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ON PERMUTABILITY AND SUBMULTIPLICATIVITY OF SPECTRAL RADIUS

W. E. LONGSTAFF AND H. RADJAVI

ABSTRACT. Let $r(T)$ denote the spectral radius of the operator T acting on a complex Hilbert space H. Let $\mathcal{S}$ be a multiplicative semigroup of operators on H. We say that r is *permutable* on $\mathcal{S}$ if $r(ABC) = r(BAC)$, for every $A, B, C \in \mathcal{S}$. We say that r is *submultiplicative* on $\mathcal{S}$ if $r(AB) \leq r(A)r(B)$, for every $A, B \in \mathcal{S}$. It is known that, if r is permutable on $\mathcal{S}$, then it is submultiplicative. We show that the converse holds in each of the following cases: (i) $\mathcal{S}$ consists of compact operators (ii) $\mathcal{S}$ consists of normal operators (iii) $\mathcal{S}$ is generated by orthogonal projections.

1. **Introductions and preliminaries.** Let $r(T)$ denote the spectral radius of the operator T acting on a complex Hilbert space H. Let $\mathcal{S}$ be a multiplicative semigroup of operators on H. We say that r is *multiplicative* on $\mathcal{S}$ if $r(AB) = r(A)r(B)$, for every $A, B \in \mathcal{S}$. We say that r is *submultiplicative* on $\mathcal{S}$ if $r(AB) \leq r(A)r(B)$, for every $A, B \in \mathcal{S}$. In [4], the effects of certain spectral conditions, including the submultiplicativity of r, on reducibility were the central consideration. Here we consider the question of whether or not r must be permutable on $\mathcal{S}$ if it is known to be submultiplicative. By *permutability* of r on $\mathcal{S}$ is meant the condition: $r(ABC) = r(BAC)$ for every $A, B, C \in \mathcal{S}$. (This condition is equivalent to the requirement that the spectral radius of any word of finite length, in letters from $\mathcal{S}$, be independent of the order of the letters. This can be verified by induction and using the fact that $r(AB) = r(BA)$ for all A and B.) For any semigroup of operators, Theorem 9 of [4] shows that the permutability of r implies its submultiplicativity. Whether or not the reverse implication must always hold has still not been settled. However, we show that it does hold in each of the following cases: (i) $\mathcal{S}$ consists of compact operators (ii) $\mathcal{S}$ consists of normal operators (iii) $\mathcal{S}$ is generated by orthogonal projections.

In what follows the underlying field of scalars is $\mathbb{C}$, all operators are linear and bounded and all subspaces are closed. We shall frequently identify operators with their matrices (relative to a tacit fixed basis or orthogonal decomposition of the underlying space) when no confusion is likely to arise. For any Hilbert space H, the inner product on H will be denoted $(\cdot|\cdot)$, and $\mathcal{B}(H)$ (respectively, $\mathcal{K}(H)$) will denote the algebra of all operators (respectively, compact operators) on H. Continuity of r on $\mathcal{K}(H)$ (see [1]) will be frequently used to replace a semigroup $\mathcal{S} \subseteq \mathcal{K}(H)$, on which r is submultiplicative, with its norm closure $\bar{\mathcal{S}}$, with no loss of generality. For any operator $T \in \mathcal{B}(H)$, $\sigma(T)$ denotes its spectrum and $\sigma_{\mathrm{ap}}(T)$ is approximate point spectrum. Also $\mathcal{R}(T)$ denotes the range of T and tr T denotes the trace of T if T is of trace class.

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The following lemmas, whose proofs are contained in [6], will be used more than once. A subset $\mathcal{J}$ of a semigroup $\mathcal{S}$ is called an *ideal* of $\mathcal{S}$ if JS and SJ belong to $\mathcal{J}$, for every $J \in \mathcal{J}$ and $S \in \mathcal{S}$. A semigroup $\mathcal{S} \subseteq \mathcal{B}(H)$ is *reducible* if it has a non-trivial invariant subspace; otherwise it is *irreducible*.

LEMMA 1. *If $\mathcal{S}$ is a semigroup of compact operators on a Hilbert space H and there is a non-zero continuous linear functional on $\mathcal{K}(H)$ which is constant on $\mathcal{S}$, then $\mathcal{S}$ is reducible.*

LEMMA 2. *Every non-zero ideal of an irreducible semigroup of operators on a Hilbert space is irreducible.*

LEMMA 3. *Let K be a compact operator on a Hilbert space H with $r(K) = 1$. Let m be the (finite) rank of the Riesz projection P of K corresponding to the non-empty set*

$$\{z \in \sigma(K) : |z| = 1\},$$

and let $\mathcal{C}$ be the norm closure of

$$\{cK^n : c \in \mathbb{C}, n \in \mathbb{Z}^+\}.$$

Then

(i) *If K is not similar to a contraction, then $\mathcal{C}$ contains a non-zero nilpotent operator of rank less than m.*

(ii) *If K is similar to a contraction, then $P \in \mathcal{C}$ and the restrictions of K to $\mathcal{R}(P)$ and $\mathcal{R}(1 - P)$ are similar, respectively, to a unitary operator and to a strict contraction.*

We will also need the following lemma taken from [5].

LEMMA 4. *A semigroup of quasinilpotent trace class operators on a Hilbert space of dimension greater than one is reducible.*

2. **Semigroups of compact operators.**

We begin by considering semigroups, consisting of compact operators, on which r is submultiplicative. Our main result in this context (Theorem 2.6) is that r is permutable on such semigroups. For irreducible semigroups of this type an even stronger result holds.

THEOREM 2.1. *Let $\mathcal{S}$ be an irreducible semigroup of compact operators on a Hilbert space H. If r is submultiplicative on $\mathcal{S}$, then r is multiplicative, so permutable, on $\mathcal{S}$.*

PROOF. Assume that r is submultiplicative on $\mathcal{S}$. We can assume that $\dim H \geq 2$ and $\mathcal{S} = \bar{\mathcal{S}}$ (by the continuity of r on $\mathcal{K}(H)$). We can also assume that $\mathcal{S}$ is closed under scalar multiplication, that is, $\mathcal{S} = \mathbb{C}\mathcal{S}$. It is enough to show that, if $A, B \in \mathcal{S}$ and $r(A) = r(B) = 1$, then $r(AB) = 1$.

First, suppose that $A^2 = A$ and $B^2 = B$. Then $r(AB) \neq 0$. For, suppose that $r(AB) = 0$. Since $\mathcal{S}$ contains no non-zero finite-rank nilpotent elements by Lemmas 2 and 4, $AB = 0$.

The semigroup $B\mathcal{S}A$ then consists of finite rank nilpotent operators so $B\mathcal{S}A = (0)$. Let $x, y \in H$ satisfy $Ax \neq 0$, $B^*y \neq 0$. The linear functional $f\colon \mathcal{K}(H) \longrightarrow \mathbb{C}$ defined by $f(K) = (KAx|B^*y)$ is continuous and non-zero and $f|_{\mathcal{S}} = 0$. By Lemma 1, this contradicts the irreducibility of $\mathcal{S}$. Hence we must have $r(AB) \neq 0$. Then

$$r(AB) = r(A^2B^2) = r(BA \cdot AB) \leq r(BA)r(AB) = r(AB)^2,$$

so $r(AB) \geq 1$. Since $r(AB) \leq r(A)r(B) = 1$, $r(AB) = 1$.

