii) $\langle(\tau,\xi)\rangle^r \langle\tau - |\xi|\rangle^\beta \widehat{\varphi u}(\tau,\xi) \in L^2(\Gamma)$

we call $u \in (H^s)^\alpha_{P_1} \cap (H^r)^\beta_{P_1}(t_0, x_0, \tau_0, \xi_0)$, $P_1 = D_t - |D_x|$. It is the same to define $(H^s)^\alpha_{P_2} \cap (H^r)^\beta_{P_2}(t_0, x_0, \tau_0, \xi_0)$ for $P_2 = D_t + |D_x|$.

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THEOREM 1.2 [9]. *Let* $u_i \in (H^{s_i})^{\alpha}_{P_\mu} \cap (H^r)^{\beta}_{P\mu}(t_0, x_0, \tau_0, \xi_0)$, $\mu = 1$ *or* 2. *If*

$$\begin{cases} s+\alpha > \frac{n}{2},\ s-\beta > \frac{n-1}{2} \\ r+\beta < s_1+s_2+2\alpha-\frac{n}{2} \\ r+\beta < s_1+s_2-\frac{n-1}{2},\ (\tau_0,\xi_0)\in N_\mu \\ r+\beta < s_1+s_2+\alpha-\frac{n-1}{2},\ (\tau_0,\xi_0)\notin N_\mu \end{cases}$$

then $u_1u_2 \in (H^r)^{\beta}_{P_\mu}(t_0, x_0, \tau_0, \xi_0)$. *If* $s-\alpha > \frac{n-1}{2}$ *also*, $u_1u_2 \in (H^s)^{\alpha}_{P_\mu} \cap (H^r)^{\beta}_{P_\mu}(t_0, x_0, \tau_0, \xi_0)$ *where* $N_1 = \{(\tau, \xi)\ ;\ \tau > |\xi|\}, N_2 = \{(\tau, \xi)\ ;\ -\tau > |\xi|\}$.

DEFINITION 1.3. If $u = v_1 + v_2$, $\Pi WF(v_i) = E_i$, $v_i \in (H^{s_i})^{\alpha_i}_{P_i}$, $i = 1, 2$, $P_1 = D_t - |D_x|$, $P_2 = D_t + |D_x|$, $E_1 = \{(\tau, \xi)\ ;\ \tau \geq 0\}$, $E_2 = \{(\tau, \xi)\ ;\ \tau \leq 0\}$ we say $u \in (H^{s_1})^{\alpha_1}_{P_1} \oplus (H^{s_2})^{\alpha_2}_{P_2}$.

DEFINITION 1.4. If there exists $\varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^{n-1})$, supported near (t_0, x_0) and $\varphi(t_0, x_0) = 1$ such that $\varphi u \in (H^{s_1})^{\alpha_1}_{P_1} \oplus (H^{s_2})^{\alpha_2}_{P_2}$, then we say $u \in (H^{s_1})^{\alpha_1}_{P_1} \oplus (H^{s_2})^{\alpha_2}_{P_2}(t_0, x_0)$.

THEOREM 1.5 [9]. *Let* $f\colon \mathbb{C}^N \longrightarrow \mathbb{C}$, $f \in C^\infty$, $u_j \in (H^s)^{\alpha}_{P_1} \oplus (H^s)^{\alpha}_{P_2}(t, x)$ *and* $u_j \in (H^r)^{\beta}_{P_\mu}(t_0, x_0, \tau_0, \xi_0)$, $j = 1, 2, \ldots, N$. *If*

$$\begin{cases} s-\beta > \frac{n-1}{2},\ s-\alpha > \frac{n-1}{2},\ s > \frac{n}{2} \\ r+\beta < 2s+2\alpha-\frac{n}{2} \\ r+\beta < 2s-\frac{n-1}{2},\ (\tau_0,\xi_0)\notin \operatorname{Char} P_\mu \\ r+\beta < 2s+\alpha-\frac{n-1}{2},\ (\tau_0,\xi_0)\in \operatorname{Char} P_\mu \end{cases}$$

then

$$f(u_1, u_2, \ldots, u_N) \in (H^s)^{\alpha}_{P_1} \oplus (H^s)^{\alpha}_{P_2}(t_0, x_0)$$

and

$$f(u_1, u_2, \ldots, u_N) \in (H^r)^{\beta}_{P_\mu}(t_0, x_0, \tau_0. \xi_0).$$

Let g_1 and g_2 be positive functions in some domain. If there exists a constant $c > 0$ such that $g_1 \leq cg_2$, we write $g_1 \lesssim g_2$. If $c_1, c_2 > 0$, such that $c_1g_1 \leq g_2 \leq c_2g_1$, we write $g_1 \sim g_2$.

We often use the following lemmas.

LEMMA. *Let* $K\colon \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{C}$ *be a locally integrable measurable function and there exists a constant* $c > 0$ *such that either* $\sup_\xi \int |K(\xi, \eta)|^2\, d\eta \leq c^2$ *or* $\sup_\eta \int |K(\xi, \eta)|^2\, d\xi \leq c^2$. *Then*

$$\left\| \int K(\xi, \eta) g(\xi - \eta) h(\eta)\, d\eta \right\|_{L^2} \leq c\|g\|_{L^2} \cdot \|h\|_{L^2}$$

for all $g, h \in L^2$.

LEMMA [9]. *Let* $s_i \geq 0$, $\alpha_i \geq 0$, $i = 1, 2$. *If*

$$\begin{cases} s_1+s_2 > \frac{n-1}{2} \\ s_1+s_2+\alpha_1+\alpha_2 > \frac{n}{2} \end{cases}$$

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then

$$\sup_{(\tau,\xi)} \int_{\mathbb{R}^n\times\mathbb{R}^{n-1}} \frac{d\lambda\, d\eta}{\langle(\lambda,\eta)\rangle^{2s_1}\langle(\lambda-|\eta|)\rangle^{2\alpha_1}\langle(\tau-\lambda,\xi-\eta)\rangle^{2s_2}\langle\tau-\lambda-|\xi-\eta|\rangle^{2\alpha_2}} < \infty.$$

Notice that if $\langle\lambda+|\eta|\rangle$ replaces $\langle\lambda-|\eta|\rangle$, or $\langle(\tau-\lambda+|\xi-\eta|\rangle$ replaces $\langle\tau-\lambda-|\xi-\eta|\rangle$, or both replace them, the convergence of the integral does not change.

The main results in this paper are the following.

THEOREM 1.6. *Let* $u \in H^s_{\mathrm{loc}}(\Omega)$, $s > \frac{n}{2}+1$, *be a solution in* $\Omega \subset \mathbb{R}^n$ *for* $\Box u = f(t,x,u,Du), f \in C^\infty$, γ *be a null-bicharacteristic of* $\Box$ *passing through* (t_0,x_0,τ_0,ξ_0). *If* $u \in (H^{r_\mu})^{\beta_u}_{P_\mu}(t_0,x_0,\tau_0,\xi_0)$

(i) $\beta_\mu < s - \frac{n+1}{2}$

(ii) $r_\mu + \beta_\mu < 3s - n - 1$

then $u \in (H^{r_\mu})^{\beta_\mu}_{P_\mu}(\gamma)$.