In general, if $r(A) = r(B) = 1$, since $\mathcal{S}$ contains no non-zero finite-rank nilpotent elements, there exists by Lemma 3 a sequence (a_j) of scalars and an increasing sequence (n_j) of positive integers such that $(a_jA^{n_j})$ converges to a non-zero idempotent $P \in \mathcal{S}$. Since $r(a_jA^{n_j}) = |a_j| \longrightarrow r(P) = 1$, we may suppose that $a_j \longrightarrow a$ with $|a| = 1$. Then $A^{n_j} \longrightarrow P/a$. Similarly, $B^{m_j} \longrightarrow Q/b$ for some increasing sequence (m_j) of positive integers where Q is a non-zero idempotent element of $\mathcal{S}$ and $|b| = 1$. Then

$$r(A^{n_j}B^{m_j}) = r(AB \cdot B^{m_j-1}A^{n_j-1}) \leq r(AB)r(A^{n_j-1})r(B^{m_j-1}) = r(AB).$$

Taking limits gives $r(PQ) \leq r(AB)$. By the first part of the proof $r(PQ) = 1$. Hence $r(AB) \geq 1$ so $r(AB) = 1$. ∎

Before proving the main result of this section, we need some results concerning the transmission of certain properties of spectral radius from non-zero ideals to the overlying semigroup.

PROPOSITION 2.2. *Let A be an $n \times n$ matrix such that, for some $L \geq 0$, $|\operatorname{tr}(A^k)| \leq L$, for every $k \geq 1$. Then $r(A) \leq 1$.*

PROOF. We use the fact that, for every finite set $\mu_1, \mu_2, \ldots, \mu_N$ of complex numbers of modulus one, there exists an increasing sequence (m_k) of positive integers such that $\mu_i^{m_k} \longrightarrow 1$ as $k \longrightarrow \infty$, for every $i = 1, 2, \ldots, N$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A, counted according to multiplicity, and suppose that $\rho = r(A) > 1$. Let $|\lambda_1| = |\lambda_2| = \cdots = |\lambda_j| = \rho$ and $|\lambda_i| < \rho$ for $i = j+1, \ldots, n$. Then, for every $k \geq 1$,

$$|\operatorname{tr}(A^k)| = \Big|\sum_{i=1}^{n} \lambda_i^k\Big| \leq L,$$

so

$$\Big|\sum_{i=1}^{j}\Big(\frac{\lambda_i}{\rho}\Big)^k\Big| \leq \frac{L}{\rho^k} + \Big|\sum_{i=j+1}^{n}\Big(\frac{\lambda_i}{\rho}\Big)^k\Big| \leq \frac{L}{\rho^k} + \sum_{i=j+1}^{n}\Big(\frac{|\lambda_i|}{\rho}\Big)^k.$$

It follows that $\sum_{i=1}^{j}(\frac{\lambda_i}{\rho})^k \longrightarrow 0$ as $k \longrightarrow \infty$. However, for $i = 1, 2, \ldots, j$ we have $|\lambda_i/\rho| = 1$ so there exists an increasing sequence (m_k) of positive integers such that $(\lambda_i/\rho)^{m_k} \longrightarrow 1$. So $\sum_{i=1}^{j}(\lambda_i/\rho)^{m_k} \longrightarrow j$. This contradiction shows that $r(A) \leq 1$. ∎

COROLLARY 2.2.1. *Let $n \geq 2$, let $\mathcal{S}$ be an irreducible semigroup of $n \times n$ matrices and let $\mathcal{J}$ be a non-zero ideal of $\mathcal{S}$. If $r(J) \leq 1$, for every $J \in \mathcal{J}$, then $r(S) \leq 1$, for every $S \in \mathcal{S}$.*

PROOF. Suppose that $r(J) \leq 1$, for every $J \in \mathcal{J}$. By Lemma 2, $\mathcal{J}$ is irreducible; so then is its linear span. The latter is an algebra, so by Burnside's Theorem, $\mathcal{J}$ contains a basis $\{J_i : 1 \leq i \leq n^2\}$ of $\mathcal{B}(\mathbb{C}^n)$. In particular, there exist scalars α_i, $i = 1, 2, \ldots, n^2$ such that $\sum_{i=1}^{n^2} \alpha_i J_i = 1$. Let $A \in \mathcal{S}$. Then $A = \sum_{i=1}^{n^2} \alpha_i A J_i$ so $\operatorname{tr} A = \sum_{i=1}^{n^2} \alpha_i \operatorname{tr}(AJ_i)$. But, for every i, $|\operatorname{tr}(AJ_i)| \leq nr(AJ_i) \leq n$ so $|\operatorname{tr} A| \leq n(\sum_{i=1}^{n^2} |\alpha_i|)$. Thus, for every $S \in \mathcal{S}$ and every $k \geq 1$ we have $|\operatorname{tr}(S^k)| \leq n(\sum_{i=1}^{n^2} |\alpha_i|)$. By the preceding proposition, $r(S) \leq 1$, for every $S \in \mathcal{S}$. ∎

COROLLARY 2.2.2. *Let $n \geq 2$, let $\mathcal{S}$ be an irreducible semigroup of $n \times n$ matrices and let $\mathcal{J}$ be a non-zero ideal of $\mathcal{S}$. If $r(J) = 1$, for every $J \in \mathcal{J}$, then $r(S) = 1$, for every $S \in \mathcal{S}$.*

PROOF. Suppose that $r(J) = 1$, for every $J \in \mathcal{J}$. Let $S \in \mathcal{S}$ and suppose that $r(S) \neq 1$. By the preceding corollary, $r(S) < 1$. By first performing a similarity transformation on $\mathcal{S}$, if necessary, we can assume that $\|S\| < 1$. (By replacing the off-diagonal 1's in the Jordan canonical form of S by sufficiently small positive ε we obtain a matrix, similar to S, but with norm strictly less than one.) Let $J \in \mathcal{J}$. Then $\|S^n\| < 1/\|J\|$, for n sufficiently large. Then $\|S^n J\| < 1$ so $r(S^n J) < 1$. This is a contradiction because $S^n J \in \mathcal{J}$. ∎

It is clear that the requirement of irreducibility cannot be dropped in the statements of the corollaries above. Consider, for example, the semigroup $\left\{\begin{pmatrix} 1 & 0 \\ 0 & aI_{n-1} \end{pmatrix} : a \in \mathbb{C}\right\}$ and its ideal $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right\}$. This example also shows that the requirement of irreducibility cannot be dropped in the statement of Theorem 2.1.

PROPOSITION 2.3. *Let $\mathcal{S}$ a semigroup of operators acting on a (possibly infinite-dimensional) Hilbert space and let $\mathcal{J}$ be an ideal of $\mathcal{S}$. If r is multiplicative on $\mathcal{J}$ then $r(JS) \leq r(J)r(S)$, for every $J \in \mathcal{J}$ and every $S \in \mathcal{S}$.*

PROOF. We may suppose that $\mathcal{S} = \mathbb{C}\mathcal{S}$ and $\mathcal{J} = \mathbb{C}\mathcal{J}$. Let $S \in \mathcal{S}$ and $J \in \mathcal{J}$.

If $r(J) = 1$, then $r(JSJ) = r(JS)r(J) = r(JS)$. By induction, $r(JS^kJ) = r(JS)^k$, for every $k \geq 1$. Thus $r(JS)^k = r(JS^k)r(J) = r(JS^k) \leq \|JS^k\| \leq \|J\|\,\|S^k\|$, so $r(JS) \leq \|J\|^{1/k}\|S^k\|^{1/k}$, for every $k \geq 1$. Hence $r(JS) \leq r(S)$.

If $r(J) \neq 0$, then, by what has just been proved, $r\Big(\big(J/r(J)\big)S\Big) \leq r(S)$, so $r(JS) < r(J)r(S)$.

If $r(J) = 0$, then $r(JS)^2 = r(JS)r(SJ) = r(JS^2J) = r(J)r(S^2J) = 0$, so $r(JS) = 0$ and again $r(JS) \leq r(J)r(S)$. ∎

THEOREM 2.4. *Let $\mathcal{S}$ be an irreducible semigroup of $n \times n$ matrices and let $\mathcal{J}$ be a non-zero ideal of $\mathcal{S}$. If r is submultiplicative on $\mathcal{J}$, then it is submultiplicative on $\mathcal{S}$.*

PROOF. Let r be submultiplicative on $\mathcal{J}$. We can suppose that $n \geq 2$ and that $\mathcal{S} = \mathbb{C}\mathcal{S}$, $\mathcal{J} = \mathbb{C}\mathcal{J}$. By Lemma 2, $\mathcal{J}$ is irreducible. By the same lemma and Lemma 4, $\mathcal{J}$ contains no non-zero nilpotent elements. By Theorem 2.1, r is multiplicative on $\mathcal{J}$ so, by Proposition 2.3, $r(JS) \leq r(J)r(S)$, for every $J \in \mathcal{J}$ and every $S \in \mathcal{S}$. By Burnside's Theorem, the linear span of $\mathcal{J}$ is $\mathcal{B}(\mathbb{C}^n)$ so there exist scalars α_i and elements $J_i \in \mathcal{J}$, with $r(J_i) = 1$, $i = 1, 2, \ldots, n^2$ such that $\sum_{i=1}^{n^2} \alpha_i J_i = 1$.

Let $S, T \in \mathcal{S}$ with $r(S) = r(T) = 1$. Let $J \in \mathcal{J}$ with $r(J) = 1$. Then

$$r\big((ST)J\big) = r(J)r\big((ST)J\big) = r\big(J(ST)J\big) = r(JS)r(JT).$$

By induction, for every $k \geq 1$, $r\big((ST)^k J\big) = r(JS)^k r(JT)^k$ (note that $r\big((ST)^{k+1}J\big) = r(J)r\big((ST)^{k+1}J\big) = r\big(J(ST)^{k+1}J\big) = r\big(J(ST)\big)r\big((ST)^k J\big) = r\big((ST)J\big)r\big((ST)^k J\big)$). Since

$$(ST)^k = \sum_{i=1}^{n^2} \alpha_i (ST)^k J_i$$

we have, for every $k \geq 1$,

$$\begin{aligned}
|\operatorname{tr}(ST)^k| &\leq \sum_{i=1}^{n^2} |\alpha_i|\,|\operatorname{tr}(ST)^k J_i| \leq n\sum_{i=1}^{n^2} |\alpha_i| r\big((ST)^k J_i\big)\\
&= n\sum_{i=1}^{n^2} |\alpha_i| r(J_iS)^k r(J_iT)^k \leq n\sum_{i=1}^{n^2} |\alpha_i| r(J_i)^k r(S)^k r(J_i)^k r(T)^k\\
&= n\sum_{i=1}^{n^2} |\alpha_i|.
\end{aligned}$$

By Proposition 2.2, $r(ST) \leq 1$.

Now $\mathcal{S}$, like $\mathcal{J}$, contains no non-zero nilpotent elements. For if $A \in \mathcal{S}$ and $r(A) = 0$ then, with $\{J_i : 1 \leq i \leq n^2\}$ as above, $r(J_iA) \leq r(J_i)r(A) = 0$ so $J_iA = 0$, for every i. Thus $A = \sum_{i=1}^{n^2} \alpha_i J_i A = 0$.

Finally, let $B, C \in \mathcal{S}$. If $r(B) = 0$ or $r(C) = 0$ certainly $r(BC) \leq r(B)r(C)$. Otherwise $r\Big(\big(B/r(B)\big)\big(C/r(C)\big)\Big) \leq 1$ so $r(BC) \leq r(B)r(C)$. ∎

The semigroup $\left\{\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} : A \in \mathcal{B}(\mathbb{C}^{n-1})\right\}$ of $n \times n$ matrices, where $n \geq 3$, shows that the requirement of irreducibility cannot be dropped from the statement of the preceding theorem. Less trivially, the following example shows that the requirement of finite-dimensionality cannot be dropped. (However, see Theorem 2.5.)

EXAMPLE. There exists an irreducible semigroup $\mathcal{S}$ of operators on infinite-dimensional Hilbert space such that r is multiplicative on a non-zero ideal $\mathcal{J}$ of $\mathcal{S}$ yet is not submultiplicative on $\mathcal{S}$.

Let $\dim H = 2$ and let $\mathcal{H} = H^{(\infty)}$. Let $\mathcal{S}_0 = \{A^{(\infty)} : A \in \mathcal{B}(H)\} \subseteq \mathcal{B}(\mathcal{H})$ be the set of all inflations of operators in $\mathcal{B}(H)$. For every $n \geq 1$ define the subset $\mathcal{J}_n$ of $\mathcal{B}(\mathcal{H})$ by:

$$\mathcal{J}_1 = \left\{\begin{pmatrix} A & A \\ -A & -A \end{pmatrix}^{(\infty)} : A \in \mathcal{B}(H)\right\}$$

$$\mathcal{J}_2 = \left\{\begin{pmatrix} B & B \\ -B & -B \end{pmatrix}^{(\infty)} : B \in \mathcal{B}(H^{(2)})\right\}$$

$$\vdots$$

$$\mathcal{J}_n = \left\{\begin{pmatrix} Z & Z \\ -Z & -Z \end{pmatrix}^{(\infty)} : Z \in \mathcal{B}(H^{(2^{n-1})})\right\}.$$

Put $\mathcal{J} = \bigcup_{n=1}^{\infty} \mathcal{J}_n$. Then, for every $J \in \mathcal{J}$, $J^2 = 0$ so $r(J) = 0$. Also, $\mathcal{S} = \mathcal{J} \cup \mathcal{S}_0$ is a semigroup and $\mathcal{J}$ is an ideal of $\mathcal{S}$. Although $r \equiv 0$ on $\mathcal{J}$, r is not submultiplicative on $\mathcal{S}$. For, let A and B be the elements of $\mathcal{B}(H)$ defined by $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then $A^{(\infty)}, B^{(\infty)} \in \mathcal{S}$ with $r(A^{(\infty)}) = r(B^{(\infty)}) = r(A) = r(B) = 0$. However, $r(A^{(\infty)}B^{(\infty)}) = r(AB) = 1$.

The semigroup $\mathcal{S}$ described above is a slight modification of an example given in [3], which contains a proof of its irreducibility. (In fact the linear span of $\mathcal{S}$ is dense in $\mathcal{B}(\mathcal{H})$ in the strong operator topology.)

The following result extends Theorem 2.4 to semigroups of compact operators.