COROLLARY 1.7. *Let* $u \in H^s_{\mathrm{loc}}(\Omega)$, $s > \frac{n}{2}+1$, *be a solution in* $\Omega \subset \mathbb{R}^n$ *for* $\Box u = f(t,x,u,Du), f \in C^\infty$, γ *be a null-bicharacteristic of* $\Box$ *passing through* (t_0,x_0,τ_0,ξ_0). *If* $u \in H^r(t_0,x_0,\tau_0,\xi_0)$ *and* $r < 3s-n-1$, *then* $u \in H^r(\gamma)$.

2. **Local regularity of solution.**

PROPOSITION 2.1. *Let* $P_m(t,x,\tau,\xi) \in S^m_{1,0}$. $P_m = P_m(t,x,D)$ *is a proper pseudodifferential operator,* $u \in \mathcal{D}'(\Omega)$, $(t,x) \in \overset{o}{\Omega}$, $u \in (H^r)^\beta_{P_\mu}(t_0,x_0,\tau_0,\xi_0)$, *where* r,β *are any real number. Then* $P_m u \in (H^{r-m})^\beta_{P_\mu}(t_0,x_0,\tau_0,\xi_0)$.

And if $P_m(t,x,\tau,\xi)$ *has compact support for* (t,x), $\Gamma \subset\subset \Gamma_1 \subset \mathbb{R}^n\setminus 0$ *are conic neighbourhoods of* (τ_0,ξ_0), s,α,r,β *are any real numbers, then for* $\forall u \in (H^s)^\alpha_{P_\mu} \cap (H^r)^\beta_{P_\mu}(\Gamma_1)$, *there exists a constant* $c>0$ *such that*

$$\|P_m u\|_{(H^{r-m})^\beta_{P_m}(\Gamma)} \le c\|u\|_{(H^s)^\alpha_{P_\mu}\cap(H^r)^\beta_{P_\mu}(\Gamma_1)}$$
$$\|P_m u\|_{(H^{s-m})^\alpha_{P_\mu}} \le c\|u\|_{(H^s)^\alpha_{P_\mu}}$$

PROOF. We prove it only for $\mu=1$, that is $P_1 = D_t - |D_x|$. We suppose that $P_m(t,x,\tau,\xi)$ and $u(t,x)$ have compact support for (t,x).

$$\widehat{P_m u}(\tau,\xi) = \frac{1}{(2\pi)^n}\int P_m^\wedge(\lambda,\eta,\tau-\lambda,\xi-\eta)\hat{u}(\tau-\lambda,\xi-\eta)\,d\lambda\,d\eta.$$

Notice that $P_m(t,x,\tau,\xi) \in S^m_{1,0}$ has compact support for (t,x), then there exists a constant $c_M > 0$ for any $M > 0$ such that

$$|P_m^\wedge(\lambda,\eta,\tau-\lambda,\xi-\eta)| \le \frac{c_M\langle(\tau-\lambda,\xi-\eta)\rangle^m}{\langle(\lambda,\eta)\rangle^M}.$$

Notice that if $(\tau,\eta) \in \Gamma \subset\subset \Gamma_1$, then there exists a constant $\varepsilon_0 > 0$ such that $(\tau-\lambda,\xi-\eta) \in \Gamma_1$ for $|(\lambda,\eta)| < \varepsilon_0|(\tau,\xi)|$.

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(1) If $|(\lambda,\eta)| < \varepsilon_0|(\tau,\xi)|$. So $(\tau-\lambda,\xi-\eta)\in\Gamma_1$, and $u\in (H^r)^\beta_{P_1}$, we have

$$\begin{aligned}
&\langle(\tau,\xi)\rangle^{r-m}\langle\tau-|\xi|\rangle^\beta\widehat{P_m u}(\tau,\xi)\\
&\quad=\frac{1}{(2\pi)^n}\int\frac{\langle(\tau,\xi)\rangle^{r-m}\langle\tau-|\xi|\rangle^\beta P_m^\wedge(\lambda,\eta,\tau-\lambda,\xi-\eta)\langle(\lambda,\eta)\rangle^{\frac{n+1}{2}}}{\langle(\tau-\lambda,\xi-\eta)\rangle^r\langle\tau-\lambda-|\xi-\eta|\rangle^\beta}\cdot\\
&\qquad\frac{1}{\langle(\lambda,\eta)\rangle^{\frac{n+1}{2}}}\langle(\tau-\lambda,\xi-\eta)\rangle^r\langle\tau-\lambda-|\xi-\eta|\rangle^\beta\hat{u}(\tau-\lambda,\xi-\eta)\,d\lambda\,d\eta\\
&\quad=\frac{1}{(2\pi)^n}\int K_1 g_1 f_1\,d\lambda\,d\eta
\end{aligned}$$

where K_1, g_1 and f_1 denote the obvious factors respectively, and $g_1, f_1 \in L^2$.

$$K_1 \lesssim \frac{\langle(\tau,\xi)\rangle^{r-m}\langle\tau-|\xi|\rangle^\beta}{\langle(\tau-\lambda,\xi-\eta)\rangle^{r-m}\langle\tau-\lambda-|\xi-\eta|\rangle^\beta\langle(\lambda,\eta)\rangle^M}.$$

By

$$\begin{aligned}
&\langle\tau-|\xi|\rangle \le \langle\tau-\lambda-|\xi-\eta|\rangle+\langle(\lambda,\eta)\rangle,\quad \langle\tau-\lambda-|\xi-\eta|\rangle\le\langle\tau-|\xi|\rangle+\langle(\lambda,\eta)\rangle,\\
&\langle(\tau,\xi)\rangle\le\langle(\tau-\lambda,\xi-\eta)\rangle+\langle(\lambda,\eta)\rangle,\quad \langle(\tau-\lambda,\xi-\eta)\rangle\le\langle(\tau,\xi)\rangle+\langle(\lambda,\eta)\rangle,
\end{aligned}$$

for $-\infty < r-m < \infty$, $-\infty < \beta < \infty$, we always have

$$K_1 \lesssim \frac{1}{\langle(\lambda,\eta)\rangle^{M-|r-m|-|\beta|}}.$$

(2) If $|(\lambda,\eta)| > \varepsilon_0|(\tau,\xi)|$.

Using $u\in (H^s)^\alpha_{P_1}$ and $|(\tau-\lambda,\xi-\eta)|\le c|(\lambda,\eta)|$, we have

$$\begin{aligned}
&\langle(\tau,\xi)\rangle^{r-m}\langle\tau-|\xi|\rangle^\beta\widehat{P_m u}(\tau,\xi)\\
&\quad=\frac{1}{(2\pi)^n}\int\frac{\langle(\tau,\xi)\rangle^{r-m}\langle\tau-|\xi|\rangle^\beta P_m^\wedge(\lambda,\eta,\tau-\lambda,\xi-\eta)\langle(\lambda,\eta)\rangle^{\frac{n+1}{2}}}{\langle(\tau-\lambda,\xi-\eta)\rangle^s\langle\tau-\lambda-|\xi-\eta|\rangle^\alpha}\\
&\qquad\cdot\frac{1}{\langle(\lambda,\eta)\rangle^{\frac{n+1}{2}}}\cdot\langle(\tau-\lambda,\xi-\eta)\rangle^s\langle\tau-\lambda-|\xi-\eta|\rangle^\alpha\,\hat{u}(\tau-\lambda,\xi-\eta)\,d\lambda\,d\eta\\
&\quad=\frac{1}{(2\pi)^n}\int K_2 g_1 f_2\,d\lambda\,d\eta
\end{aligned}$$

where K_2, g_1 and f_2 denote the obvious factors respectively, and $g_1, f_2 \in L^2$.

$$\begin{aligned}
K_2 &\lesssim \frac{\langle(\tau,\xi)\rangle^{r-m}\langle\tau-|\xi|\rangle^\beta}{\langle(\tau-\lambda,\xi-\eta)\rangle^{s-m}\langle\tau-\lambda-|\xi-\eta|\rangle^\alpha\langle(\lambda,\eta)\rangle^M}\\
&\lesssim \frac{1}{\langle(\lambda,\eta)\rangle^{M-(|r-m|+|\beta|+|s-m|+|\alpha|)}}.
\end{aligned}$$

Notice that $M > 0$ is any constant, so $\sup_{(\tau,\xi)}\int K_i^2\,d\lambda\,d\eta < \infty$, $i = 1, 2$.

$$\therefore\qquad \|P_m u\|_{(H^{r-m})^\beta_{P_1}(\Gamma)} \le c\|u\|_{(H^s)^\alpha_{P_1}\cap(H^r)^\beta_{P_1}(\Gamma_1)}$$

Let $r = s$, $\alpha = \beta$, so $\|P_m u\|_{(H^{s-m})^\alpha_{P_1}} \le c\|u\|_{(H^s)^\alpha_{P_1}}$.

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PROPOSITION 2.2. *Let* $(t_0, x_0, \tau_0, \xi_0) \in T^*(\Omega) \setminus 0$, $\sigma_q \in S^m$ *is elliptic at* $(t_0, x_0, \tau_0, \xi_0)$, $u \in \mathcal{D}'(\Omega)$, $qu \in (H^s)^\alpha_{P_\mu}(t_0, x_0, \tau_0, \xi_0)$. *Then* $u \in (H^{s+m})^\alpha_{P_\mu}(t_0, x_0, \tau_0, \xi_0)$.

PROOF. Set $\sigma_Q \in S^{-m}$ such that $\sigma_{Qq}(t, x, \tau, \xi) = 1$ in a conic neighbourhood of $(t_0, x_0, \tau_0, \xi_0)$, so $(t_0, x_0, \tau_0, \xi_0) \notin WF(u - Qqu)$. By Proposition 2.1, $Qqu \in (H^{s+m})^\alpha_{P_\mu}(t_0, x_0, \tau_0, \xi_0)$, so $u \in (H^{s+m})^\alpha_{P_\mu}(t_0, x_0, \tau_0, \xi_0)$.

LEMMA 2.3. *Let* $u \in H^s_{\text{loc}}(\Omega)$ *is a solution of* $\Box u = f(t, x, u, Du)$ *in* $\Omega \subset \mathbb{R}^n$, *where* f *is a* c^∞ *function of* u *and* Du, $u \in (H^s)^\alpha_{P_1} \oplus (H^s)^\alpha_{P_2}(t_0, x_0)$, $(t_0, x_0) \in \overset{o}{\Omega}$ *(inner of* Ω*). If* $s - \alpha > \frac{n+1}{2}$, $s > \frac{n}{2} + 1$, *then* $u \in (H^s)^{\alpha+1}_{P_1} \oplus (H^s)^{\alpha+1}_{P_2}(t_0, x_0)$.

PROOF. We suppose that u and f have compact support already. We know $f(u, Du) \in (H^{s-1})^\alpha_{P_1} \oplus (H^{s-1})^\alpha_{P_2}(t_0, x_0)$.

Let $(t_0, x_0, \tau_0, \xi_0)$ be elliptic for $\Box = D_t^2 - \sum D^2_{x_i}$, and $\Box u = f(t, x, u, Du) \in H^{s-1+\alpha}(t_0, x_0, \tau_0, \xi_0)$, so $u \in H^{s+1+\alpha}(t_0, x_0, \tau_0, \xi_0)$.

Let $(t_0, x_0, \tau_0, \xi_0)$ be a characteristic point of P_μ, so $u \in (H^s)^{\alpha+1}_{P_\mu}(t_0, x_0, \tau_0, \xi_0)$, $\mu = 1, 2$.

$$\therefore u \in (H^s)^{\alpha+1}_{P_1} \oplus (H^s)^{\alpha+1}_{P_2}(t_0, x_0).$$

THEOREM 2.4. *Let* $u \in H^s_{\text{loc}}(\Omega)$, $s > \frac{n}{2} + 1$ *be a solution to* $\Box u = f(t, x, u, Du)$ *in* $\Omega \subset \mathbb{R}^n$, *where* f *is a* c^∞ *function of* t, x, u *and* Du. *Then* $u \in (H^s)^\alpha_{P_1} \oplus (H^s)^\alpha_{P_2}(t, x), \forall (t, x) \in \overset{o}{\Omega}$ *for any* $0 \leq \alpha < s - \frac{n-1}{2}$, *where* $P_1 = D_t - |D_x|$, $P_2 = D_t + |D_x|$.

3. **Propagation of singularities.**

LEMMA 3.1 (COMMUTATOR). *Suppose* $P(\tau, \xi) \in S^1_{1,0}$, $P = P(D)$, $b_0(t, x, \tau, \xi) \in S^0_{1,0}$, $B_0 = b_0(t, x, D)$, $Au(t, x) = a(t, x)u(t, x)$.