THEOREM 2.5. *Let $\mathcal{S}$ be an irreducible semigroup of compact operators on an infinite-dimensional Hilbert space H and let $\mathcal{J}$ be a non-zero ideal of $\mathcal{S}$. If r is submultiplicative on $\mathcal{J}$, then it is submultiplicative on $\mathcal{S}$.*

PROOF. Let r be submultiplicative on $\mathcal{J}$. We can suppose that $\mathcal{S} = \mathbb{C}\mathcal{S} = \bar{\mathcal{S}}$ and that r is not identically zero on $\mathcal{S}$. Then $r(A) = 1$, for some $A \in \mathcal{S}$. By Lemma 3, $\mathcal{S}$ contains a non-zero finite rank operator, F say. By Lemma 2, $\mathcal{J}$ is irreducible so $JF \neq 0$, for some $J \in \mathcal{J}$. Thus $\mathcal{J}$ contains a non-zero finite rank operator. Let

$$m = \min\{\operatorname{rank} J : J \in \mathcal{J} \setminus \{0\} \text{ and } J \text{ has finite rank}\}.$$

Then $m \geq 1$ and the set $\mathcal{J}_0$ of elements of $\mathcal{J}$ of rank m or 0 is a non-zero ideal of $\mathcal{S}$. By Lemma 2, $\mathcal{J}_0$ is irreducible. Since r is submultiplicative on $\mathcal{J}_0$, the set of nilpotent elements of $\mathcal{J}_0$ forms an ideal of $\mathcal{J}_0$. This ideal must be zero by Lemmas 2 and 4. Thus $\mathcal{J}_0$ contains no non-zero nilpotent elements. It follows that $\mathcal{S}$ contains no non-zero quasinilpotent elements. For, suppose that the element $B \in \mathcal{S}$ is quasinilpotent and non-zero. By Theorem 2.1 and Proposition 2.3, $r(J_0B) \leq r(J_0)r(B)$, for every $J_0 \in \mathcal{J}_0$, so $r(J_0B) = 0$ and $J_0B = 0$. Then $BJ_0B = 0$, for every $J_0 \in \mathcal{J}_0$. If $x, y \in H$ satisfy $Bx \neq 0$, $B^*y \neq 0$, then the linear functional $f: \mathcal{K}(H) \longrightarrow \mathbb{C}$ defined by $f(K) = (KBx|B^*y)$ is continuous and non-zero and $f|_{\mathcal{J}_0} = 0$. By Lemma 1, $\mathcal{J}_0$ is reducible and this is a contradiction. Thus $\mathcal{S}$ contains no non-zero quasinilpotent elements, and to prove that r is submultiplicative on $\mathcal{S}$ it is therefore enough to show that, if $S, T \in \mathcal{S}$ satisfy $r(S) = r(T) = 1$, then $r(ST) \leq 1$.

We can assume that $r(ST) \neq 0$. Let P be the non-zero finite rank Riesz projection of ST corresponding to the non-empty set $\{z \in \sigma(ST) : |z| = r(ST)\}$. By Lemma 3, since $\mathcal{S}$ contains no non-zero nilpotent elements, $P \in \mathcal{S}$. Of course $P(ST) = (ST)P$.

Now $P\mathcal{S}P|_{\mathcal{R}(P)}$ is a semigroup on $\mathcal{R}(P)$ having $P\mathcal{J}_0P|_{\mathcal{R}(P)}$ as an ideal. Also, $P\mathcal{S}P|_{\mathcal{R}(P)}$ is irreducible. For, suppose it were reducible. Then, there exist non-zero functional vectors $u, v \in \mathcal{R}(P)$ such that $(PXPu|v) = 0$, for every $X \in \mathcal{S}$. The linear functional $g: \mathcal{K}(H) \longrightarrow \mathbb{C}$ defined by $g(K) = (KPu|P^*v)$ is continuous and non-zero and $g|_{\mathcal{S}} = 0$. By Lemma 1 this contradicts the irreducibility of $\mathcal{S}$. Thus $P\mathcal{S}P|_{\mathcal{R}(P)}$ is irreducible. A similar argument gives that $P\mathcal{J}_0P|_{\mathcal{R}(P)}$ is non-zero, since $\mathcal{J}_0$ is irreducible. Since $P\mathcal{J}_0P \subseteq \mathcal{J}$ and r is submultiplicative on $\mathcal{J}$, r is submultiplicative on $P\mathcal{J}_0P|_{\mathcal{R}(P)}$. Hence, by Theorem 2.4, r is submultiplicative on $P\mathcal{S}P|_{\mathcal{R}(P)}$ and so is submultiplicative on $P\mathcal{S}P$.

By Lemma 2, $P\mathcal{J}_0P|_{\mathcal{R}(P)}$ is irreducible, so by Burnside's Theorem there exist scalars α_i and operators $J_i \in \mathcal{J}_0$, with $r(PJ_iP) = 1, i = 1, 2, \ldots N^2$ (where $N = \dim \mathcal{R}(P)$) such that

$$P = \sum_{i=1}^{N^2} \alpha_i PJ_iP.$$

Then, for every $k \geq 1$, we have

$$P(ST)^kP = (ST)^kP = \sum_{i=1}^{N^2} \alpha_i (ST)^k PJ_iP$$

so

$$\left|\operatorname{tr}\left(P(ST)P\right)^k\right| = |\operatorname{tr} P(ST)^kP| \leq \sum_{i=1}^{N^2} |\alpha_i|\,|\operatorname{tr}(ST)^kPJ_iP| \leq N\sum_{i=1}^{N^2} |\alpha_i| r\left((ST)^kPJ_iP\right).$$

Now $r(XJ_0) \leq r(X)r(J_0)$, for every $X \in \mathcal{S}$ and $J_0 \in \mathcal{J}_0$ so

$$\begin{aligned} r\left((ST)^kPJ_iP\right) &= r\left(ST(ST)^{k-1}PJ_iP\right) \leq r(S)r\left(T(ST)^{k-1}PJ_iP\right) \\ &= r\left(T(ST)^{k-1}PJ_iP\right) \leq r(T)r\left((ST)^{k-1}PJ_iP\right) \\ &= r\left((ST)^{k-1}PJ_iP\right) \cdots \leq r(PJ_iP) = 1, \end{aligned}$$

and hence $r\left((ST)^kPJ_iP\right) \leq 1$, for every $k \geq 1$. Thus

$$\left|\operatorname{tr}\left(P(ST)P\right)^k\right| \leq N\sum_{i=1}^{N^2} |\alpha_i|,$$

for every $k \geq 1$ and so, by Proposition 2.2, $r\left(P(ST)P\right) = r(ST) \leq 1$. ∎

We can now prove the main result of this section. Recall that a chain $\mathcal{N}$ of subspaces of a Hilbert space is called *complete* if $\bigcap_\Gamma N_\gamma \in \mathcal{N}$ and $\bigvee_\Gamma N_\gamma \in \mathcal{N}$, for every family $\{N_\gamma\}_\Gamma$ of elements of $\mathcal{N}$. Also, if $\mathcal{N}$ is a complete chain and $N \in \mathcal{N}$, the element N_- of $\mathcal{N}$ is defined by $N_- = \bigvee\{M \in \mathcal{N} : M \subsetneq N\}$ (where, by convention $\bigvee \emptyset = (0)$ so that $(0)_- = (0)$). A complete chain $\mathcal{N}$ is called *continuous* if $N_{--} = N$, for every $N \in \mathcal{N}$.