If

$$\begin{cases} 0 \leq \varepsilon \leq \frac{r_2 - r + 1}{2}, \; s_1 - 1 + \varepsilon \geq 0 \\ s + \alpha - \varepsilon > \frac{n}{2}, \; s - \beta - \varepsilon > \frac{n-1}{2}, \; r + \beta - \varepsilon > \frac{n}{2} \\ r + \beta + \varepsilon < s_1 + s_2 + 2\alpha - \frac{n}{2} \\ r + \beta + \varepsilon < s_1 + s_2 - \frac{n-1}{2}, \; (\tau_0, \xi_0) \in N_\mu \\ r + \beta + \varepsilon < s_1 + s_2 + \alpha - \frac{n-1}{2}, \; (\tau_0, \xi_0) \notin N_\mu \end{cases}$$

and

$$a(t, x) \in (H^{s_1})^\alpha_{P_\mu} \cap (H^{r_1})^\beta_{P_\mu}(t_0, x_0, \tau_0, \xi_0)$$
$$v(t, x) \in (H^{s_2})^\alpha_{P_\mu} \cap (H^{r_2})^\beta_{P_\mu}(t_0, x_0 \tau_0, \xi_0),$$

then $[B_0, AP]v \in (H^{r-1+\varepsilon})^\beta_{P_\mu}(t_0, x_0, \tau_0, \xi_0)$, $r = \min\{r_1, r_2\}$, and if $s + \alpha - \varepsilon > \frac{n-1}{2}$, $\varepsilon \leq \frac{1}{2}(s_2 - s + 1)$, then

$$[B_0, AP]v \in (H^{s-1+\varepsilon})^\alpha_{P_\mu} \cap (H^{r-1+\varepsilon})^\beta_{P_\mu}(t_0, x_0, \tau_0, \xi_0).$$

PROOF. It is assumed that $a(t,x)$, $u(t,x)$ and $b_0(t,x,D)$ are supported sufficiently near (t_0,x_0).

$$\begin{aligned}&[B_0,AP]v(t,x)\\ &=\frac{1}{(2\pi)^{2n}}\int\!\!\int e^{i(t\sigma+x\zeta)}\hat a(\lambda,\eta)[b_0(t,x,\sigma,\zeta)p(\sigma-\lambda,\xi-\eta)-\sigma_{P_0B_0}(t,x,\sigma-\lambda,\zeta-\eta)]\\ &\qquad \hat v(\sigma-\lambda,\zeta-\eta)\,d\lambda\,d\eta\,d\sigma\,d\zeta.\end{aligned}$$

By the calculus of pseudodifferential operators,

$$\sigma_{P_0B_0}(t,x,\sigma-\lambda,\zeta-\eta)=p(\sigma-\lambda,\zeta-\eta)b_0(t,x,\sigma-\lambda,\zeta-\eta)+R_0(t,x,\sigma-\lambda,\zeta-\eta)$$

where $\sigma_{R_0}\in S^0_{1,0}$. So

$$\begin{aligned}\widehat{[B_0,AP]v}\,(\tau,\xi)&=\frac{1}{(2\pi)^n}\int\langle(\tau-\lambda,\xi-\eta)\rangle^{1-\varepsilon}\hat a(\tau-\lambda,\xi-\eta)\hat v_1(\lambda,\eta)\,d\lambda\,d\eta\\ &\quad+\frac{1}{(2\pi)^n}\int\langle(\tau-\lambda,\xi-\eta)\rangle^{1-\varepsilon}\hat a(\tau-\lambda,\xi-\eta)\hat v_2(\lambda,\eta)\,d\lambda\,d\eta\end{aligned}$$

where

$$\begin{aligned}\hat v_1(\lambda,\eta)&=\frac{1}{(2\pi)^n}\int\frac{b_0^\wedge(\sigma,\zeta,\tau-\sigma,\xi-\zeta)-b_0^\wedge(\sigma,\zeta,\lambda-\sigma,\eta-\zeta)}{\langle(\tau-\lambda,\xi-\eta)\rangle^{1-\varepsilon}}\\ &\qquad p(\lambda-\sigma,\eta-\zeta)\hat v(\lambda-\sigma,\eta-\zeta)\,d\sigma\,d\zeta\\ \hat v_1(\lambda,\eta)&=\frac{1}{(2\pi)^n}\int\frac{R_0^\wedge(\sigma,\zeta,\lambda-\sigma,\eta-\zeta)}{\langle(\tau-\lambda,\xi-\eta)\rangle^{1-\varepsilon}}\hat v(\lambda-\sigma,\eta-\zeta)\,d\sigma\,d\zeta.\end{aligned}$$

It is easy to prove

$$\frac{b_0^\wedge(\sigma,\zeta,\tau-\sigma,\xi-\zeta)-b_0^\wedge(\sigma,\zeta,\lambda-\sigma,\eta-\zeta)}{\langle(\tau-\lambda,\xi-\eta)\rangle^{1-\varepsilon}}\lesssim\frac{1}{\langle(\sigma,\zeta)\rangle^{M_1}\langle(\lambda-\sigma,\eta-\zeta)\rangle^{1-\varepsilon}}.$$

We prove the lemma only for $\mu=1$, *i.e.* $P_1=D_t-|D_x|$. Let ε_0, Γ_1 and Γ be the same as the ones in the proof of Theorem 2.1.

We suppose $v\in(H^{r_2})^\beta_{P_1}(\Gamma_1)$. It is sufficient to prove $v_1\in(H^{r_2-\varepsilon})^\beta_{P_1}(\Gamma)$, $v_2\in(H^{r_2})^\beta_{P_1}(\Gamma)$.

(1) If $|(\sigma,\zeta)|\le\varepsilon_0|(\lambda,\eta)|$, $\varepsilon_0<1$.

Notice that $(\lambda-\sigma,\eta-\zeta)\in\Gamma_1$, $v\in(H^{r_2})^\beta_{P_1}(\Gamma_1)$ and $|(\lambda-\sigma,\eta-\zeta)|>(1-\varepsilon_0)|(\lambda,\eta)|$.

$$\begin{aligned}&\langle(\lambda,\eta)\rangle^{r_2-\varepsilon}\langle\lambda-|\eta|\rangle^\beta\hat v_1(\lambda,\eta)\\ &=\frac{1}{(2\pi)^n}\int\frac{\langle(\lambda,\eta)\rangle^{r_2-\varepsilon}\langle\lambda-|\eta|\rangle^\beta[b_0^\wedge(\sigma,\zeta,\tau-\sigma,\xi-\zeta)-b_0^\wedge(\sigma,\zeta,\lambda-\sigma,\eta-\zeta)]}{\langle(\tau-\lambda,\xi-\eta)\rangle^{1-\varepsilon}\langle(\lambda-\sigma,\eta-\zeta)\rangle^{r_2}\langle\lambda-\sigma-|\eta-\zeta|\rangle^\beta}\\ &\qquad p(\lambda-\sigma,\eta-\zeta)\langle(\sigma,\zeta)\rangle^{\frac{n+1}{2}}\cdot\frac{1}{\langle(\sigma,\zeta)\rangle^{\frac{n+1}{2}}}\cdot\langle(\lambda-\sigma,\eta-\zeta)\rangle^{r_2}\\ &\qquad\langle\lambda-\sigma-|\eta-\zeta|\rangle^\beta\hat v(\lambda-\sigma,\eta-\zeta)\,d\sigma\,d\zeta\\ &=\frac{1}{(2\pi)^n}\int K_{11}g_1f_1\,d\sigma\,d\zeta\end{aligned}$$

where K_{11}, g_1, f_1 denote the obvious factors respectively and $g_1, f_1 \in L^2$.

$$K_{11} \lesssim \frac{\langle(\lambda,\eta)\rangle^{r_2-\varepsilon}\langle\lambda-|\eta|\rangle^{\beta}}{\langle(\lambda-\sigma,\eta-\zeta)\rangle^{r_2-\varepsilon}\langle\lambda-\sigma-|\eta-\zeta|\rangle^{\beta}\langle(\sigma,\zeta)\rangle^{M}} \lesssim \frac{1}{\langle(\sigma,\zeta)\rangle^{M-\beta}}.$$

We also have $\langle(\lambda,\eta)\rangle\langle\lambda-|\eta|\rangle^{\beta}\hat{v}_2(\lambda,\eta) = \frac{1}{(2\pi)^n}\int K_{21}g_1f_1\,d\sigma\,d\zeta$

$$K_{21} \lesssim \frac{\langle(\lambda,\eta)\rangle^{r_2}\langle\lambda-|\eta|\rangle^{\beta}}{\langle(\tau-\lambda,\xi-\eta)\rangle^{1-\varepsilon}\langle(\lambda-\sigma,\eta-\zeta)\rangle^{r_2}\langle\lambda-\sigma-|\eta-\zeta|\rangle^{\beta}\langle(\sigma,\zeta)\rangle^{M}} \lesssim \frac{1}{\langle(\sigma,\zeta)\rangle^{M-\beta}}.$$

(2) If $|(\sigma,\zeta)| > \varepsilon_0|(\lambda,\eta)|$

$$\begin{aligned}&\langle(\lambda,\eta)\rangle^{r_2-\varepsilon}\langle\lambda-|\eta|\rangle^{\beta}\hat{v}_1(\lambda,\eta)\\ &= \frac{1}{(2\pi)^n}\int \frac{\langle(\lambda,\eta)\rangle^{r_2-\varepsilon}\langle\lambda-|\eta|\rangle^{\beta}[b_0^{\wedge}(\sigma,\zeta,\tau-\sigma,\xi-\zeta)-b_0^{\wedge}(\sigma,\zeta,\lambda-\sigma,\eta-\zeta)]}{\langle(\tau-\lambda,\xi-\eta)\rangle^{1-\varepsilon}\langle(\lambda-\sigma,\eta-\zeta)\rangle^{s_2}\langle\lambda-\sigma-|\eta-\zeta|\rangle^{\alpha}}\\ &\quad p(\lambda-\sigma,\eta-\zeta)\langle(\sigma,\zeta)\rangle^{\frac{n+1}{2}}\cdot\frac{1}{\langle(\sigma,\zeta)\rangle^{\frac{n+1}{2}}}\cdot\langle(\lambda-\sigma,\eta-\zeta)\rangle^{s_2}\\ &\quad \langle\lambda-\sigma-|\eta-\zeta|\rangle^{\alpha}\hat{v}(\lambda-\sigma,\eta-\zeta)\,d\sigma\,d\zeta\end{aligned}$$

where K_{12}, g_1 and g_2 denote the obvious factors respectively and $g_1, g_2 \in L^2$.

$$K_{12} \lesssim \frac{\langle(\lambda,\eta)\rangle^{r_2-\varepsilon}\langle\lambda-|\eta|\rangle^{\beta}}{\langle(\lambda-\sigma,\eta-\zeta)\rangle^{s_2-\varepsilon}\langle\lambda-\sigma-|\eta-\zeta|\rangle^{\alpha}\langle(\sigma,\zeta)\rangle^{M}} \lesssim \frac{1}{\langle(\sigma,\zeta)\rangle^{M-r_2-\beta}}.$$

It is the same that $\langle(\lambda,\eta)\rangle^{r_2}\langle\lambda-|\eta|\rangle^{\beta}\hat{v}_2(\lambda,\eta) = \frac{1}{(2\pi)^n}\int K_{22}g_1g_2\,d\sigma\,d\zeta$

$$K_{22} \lesssim \frac{\langle(\lambda,\eta)\rangle^{r_2}\langle\lambda-|\eta|\rangle^{\beta}}{\langle(\tau-\lambda,\xi-\eta)\rangle^{1-\varepsilon}\langle(\lambda-\sigma,\eta-\zeta)\rangle^{s_2}\langle\lambda-\sigma-|\eta-\zeta|\rangle^{\alpha}\langle(\sigma,\zeta)\rangle^{M}} \lesssim \frac{1}{\langle(\sigma,\zeta)\rangle^{M-r_2-\beta}}.$$

$$\therefore \sup_{(\lambda,\eta)}\int K_{ij}^2\,d\sigma\,d\zeta < +\infty, \quad i,j = 1,2.$$

$$\therefore v_1 \in (H^{r_2-\varepsilon})^{\beta}_{P_1}(\Gamma), \quad v_2 \in (H^{r_2})^{\beta}_{P_1}(\Gamma).$$

It is not difficult to prove several following commutator lemmas.

LEMMA 3.2. *Suppose $B_0 = b_0(D)$, $b_0(\tau,\xi) \in S^0_{1,0}$, $0 \le \varepsilon \le 1$, the other conditions are the same as the ones in Lemma 1.4, then the result in Lemma 3.1 is valid.*

LEMMA 3.3. *Suppose $p(\tau,\xi) \in S^1_{1,0}$, $b_0(\tau,\xi) \in S^0_{1,0}$, and*

$$\begin{cases} 0 \le \varepsilon \le 1 \\ s - \beta - \varepsilon > \frac{n-1}{2},\ s + \alpha - \varepsilon > \frac{n}{2} \\ r + \beta + \varepsilon < s_1 + s_2 + 2\alpha - \frac{n}{2} \\ r + \beta + \varepsilon < s_1 + s_2 + \alpha - \frac{n-1}{2} \end{cases}$$

(1) *If $a(t,x) \in (H^{s_1})^\alpha_{P_1} \cap (H^r)^\beta_{P_1}(t_0,x_0,\tau_0,\xi_0)$, $(\tau_0,\xi_0) \in \operatorname{Char} P_1 \cup N_1$, $v(t,x) \in (H^{s_2})^\alpha_{P_2}, \Pi WF(v) \subset E_2$, then $[b_0(D), a(t,x)P(D)]v \in (H^{r-1+\varepsilon})^\beta_{P_1}(t_0,x_0,\tau_0,\xi_0)$,*