THEOREM 2.6. *Let $\mathcal{S}$ be a semigroup of compact operators acting on a Hilbert space H. If r is submultiplicative on $\mathcal{S}$, then r is permutable on $\mathcal{S}$.*

PROOF. Let r be submultiplicative on $\mathcal{S}$. We can suppose that r is not identically zero on $\mathcal{S}$. The set of all invariant chains of $\mathcal{S}$ is non-empty (it contains $\{(0), H\}$) and has a maximal element, $\mathcal{N}$ say, by Zorn's Lemma. By maximality, $(0), H \in \mathcal{N}$ and $\mathcal{N}$ is complete. Since every compact operator leaving a continuous complete chain, containing (0) and H, invariant is quasinilpotent [7], $\mathcal{N}$ is not continuous. Thus the set of mutually orthogonal projections

$$\mathcal{P} = \{P_N \ominus P_{N_-} : N \in \mathcal{N}, P_N \neq P_{N_-}\}$$

is non-empty. By compactness, for every $S \in \mathcal{S}$ there is a member P of $\mathcal{P}$ such that $r(S) = r(PSP)$. Note that, for each $P \in \mathcal{P}$, $P\mathcal{S}P|_{\mathcal{R}(P)}$ is irreducible. (It can be zero only if P has rank 1.)

Consider the mapping $\varphi: \mathcal{S} \longrightarrow \mathcal{B}(H_0)$ defined by

$$\varphi(S) = \bigoplus_{P\in\mathcal{P}} S_P,$$

where $H_0 = \oplus_{P\in\mathcal{P}} \mathcal{R}(P)$ and where $S_P = PSP|_{\mathcal{R}(P)}$. Since for every $N \in \mathcal{N}$ and $S \in \mathcal{S}$ we have

$$(P_N - P_{N_-})S(P_N - P_{N_-}) = (P_N - P_{N_-})SP_N,$$

it is easily verified that φ is multiplicative. By the construction of $\mathcal{P}$, φ preserves spectral radii, so we can assume with no loss of generality that $\mathcal{S} = \varphi(\mathcal{S})$.

It is convenient to reduce the size of $\mathcal{P}$. Let us call a subset $\mathcal{A}$ of $\mathcal{P}$ *admissible* if it is non-empty and

$$r\Big(\bigoplus_{P\in\mathcal{P}} S_P\Big) = r\Big(\bigoplus_{P\in\mathcal{A}} S_P\Big)$$

for all $S \in \mathcal{S}$. Choose a maximal chain $\mathcal{C}$ of admissible subsets (ordered by set inclusion) and let $\mathcal{P}_0$ be the intersection of all members of $\mathcal{C}$. To see that $\mathcal{P}_0$ is admissible let S be any member of $\mathcal{S}$ with $r(S) \neq 0$. By compactness, the set $\{P \in \mathcal{P} : r(S_P) = r(S)\}$ is finite. Since every member of $\mathcal{C}$ has a non-empty intersection with this set, so does $\mathcal{P}_0$. Thus $\mathcal{P}_0$ is a minimal admissible subset of $\mathcal{P}$. The admissibility of $\mathcal{P}_0$ allows us now to assume that $\mathcal{P} = \mathcal{P}_0$ with no loss of generality. We also assume $\mathcal{S} = \bar{\mathcal{S}} = \mathbb{C}\mathcal{S}$ as usual.

The new minimality property of $\mathcal{P}$ implies that for every $P \in \mathcal{P}$ there exists an element $S \in \mathcal{S}$ such that $r(S_P) = r(S) = 1$ and

$$\sup_{\substack{Q\neq P\\ Q\in\mathcal{P}}} r(S_Q) < 1.$$

By Lemma 3, there is a sequence $\{n_j\}$ of positive integers and a sequence of complex numbers $\{c_j\}$ with $\{|c_j|\}$ converging such that

$$K = \lim c_j S_P^{n_j}$$

is non-zero and finite rank. Then it is easily verified that $F = \lim c_j S^{n_j} \in \mathcal{S}$, $F_P = K$, and $F_Q = 0$ for all $Q \in \mathcal{P}$ other than P.

Let $P \in \mathcal{P}$, let F and F_P be the operators just described and let $\mathcal{J}$ be the ideal of $\mathcal{S}$ generated by F. Then $\mathcal{J}_P = \{J_P : J \in \mathcal{J}\}$ is a non-zero ideal of the corresponding irreducible block $\mathcal{S}_P$ of $\mathcal{S}$. The submultiplicativity of r on $\mathcal{J}$ and thus on $\mathcal{J}_P$ implies that it is submultiplicative on $\mathcal{S}_P$ by Theorems 2.4 and 2.5 and multiplicative on $\mathcal{S}_P$ by Theorem 2.1.

To complete the proof, let A, B, C be any members of $\mathcal{S}$. Then

$$\begin{aligned} r(ABC) &= \max_{P\in\mathcal{P}} r[(ABC)_P] = \max_{P\in\mathcal{P}} r[A_P B_P C_p] \\ &= \max_{P\in\mathcal{P}}[r(A_P)r(B_P)r(C_P)] \\ &= \max_{P\in\mathcal{P}} r[B_P A_P C_P] = \max_{P\in\mathcal{P}} r[(BAC)_P] \\ &= r(BAC). \end{aligned}$$

■

3. **Semigroups satisfying** $r(A) = \|A\|$. If $\mathcal{S}$ is a semigroup of operators on a Hilbert space and $r(A) = \|A\|$, for every $A \in \mathcal{S}$, then r is, of course, submultiplicative on $\mathcal{S}$. The class of such semigroups contains those semigroups which consist of normal operators. It also contains those $*$-semigroups on which r is submultiplicative. Indeed, let $\mathcal{S}$ be a $*$-semigroup, that is, $\mathcal{S} = \mathcal{S}^*$, and suppose that r is submultiplicative on $\mathcal{S}$. For every $A \in \mathcal{S}$ we have

$$\|A\|^2 = r(A^*A) \le r(A^*)r(A) = r(A)^2 \le \|A\|^2,$$

so $r(A) = \|A\|$. Although it is not known whether or not r, equivalently $\|\cdot\|$, is permutable on every member of this class of semigroups, we show below that it is permutable on all normal semigroups (Theorem 3.2) and on certain $*$-semigroups in this class, namely, those generated by (orthogonal) projections (Theorem 3.4).

We begin by considering irreducible semigroups of normal operators.

THEOREM 3.1. *If $\mathcal{S}$ is an irreducible semigroup of normal operators, then every element of $\mathcal{S}$ is a scalar multiple of a unitary operator. Consequently $\|\cdot\|$ is multiplicative, so permutable, on $\mathcal{S}$.*

PROOF. We use the fact that, if a normal operator A with $\|A\| \le 1$ has no unitary direct summand, then $A^n \longrightarrow 0$ in the strong operator topology.

Let $\mathcal{S}$ be a semigroup of normal operators on a Hilbert space H with a member A that is not a scalar multiple of a unitary operator. We show that, for some non-trivial subspace M of H, both M and $M^\perp$ are invariant under $\mathcal{S}$, contrary to the irreducibility hypothesis.