(2) *If $a(t,x) \in (H^{s_1})^\alpha_{P_1}, \Pi WF(a) \subset E_1$, $v(t,x) \in (H^{s_2})^\alpha_{P_2} \cap (H^r)^\beta_{P_2}(t_0,x_0,\tau_0,\xi_0)$, $(\tau_0,\xi_0) \in \operatorname{Char} P_2 \cup N_2$, then $[b_0(D), a(t,x)P(D)v \in (H^{r-1+\varepsilon})^\beta_{P_2}(t_0,x_0\tau_0,\xi_0)$.*

(3) *If we substitute P_2, N_2, E_2 and s_2 for P_1, N_1, E_1 and s_1 respectively at the same time, the results are valid also.*

THEOREM 3.4 (COMMUTATOR). *Let $p(\tau,\xi) \in S^1_{1,0}$, $b_0(\tau,\xi) \in S^0_{1,0}$ and v, $a(t,x) \in (H^{s_1})^\alpha_{P_1} \oplus (H^{s_2})^\alpha_{P_2}(t_0,x_0)$; $v, a(t,x) \in (H^r)^\beta_{P_\mu}(t_0,x_0,\tau_0,\xi_0), (\tau_0,\xi_0) \in \operatorname{Char} P_\mu$, $s_i \le r$, $i = 1,2$. If*

$$\begin{cases} 0 \le \varepsilon \le 1,\ r + \beta - \varepsilon > \frac{n}{2} \\ s - \beta - \varepsilon > \frac{n-1}{2},\ s + \alpha - \varepsilon > \frac{n}{2},\ s = \min\{s_1,s_2\} \\ r + \beta + \varepsilon < s_1 + s_2 + 2\alpha - \frac{n}{2} \\ r + \beta + \varepsilon < s_1 + s_2 + \alpha - \frac{n-1}{2} \end{cases}$$

then $[b_0(D), a(t,x)P(D)]v \in (H^{r-1+\varepsilon})^\beta_{P_\mu}(t_0,x_0,\tau_0,\xi_0)$.

PROOF. It is supposed that v and $a(t,x)$ have compact support already, and $\mu = 1$, $\tau_0 > 0$. Let v_i, a_i, $i = 1,2$ satisfy $v_i, a_i \in (H^s)^\alpha_{P_i}$; $\Pi WF(v_i)$, $\Pi WF(a_i) \subset E_i$ and $v = v_1 + v_2$, $a = a_1 + a_2$. Notice that $[b_0(D), a(t,x)P(D)]v = [b_0, a_1P]v_1 + [b_0, a_1P]v_2 + [b_0, a_2P]v_1 + [b_0, a_2P]v_2$. By commutator Lemmas 3.2 and 3.3, each term belongs to $(H^{r-1+\varepsilon})^\beta_{P_1}(t_0,x_0,\tau_0,\xi_0)$.

PROPOSITION 3.5. *Let real $k_1 \in S^1_{1,0}(\mathbb{R}^{n-1})$, $p_0(t,x,\tau,\xi) \in S^0_{1,0}$. Suppose projection of compact set K on the space $(\tau,\xi) \in \mathbb{R}^n$ is independent of (t,x). If $s - \beta > \frac{n-1}{2}$, $s + \alpha > \frac{n}{2}$, $r + \beta > \frac{n}{2}$, and*

(i) *$w \in (H^{r-1})^\beta_{P_\mu}$, $\Pi WF(w) \subset\subset K$*

(ii) *$a \in (H^s)^\alpha_{P_\mu} \cap (H^r)^\beta_{P_\mu}(K)$, $r \ge s$*

(iii) *$g \in (H^{r-1+\varepsilon})^\beta_{P_\mu}(K)$, $0 \le \varepsilon \le 1$*

(iv) *$w \in (H^{r-1+\varepsilon})^\beta_{P_\mu}$, near $t = 0$.*

and $\big(D_t - k_1(D_x)\big)w = a(t,x)P_0(t,x,D)w + g$, then $w \in (H^{r-1+\varepsilon})^\beta_{P_\mu}$.

PROOF. Suppose that $\alpha(t) \geq 0$ with $\alpha(0) = 1$ is smooth, supported sufficiently near 0, such that $\alpha w \in (H^{r-1+\varepsilon})^{\beta}_{P_\mu}$. Let $\alpha_t(s) = \alpha(t-s)$, and

$$F(t) = \|\alpha_t w\|^2_{(H^{r-1+\varepsilon})^{\beta}_{P_\mu}} = \big(q\alpha(s)w(t-s),\ q\alpha(s)w(t-s)\big)$$

where $q = \Lambda^{r-1+\varepsilon}\langle P_\mu\rangle^{\beta}$, $\Lambda = \langle D\rangle$, the inner product is for $(s,x) \in \mathbb{R}^n$.

$$\frac{dF}{dt} = (iq\alpha_t k_1 w, q\alpha_t w) + (q\alpha_t w, iq\alpha_t k_1 w) + 2\,\mathrm{Re}\{(iq\alpha_t a p_0 w, q\alpha_t w) + (iq\alpha_t g, q\alpha_t w)\} = I_1 + I_2 + I_3 + I_4$$

where I_i, $i = 1,2,3,4$ denote the obvious factors respectively. By the calculus of pseudodifferential operators, we know

$$|I_4| \leq c + \|\alpha_t w\|^2_{(H^{r-1+\varepsilon})^{\beta}_{P_\mu}}$$

$$\begin{aligned} |I_3| &\leq c[|(B_0^*\varphi q\alpha_t a p_0 w, q\alpha_t w)| + |(R^* q\alpha_t a p_0 w, q\alpha_t w)|] \\ &\leq c(\|B_0^*\alpha_t a p_0 w\|^2_{(H^{r-1+\varepsilon})^{\beta}_{P_\mu}} + \|\alpha_t w\|^2_{(H^{r-1+\varepsilon})^{\beta}_{P_\mu}}) + c \\ &\leq c(\|\alpha_t w\|^2_{(H^{r-1+\varepsilon})^{\beta}_{P_\mu}} + \|w\|^2_{(H^{r-1})^{\beta}_{P_\mu}}) + c \end{aligned}$$

where $\sigma_{R^*} \in S^{-\infty}$, B_0^* is the conjugate operator for B_0.

$$|I_1 + I_2| \leq c\|\alpha_t w\|^2_{(H^{r-1+\varepsilon})^{\beta}_{P_\mu}}$$

$$\therefore \frac{dF}{dt} \leq c(\|\alpha_t w\|^2_{(H^{r-1+\varepsilon})^{\beta}_{P_\mu}} + \|w\|^2_{(H^{r-1})^{\beta}_{P_\mu}} + c).$$

By Gronwall inequality

$$\|\alpha_t w\|^2_{(H^{r-1+\varepsilon})^{\beta}_{P_\mu}} \leq c(\|\alpha_0 w\|^2_{(H^{r-1+\varepsilon})^{\beta}_{P_\mu}} + \|w\|^2_{(H^{r-1})^{\beta}_{P_\mu}} + c) < \infty.$$

So $F(0) = \|\alpha w\|^2_{(H^{r-1+\varepsilon})^{\beta}_{P_\mu}} < \infty$.

Suppose that $P_m \in S^m_{1,0}$ is homogeneous strictly hyperbolic with respect to the direction $(1,0)$,

$$P_m(t,x,\tau,\xi) = \big(\tau - k_1(t,x,\xi)\big)\cdots\big(\tau - k_m(t,x,\xi)\big)$$

where k_i, $i = 1,2,\ldots,m$ are real, homogeneous of degree 1 in ξ, distinct for $\xi \neq 0$ and $k_i \in S^1_{1,0}$, $(t,x,\tau,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$. Suppose γ is a null-bicharacteristic of P_m passing through (t_0,x_0,τ_0,ξ_0), $\gamma \subset c_{1x} = \{(t,x,\tau,\xi)\ ;\ (\tau,\xi) \neq 0,\ \tau = k_1(t,x,\xi)\}$, then we can choose $b_0(t,x,\tau,\xi) \in S^0_{1,0}$ which is elliptic near γ. Because the principal symbol of $i[P_m(t,x,D), b_0(t,x,D)]$ is $H_{P_m}b_0$, we can choose $b_0(t,x,\tau,\xi)$, whose conic support is sufficiently near γ, such that $[P_m, b_0]$ has order $m-2$. So

$$P_m(t,x,D)b_0(t,s,D) = \big(D_t - k_1(t,x,D_x)\big)q_{m-1}(t,x,D)b_0(t,x,D)$$

where $q_{m-1} \in S^{m-1}_{1,0}$ elliptic near γ.

THEOREM 3.6 (PROPAGATION OF SINGULARITIES). *Suppose $P_m(D)$ is a strictly hyperbolic homogeneous pseudodifferential operator of degree $m \geq 2$, $P_\ell(\tau,\xi) \in S^{m-1}_{1,0}$, γ is a null-bicharacteristic of P_m passing through $(t_0,x_0,\tau_0,\xi_0) \in \mathbb{R}^n \times \mathbb{R}^n \setminus 0$. If*

$$\begin{cases} 0 \leq \varepsilon \leq \frac{1}{2} \\ s-\beta-\varepsilon > \frac{n-1}{2},\ s+\alpha-\varepsilon > \frac{n}{2},\ s=\min\{s_1,s_2\} \\ r+\beta-\varepsilon > \frac{n}{2} \\ r+\beta+\varepsilon < s_1+s_2+2\alpha-\frac{n}{2} \\ r+\beta+\varepsilon < s_1+s_2-\frac{n-1}{2},\ (\tau_0,\xi_0) \in N_\mu \\ r+\beta+\varepsilon < s_1+s_2+\alpha-\frac{n-1}{2},\ (\tau_0,\xi_0) \notin N_\mu. \end{cases}$$

(i) $v \in (H^{s_2+m-2})^\alpha_{P_\mu} \cap (H^{r+m-2})^\beta_{P_\mu}(\gamma)$

(ii) $a_\ell(t,x) \in (H^{s_1})^\alpha_{P_\mu} \cap (H^r)^\beta_{P_\mu}(\gamma)$, $f \in (H^{r-1+\varepsilon})^\beta_{P_\mu}(\gamma)$

(iii) $v \in (H^{r+m-2+\varepsilon})^\beta_{P_\mu}(t_0,x_0,\tau_0,\xi_0)$.

and $\big(P_m(D)+\Sigma a_\ell(t,x)P_\ell(D)\big)v=f$, *then* $v \in (H^{r+m-2+\varepsilon})^\beta_{P_\mu}(\gamma)$.

PROOF. Choose $b_0 \in S^0_{1,0}$, as mentioned previously.

$$b_0 f = P_m(D)b_0 v + \Sigma a_\ell(t,x)P_\ell(D)b_0 v + [b_0,P_m]v + \Sigma[b_0, A_\ell P_\ell \Lambda^{-(m-2)}]\Lambda^{m-2}v - \Sigma a_\ell P_\ell[b_0,\Lambda^{-(m-2)}]\Lambda^{m-2}v.$$

Obviously $[b_0,P_m]v$, $b_0 f \in (H^{r-1+\varepsilon})^\beta_{P_m}(\gamma)$. By Theorem 1.3, it follows that

$$a_\ell(t,x)(P_\ell[b_0,\Lambda^{-(m-2)}]\Lambda^{m-2}) \in (H^s)^\alpha_{P_\mu} \cap (H^{r-1+\varepsilon})^\beta_{P_\mu}(\gamma).$$

By commutator Lemma 3.1, we have

$$[b_0, A_\ell P_\ell \Lambda^{-(m-2)}]\Lambda^{m-2}v \in (H^{s-1+\varepsilon})^\alpha_{P_\mu} \cap (H^{r-1+\varepsilon})^\beta_{P_\mu}(\gamma).$$

As mentioned previously, we have

$$P_m b_0 = \big(D_t - k_1(D_x)\big)q_{m-1}(D)b_0.$$

Let $q_{-(m-1)} \in S^{-(m-1)}_{1.0}$, such that $q_{-(m-1)}q_{m-1} = 1$ in $\operatorname{supp} b$. Write $\tilde{P}_\ell = P_\ell q_{-(m-1)}$, $w = q_{m-1}b_0 v$, so

$$\big(D_t - k_1(D_x)\big)w + \Sigma a_\ell(t,x)\tilde{P}_\ell w = g$$

where $g \in (H^{r-1+\varepsilon})^\beta_{P_\mu}(\gamma)$, $w \in (H^{r-1+\varepsilon})^\beta_{P_\mu}(t_0,x_0,\tau_0,\xi_0)$. By Proposition 3.5 $w \in (H^{r-1+\varepsilon})^\beta_{P_\mu}$, so $v \in (H^{r+m-2+\varepsilon})^\beta_{P_\mu}(\gamma)$.

Notice that if $a_\ell(t,x) = 0$, we obtain the propagation of singularities theorem in $(H^s)^\alpha_{P_\mu}$ for linear equations with smooth coefficients, whose proof is the same as the one in Theorem 3.6. Because $a_\ell(t,x) = 0$, the conditions in Theorem 1.3 and in Lemma 4.1 are not necessary. So it follows

THEOREM 3.7 (PROPAGATION OF SINGULARITIES). *Suppose that $P_m(D)$ is a smooth strictly hyperbolic homogeneous pseudodifferential operator of degree $m \geq 0$, γ is a null-bicharacteristic of $P_m(D)$ passing $(t_0, x_0, \tau_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$. If $s - \beta > \frac{n-1}{2}$, $s + \alpha > \frac{n}{2}$, $r + \beta > \frac{n}{2}$, $0 \leq \varepsilon \leq 1$,*

(i) $v \in (H^{s+m-2})^{\alpha}_{P_\mu} \cap (H^{r+m-2})^{\beta}_{P_\mu}(\gamma)$

(ii) $f \in (H^{r-1+\varepsilon})^{\beta}_{P_\mu}(\gamma)$

(iii) $v \in (H^{r+m-2+\varepsilon})^{\beta}_{P_\mu}(t_0, x_0, \tau_0, \xi_0)$.

and $P_m(D)v = f$, then $v \in (H^{r+m-2+\varepsilon})^{\beta}_{P_\mu}(\gamma)$.