We can assume that $\mathcal{S} = \mathbb{C}\mathcal{S}$ and that $\min\{|\lambda| : \lambda \in \sigma(A)\} < 1 < r(A)$. Let E be the spectral measure of A and put $H_1 = E(\{z \in \mathbb{C} : |z| \ge 1\})$ and $H_2 = E(\{z \in \mathbb{C} : |z| < 1\})$. Then $H = H_1 \oplus H_2$ and $H_2 \ne (0), H$. With respect to this decomposition $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ with A_2 a normal operator on H_2 satisfying $\|A_2\| \le 1$. Also, A_2 has no unitary direct summand. Let $B \in \mathcal{S}$. Then $B = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}$ and, for every $n \ge 1$, since A^nB is normal, we have (comparing entries in the (2,2) position)

$$A_2^n(ZZ^* + TT^*)A_2^{*n} - T^*A_2^{*n}A_2^nT = Y^*A_1^{*n}A_1^nY.$$

Now $A_2^n \longrightarrow 0$ and $A_2^{*n} \longrightarrow 0$ strongly, so $Y^*A_1^{*n}A_1^nY \longrightarrow 0$ strongly. Thus, for every $x \in H_2$, $\|A_1^nYx\| \longrightarrow 0$. Now A_1^{-1} is a contraction, so

$$\|Yx\| \le \|A_1^{-1}\|^n\|A_1^nYx\| \le \|A_1^nYx\|$$

and hence $Yx = 0$. Thus $Y = 0$ and H_2 is invariant under $\mathcal{S}$. A similar argument, considering BA^n, shows that $Z = 0$. Thus H_1 is also invariant under $\mathcal{S}$. ■

THEOREM 3.1. *Operator norm is permutable on every semigroup of normal operators.*

PROOF. Let $\mathcal{S}$ be a semigroup of normal operators on a Hilbert space H. We may suppose that $\mathcal{S} = \mathbb{C}\mathcal{S}$. The proof of the preceding theorem shows that, if $S \in \mathcal{S}$ and P is the spectral projection of S corresponding to the set $\{z \in \mathbb{C} : |z| < t\}$, for any $t > 0$, then both $\mathcal{R}(P)$ and $\mathcal{R}(1-P)$ are invariant under $\mathcal{S}$. (In the preceding proof, to make P non-trivial, t was chosen such that both $\{z \in \mathbb{C} : |z| < t\}$ and its complement gave non-zero spectral projections.)

To prove that $\|ABC\| = \|BAC\|$, for every $A, B, C \in \mathcal{S}$, it suffices to show that $\|ABC\| \leq \|BAC\|$, for every $A, B, C \in \mathcal{S}$. Assume, with no loss of generality, that $\|A\| < 1$, $\|B\| < 1$ and $\|C\| < 1$. Let $\varepsilon > 0$ be arbitrary. Choose $n > \frac{1}{\varepsilon}$ and let

$$\Omega_j = \left\{z \in \mathbb{C} : \frac{j-1}{n} \leq |z| < \frac{j}{n}\right\}, \quad j = 1, 2, \ldots, n.$$

By the remark in the paragraph above, the spectral projections of A corresponding to each Ω_j give rise to an (orthogonal) direct-sum decomposition H, relative to which every element of $\mathcal{S}$ directly decomposes. A similar comment holds for the corresponding spectral projections of B and of C. Since this set of $3n$ projections is pairwise commutative, H decomposes as $H = \bigoplus_{i=1}^{n^3} H_i$, where some H_i may be zero, such that each H_i is invariant under $\mathcal{S}$ and where, for each i, the spectra of the restrictions of A, B, C, to H_i are contained in "thin annuli". More precisely, there exist real numbers a_i, b_i, c_i, $i = 1, 2, \ldots, n^3$, belonging to $[0, 1]$ such that

$$A = \bigoplus_{i=1}^{n^3} A_i, \quad B = \bigoplus_{i=1}^{n^3} B_i, \quad \text{and} \quad C = \bigoplus_{i=1}^{n^3} C_i,$$

$$\sigma(A_i) \subseteq \left\{z \in \mathbb{C} : a_i \leq |z| < a_i + \frac{1}{n}\right\},$$

$$\sigma(B_i) \subseteq \left\{z \in \mathbb{C} : b_i \leq |z| < b_i + \frac{1}{n}\right\},$$

$$\sigma(C_i) \subseteq \left\{z \in \mathbb{C} : c_i \leq |z| < c_i + \frac{1}{n}\right\},$$

for $i = 1, 2, \ldots, n^3$. By normality, this implies that, for each i,

$$a_i b_i c_i \leq \|A_i B_i C_i\| \leq \left(a_i + \frac{1}{n}\right)\left(b_i + \frac{1}{n}\right)\left(c_i + \frac{1}{n}\right).$$

Note that these inequalities hold for any permutation of A_i, B_i, and C_i; in particular they hold for $B_i A_i C_i$. Thus

$$\begin{aligned}
\|ABC\| = \max_i \|A_i B_i C_i\| &\leq \max_i \left(a_i + \frac{1}{n}\right)\left(b_i + \frac{1}{n}\right)\left(c_i + \frac{1}{n}\right) \\
&\leq \max_i a_i b_i c_i + \left(\frac{1}{n^3} + \frac{3}{n^2} + \frac{3}{n}\right) \\
&\leq \max_i \|B_i A_i C_i\| + \frac{7}{n} \\
&\leq \|BAC\| + 7\varepsilon.
\end{aligned}$$

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Since $\varepsilon > 0$ was arbitrary, $\|ABC\| \leq \|BAC\|$. ■

Next, we consider semigroups generated by positive contractions, and more particularly, those generated by (orthogonal) projections. In the proof that follows we use the simple fact that if (y_m) and (z_m) are sequences of vectors in a Hilbert space satisfying $\|y_m\| \leq 1$, $\|z_m\| \leq 1$, for every $m \geq 1$, and $(y_m|z_m) \longrightarrow 1$, then $\|y_m - z_m\| \longrightarrow 0$ and $\|y_m\| \longrightarrow 1$, $\|z_m\| \longrightarrow 1$.

THEOREM 3.3. *Let $C_1, C_2, \ldots, C_n$ be positive contractions on a Hilbert space H. The following are equivalent.*

(i) $r(C_1C_2\cdots C_n) = 1$,

(ii) there exists a sequence (x_m) of unit vectors such that $(1 - C_j)x_m \longrightarrow 0$, for every $j = 1, 2, \ldots, n$,

(iii) $r(C_{\pi(1)}C_{\pi(2)}\cdots C_{\pi(n)}) = 1$, *for every permutation π of $1, 2, \ldots, n$.*

PROOF. (i) $\Rightarrow$ (ii): Suppose that (i) holds. Choose $\alpha \in \sigma(C_1C_2\cdots C_n)$ with $|\alpha| = 1$. Since α is in the boundary of the spectrum, it belongs to $\sigma_{\mathrm{ap}}(C_1C_2\cdots C_n)$. Hence there exists a sequence (x_m) of unit vectors such that $C_1C_2\cdots C_nx_m - \alpha x_m \longrightarrow 0$. We use induction on n to show that $\alpha = 1$ and that (x_m) satisfies the requirement in (ii). We have

$$(C_2\cdots C_nx_m|\alpha C_1x_m) = \bar{\alpha}(C_1C_2\cdots C_nx_m|x_m) \longrightarrow \bar{\alpha}\alpha = 1.$$

It follows that $\|C_1x_m\| \longrightarrow 1$ and $C_2\cdots C_nx_m - \alpha C_1x_m \longrightarrow 0$. This implies, firstly, that $\langle C_1^2x_m, x_m\rangle \longrightarrow 1$ and so $C_1^2x_m - x_m \longrightarrow 0$, giving

$$(1 - C_1)x_m = (1 + C_1)^{-1}(1 - C_1^2)x_m \longrightarrow 0;$$

secondly, that

$$C_2\cdots C_nx_m - \alpha x_m = C_2\cdots C_nx_m - \alpha C_1x_m - \alpha(1 - C_1)x_m \longrightarrow 0,$$

and we are done by induction.