4. Proof of main results.

PROOF OF THEOREM 1.6. We suppose $(\tau_0, \xi_0) \in \operatorname{Char} P_\mu$, otherwise $u \in H^{r_\mu+\beta_\mu}(t_0, x_0, \tau_0, \xi_0)$. and $\mu = 1$, *i.e.* $P_1 = D_t - |D_x|$. Differentiate the equation $\Box u = f(u, Du)$, we obtain

$$\Box \partial_t u = f_1(u, Du)\partial_t u + f_2(u, Du)D\partial_t u$$

where f_2 is a vector.

Set $\psi_1(\tau, \xi) \in C^\infty$, $\psi_1(\tau, \xi) = 1$ in a conic neighbourhood K_1 of (τ_0, ξ_0), and $\psi_1(\tau, \xi) = 0$ out of the conic neighbourhood $K_2 \supset\supset K_1$ and $\psi_1(\tau, \xi)$ is homogeneous of 0 degree, $\operatorname{supp} \psi_1 \cap \operatorname{char} P_2 = \emptyset$.

Let $v = \partial_t \psi_1(D)u$, $g = [\psi_1(D) - f_2(u, Du)D]\partial_t u + \psi_1(D)\big(f_1(u, Du)\partial_t u\big)$. It follows that

$$\Box v - f_2(u, Du)Dv = g.$$

By Theorem 2.4, $u \in (H^s)^{\alpha}_{P_1} \oplus (H^s)^{\alpha}_{P_2}(t, x)$, $\forall (t, x) \in \overset{o}{\Omega}$, for any $\alpha \geq 0$ satisfying $\alpha < s - \frac{n-1}{2}$. By Theorem 1.4, if $s - 1 - \alpha > \frac{n-1}{2}$, $f_i(u, Du) \in (H^{s-1})^{\alpha}_{P_1} \oplus (H^{s-1})^{\alpha}_{P_2}(t, x)$.

Set $\alpha = s - \frac{n+1}{2} - \delta$, where $\delta > 0$ is sufficiently small, such that $\beta \leq \alpha$, so $-\frac{n-1}{2} \leq \alpha - \frac{n}{2}$.

Set $0 \leq \varepsilon \leq \frac{1}{2}$, $\varepsilon < s - \beta - \frac{n+1}{2}$ and $\varepsilon < s + \beta - \frac{n}{2} - 1$. By the commutator Theorem 3.4, $[\psi_1(D), f_2(u, Du)D]\partial_t u \in (H^{s-2+\varepsilon})^{\alpha}_{P_1}(\gamma)$, so $g \in (H^{s-2+\varepsilon})^{\alpha}_{P_1}(\gamma)$.

Set $\varepsilon_1 = \min\{r - s, \varepsilon\}$, $v \in (H^{s-1+\varepsilon_1})^{\beta}_{P_1}(t_0, x_0, \tau_0, \xi_0)$ and by Theorem 3.6, $v \in (H^{s-1+\varepsilon_1})^{\beta}_{P_1}(\gamma)$. If $r - s \leq \varepsilon$, $u \in (H^{s+\varepsilon_1})^{\beta}_{P_1}(\gamma) = (H^r)^{\beta}_{P_1}(\gamma)$, it is proven. Otherwise, $\varepsilon_1 = \varepsilon$, so $v \in (H^{s-1+\varepsilon})^{\beta}_{P_1}(\gamma)$. $u \in (H^{s+\varepsilon})^{\beta}_{P_\mu}(\gamma)$, then we repeat the above proof again.

Suppose that we have proved $u \in (H^{r-\varepsilon})^{\beta}_{P_1}(\gamma)$ for $r - \varepsilon \geq s$ where $0 \leq \varepsilon \leq \frac{1}{2}$ and satisfying

$$\begin{cases} s - \varepsilon + \beta > \frac{n+1}{2}, & s - \varepsilon + \beta > \frac{n}{2} \\ s - \varepsilon + \alpha > \frac{n+1}{2}, & s - \varepsilon + \alpha > \frac{n}{2} \end{cases}$$

If $r + \beta < 1 + 2(s - 1) + \alpha + \min\{-\frac{n+1}{2}, \alpha - \frac{n}{2}\}$, where $\alpha = s - \frac{n+1}{2} - \delta$, *i.e.* $r + \beta < 3s - n - 1$, by Theorem 1.4 $f_1(u, Du)\partial_t u, f_2(u, Du) \in (H^{r-1-\varepsilon})^{\beta}_{P_1}(\gamma)$. By Theorem 3.4, $[\psi_1(D), f_2(u, Du)D]\partial_t u \in (H^{r-2-2\varepsilon})^{\beta}_{P_1}(\gamma)$, so $g \in (H^{r-2-2\varepsilon})^{\beta}_{P_1}(\gamma)$. And by Theorem 3.6, $v \in (H^{r-1})^{\beta}_{P_1}(\gamma)$, so $u \in (H^r)^{\beta}_{P_1}(\gamma)$.

By Theorem 4.7 we obtain

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THEOREM 4.1. *Let $u \in H^s_{\mathrm{loc}}(\Omega)$, $\Omega \subset \mathbb{R}^n$, $s > \frac{n}{2}$ be a solution of $\Box u = f(t,x,u)$ in Ω, where $f \in c^\infty$. γ is a null-bicharacteristic of $\Box$ passing through* $(t_0, x_0, \tau_0, \xi_0)$.

If $u \in (H^{r_\mu})^{\beta_\mu}_{P_\mu}(t_0, x_0, \tau_0, \xi_0)$ and

(i) $\beta_\mu < s - \frac{n-1}{2}$

(ii) $r_\mu + \beta_\mu < 3s - n + 2$,

then $u \in (H^{r_\mu})^{\beta_\mu}_{P_\mu}(\gamma)$.

COROLLARY 4.2. *Let $u \in H^s_{\mathrm{loc}}(\Omega)$, $\Omega \subset \mathbb{R}^n$, $s > \frac{n}{2}$ be a solution of $\Box u = f(t,x,u)$ in Ω, where $f \in C^\infty$, γ is a null-bicharacteristic of $\Box$ passing through $(t_0, x_0, \tau_0, \xi_0)$. If $u \in H^r(t_0, x_0, \tau_0, \xi_0)$, $r < 3s - n + 2$, then $u \in H^r(\gamma)$.*

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