(ii) $\Rightarrow$ (iii): Suppose that (ii) holds. Let $D_j = C_{\pi(j)}, j = 1, \ldots, n$. Then $D_jx_m - x_m \longrightarrow 0$ for all j. Using induction again, assume $D_2\cdots D_nx_m - x_m \longrightarrow 0$. Then

$$D_1D_2\cdots D_nx_m - x_m = D_1(D_2\cdots D_nx_m - x_m) + D_1x_m - x_m \longrightarrow 0.$$

Thus $1 \in \sigma(D_1\cdots D_n)$ and, since $\|D_1\cdots D_n\| \leq 1$, (iii) follows.

That (iii) implies (ii) is obvious. ■

REMARKS. 1. The preceding theorem shows that if $\mathcal{C}$ is a finite family of positive contractions (more particularly, projections) and some product of all the elements of $\mathcal{C}$ (in any order) has spectral radius equal to one, then $r \equiv 1$ on the semigroup generated by $\mathcal{C}$.

2. If $C_1, C_2, \ldots, C_n$ are positive contractions and $F_1, F_2, \ldots, F_n$ are projections satisfying $\mathcal{R}(C_j) \subseteq \mathcal{R}(F_j)$, $j = 1, 2, \ldots, n$, then $r(C_1C_2\cdots C_n) = 1$ implies that $r(F_1F_2\cdots F_n) = 1$. For, with (x_m) as in the statement of the preceding theorem, $(1 - C_j)x_m \longrightarrow 0$ gives $(1 - F_j)(1 - C_j)x_m = (1 - F_j)x_m \longrightarrow 0$, for every $j = 1, 2, \ldots, n$.

3. Notice that, if C_1, C_2 and C_3 are self-adjoint operators, and π is a permutation of 1, 2, 3, then $C_1C_2C_3$ and $C_{\pi(1)}C_{\pi(2)}C_{\pi(3)}$ have the same spectral radius. This follows from the facts that $r(AB) = r(BA)$ and $r(A) = r(A^*)$, for all operators A and B.

4. If $C_1, C_2, \ldots, C_n$ are positive contractions, it is clear that $r(C_1C_2\cdots C_n) = 1$ if $\bigcap_{j=1}^n \ker(1 - C_j) \neq (0)$. The converse, though not true in general (Corollary 3.3.2), is true on finite-dimensional spaces.

COROLLARY 3.3.1. *Let $C_1, C_2, \ldots, C_n$ be positive contractions on a finite-dimensional Hilbert space H. Then $r(C_1C_2\cdots C_n) = 1$ if and only if $\bigcap_{j=1}^n \ker(1 - C_j) \neq (0)$.*

PROOF. Let $r(C_1C_2\cdots C_n) = 1$. By the preceding theorem, there exists a sequence (x_m) of unit vectors such that $(1 - C_j)x_m \longrightarrow 0$, for every $j = 1, 2, \ldots, n$. Since the unit ball of H is compact, we may suppose that $x_m \longrightarrow x$, for some $x \in H$. Then for every j, $(1 - C_j)x_m \longrightarrow (1 - C_j)x$, so $C_jx = x$. Since $\|x\| = 1$, $\bigcap_{j=1}^n \ker(1 - C_j) \neq (0)$.

The reverse implication is clear. ∎

Let $E_1, E_2, \ldots, E_n$ be projections on a finite-dimensional space. The preceding corollary shows that $r(E_1E_2\cdots E_n) = 1$ if and only if $\bigcap_{j=1}^n \mathcal{R}(E_j) \neq (0)$. The following corollary shows that this is false in infinite dimensions. In it we use the fact that if M and N are non-zero subspaces of a Hilbert space satisfying $M \cap N = (0)$, then $M + N$ is not closed if and only if $\sup\{|(x|y)| : \|x\| = \|y\| = 1, x \in M, y \in N\} = 1$.

COROLLARY 3.3.2. *Let P and Q be projections on a Hilbert space with ranges M and N respectively. Then $r(PQ) = 1$ if and only if either (a) $M\cap N \neq (0)$ or (b) $M\cap N = (0)$ and $M + N$ is not closed.*

PROOF. Let $r(PQ) = 1$ and suppose that $M \cap N = (0)$. By Theorem 3.3 there exists a sequence (x_m) of unit vectors such that $(1 - P)x_M \longrightarrow 0$ and $(1 - Q)x_m \longrightarrow 0$. Then $(Px_m|Qx_m) \longrightarrow 1$ and so $\sup\{|(x|y)| : \|x\| = \|y\| = 1, x \in M, y \in N\} = 1$. Thus $M + N$ is not closed.

For the converse, by an earlier remark, it is enough to consider the case where $M\cap N = (0)$ and $M+N$ is not closed. Then there exist sequences (y_m), (z_m) of unit vectors belonging to M and N respectively such that $(y_m|z_m) \longrightarrow 1$. Then $y_m - z_m \longrightarrow 0$ so $Py_m - Qz_m = y_m - Pz_m \longrightarrow 0$. Thus

$$PQz_m - z_m = Pz_m - z_m = (Pz_m - y_m) + (y_m - z_m) \longrightarrow 0$$

and so $1 \in \sigma(PQ)$. Thus $r(PQ) \geq 1$ and since $r(PQ) \leq \|PQ\| \leq 1$, $r(PQ) = 1$. ∎

Next, we show that the submultiplicativity of spectral radius on a semigroup generated by projections is equivalent to its permutability. We have not as yet ascertained whether or not this equivalence holds when the generating elements are assumed to be only positive contractions.

THEOREM 3.4. *Let $\mathcal{P}$ be a (possibly infinite) family of projections on a Hilbert space H, and let $\mathcal{S}$ be the semigroup generated by $\mathcal{P}$. The following are equivalent.*

(i) r is submultiplicative on $\mathcal{S}$,

(ii) $r(A) = 0$ *or* 1, *for every* $A \in \mathcal{S}$,

(iii) $r(A) = 1$, *for every* $A \in \mathcal{S} \setminus \{0\}$,

(iv) r *is permutable on* $\mathcal{S}$.

PROOF. (i) $\Rightarrow$ (ii): Suppose that (i) holds. Let $A \in \mathcal{S}$. Then $A = E_1E_2 \cdots E_n$, where $n \geq 1$ and $\{E_i : 1 \leq i \leq n\}$ are (not necessarily distinct) elements of $\mathcal{P}$. If $n = 1$, clearly $r(A) = 0$ or 1. Suppose that $n > 1$. Notice that, with W the operator defined by

$$W = (E_1E_2 \cdots E_n)(E_nE_1E_2 \cdots E_{n-1})(E_{n-1}E_nE_1E_2 \cdots E_{n-2}) \cdots (E_2E_3E_4 \cdots E_nE_1),$$

we have $W = (E_1E_2 \cdots E_n)^{n-1}E_1$, so

$$r(W) = r\big(E_1(E_1E_2 \cdots E_n)^{n-1}\big) = r\big((E_1E_2 \cdots E_n)^{n-1}\big) = r(E_1E_2 \cdots E_n)^{n-1}.$$

On the other hand

$$\begin{aligned} &r(W) \\ &\leq r(E_1E_2 \cdots E_n)r(E_nE_1E_2 \cdots E_{n-1})r(E_{n-1}E_nE_1E_2 \cdots E_{n-2}) \cdots r(E_2E_3E_4 \cdots E_nE_1) \\ &= r(E_1E_2 \cdots E_n)^n. \end{aligned}$$

Hence $r(A)^{n-1} \leq r(A)^n$. Thus, if $r(A) \neq 0$, then $r(A) \geq 1$. But $r(A) \leq 1$, so $r(A) = 0$ or 1, and (ii) holds.

(ii) $\Rightarrow$ (iv): Suppose that (ii) holds. Let $A, B, C \in \mathcal{S}$. If $r(ABC) = 1$, then $r(BAC) = 1$, by Theorem 3.3. On the other hand, if $r(ABC) = 0$ then $r(BAC) = 0$ (since $r(BAC) \neq 1$, by Theorem 3.3). Thus $r(ABC) = r(BAC)$ and (iv) holds.

(iv) $\Rightarrow$ (i). This follows from Theorem 9 of [4].

Clearly (iii) $\Rightarrow$ (ii). We complete the proof by showing that (ii) $\Rightarrow$ (iii). Suppose that (ii) holds. Since (ii) $\Rightarrow$ (i) and $\mathcal{S}$ is a $*$-semigroup, $r(A) = \|A\|$, for every $A \in \mathcal{S}$. Thus (iii) holds. ∎

The following corollaries concern semigroups generated by two or three projections. In them, necessary and sufficient conditions are given on the generating set in order that r be submultiplicative on the generated semigroup. The first of these, in conjunction with Corollary 3.3.2, easily leads to a more geometric characterization.

COROLLARY 3.4.1. *Spectral radius is submultiplicative on the semigroup* $\mathcal{S}$ *generated by two projections* E *and* F *if and only if* $EF = 0$ *or* $r(EF) = 1$.

PROOF. The necessity of the condition follows immediately from Theorem 3.4. On the other hand, if $EF = 0$, then $\mathcal{S} = \{0, E, F\}$ and r is clearly submultiplicative on $\mathcal{S}$. Finally, if $r(EF) = 1$, then $r \equiv 1$ on $\mathcal{S}$ by our first remark following Theorem 3.3. ∎

COROLLARY 3.4.2. *Spectral radius is submultiplicative on the semigroup* $\mathcal{S}$ *generated by three projections* E_1, E_2 *and* E_3 *if and only if*

(i) $EF = 0$ *or* $r(EF) = 1$, *for every choice of* $E, F \in \{E_1, E_2, E_3\}$, *and*

(ii) $r(E_1E_2E_3) = 1$ *or* $E_{\pi(1)}E_{\pi(2)}E_{\pi(3)} = 0$, *for every permutation* π *of* 1, 2, 3.

PROOF. The necessity of the conditions follows immediately from Theorem 3.4 and the fact that $E_{\pi(1)}E_{\pi(2)}E_{\pi(3)}$ has the same spectral radius as $E_1E_2E_3$ for every permutation π of 1, 2, 3 (see our third remark following Theorem 3.3).

Conversely, assume that (i) and (ii) hold. By our first remark following Theorem 3.3, if $r(E_1E_2E_3) = 1$, then $r \equiv 1$ on $\mathcal{S}$. Otherwise, by (ii), the product of E_1, E_2 and E_3 in any order is zero. In this case, let $W \in S$. Since (i) holds, it follows from Corollary 3.4.1 and Theorem 3.4 that if W is a word in only one or two of $\{E_1, E_2, E_3\}$ then $W = 0$ or $r(W) = 1$. Suppose that W contains each of E_1 E_2, E_3 as a factor. Then W contains a factor of the form $E_{\pi(1)}E_{\pi(2)}E_{\pi(3)}$, for some permutation π of 1, 2, 3. (Otherwise W would have one of the forms $(PQ)^n$, $(PQ)^nP$ with P and Q distinct elements of $\{E_1, E_2, E_3\}$). Thus $W = 0$. By Theorem 3.4, r is submultiplicative on $\mathcal{S}$. ■

Next, we consider briefly the reducibility of semigroups, on which r is submultiplicative, which are generated by positive contractions (more particularly, by projections).

PROPOSITION 3.5. *Let H be a finite-dimensional Hilbert space of dimension $n \geq 2$. Let $\mathcal{S}$ be the semigroup generated by a family $\mathcal{C}$ of positive contractions on H. If r is submultiplicative on $\mathcal{S}$, then $\mathcal{S}$ is reducible.*

PROOF. Let r be submultiplicative on $\mathcal{S}$. We can suppose that $\|C\| = 1$, for every $C \in \mathcal{C}$. Assume that $\mathcal{S}$ is irreducible. By Theorem 2.1, r is multiplicative on $\mathcal{S}$, so $r \equiv 1$ on $\mathcal{S}$. By Burnside's Theorem, $\mathcal{S}$ contains a basis $S_1, S_2, \ldots, S_{n^2}$ of $\mathcal{B}(H)$. For every i, S_i is a product of elements of $\mathcal{C}$, and since $r(S_1S_2\cdots S_{n^2}) = 1$, by Corollary 3.3.1 it follows that $S_1, S_2, \ldots, S_{n^2}$ have a common non-zero fixed point. Let $x \in H$ be a non-zero vector satisfying $S_ix = x$, for $i = 1, 2, \ldots, n^2$. Since every element of $\mathcal{B}(H)$ is a linear combination of $S_1, S_2, \ldots, S_{n^2}$, x is an eigenvector of every element of $\mathcal{B}(H)$. This is a contradiction. ■

REMARK. Let $\mathcal{S}$ be as in the statement of the preceding proposition, and suppose that r is submultiplicative on $\mathcal{S}$. Since $\mathcal{S}$ is a $*$-semigroup, $M^\perp$ is invariant under $\mathcal{S}$ whenever the subspace M is.

The preceding proposition shows that, if $2 \leq \dim H < \infty$, then $\mathcal{S}$ has at least one pair of non-trivial orthogonally complemented invariant subspaces. In fact, there may be only one such pair. For example, the semigroup of operators on $H_1 \oplus H_2$ generated by the set of positive contractions $\{1 \oplus C : C \in \mathcal{B}(H_2)$ a positive contraction$\}$, where $\dim H_1 = 1$ and $\dim H_2 \geq 1$, has only one orthogonally complemented pair of non-trivial invariant subspaces.

On the other hand, if H is infinite-dimensional, $\mathcal{S}$ can be irreducible. For example, consider the semigroup generated by the family of projections $\mathcal{P} = \{1 - Q : Q \in \mathcal{Q}\}$ where $\mathcal{Q}$ is a family of finite-rank projections on H with no common non-trivial invariant subspace. Since H is infinite-dimensional, the intersection of the ranges of the elements of every finite subset of $\mathcal{P}$ is non-zero, so $r \equiv 1$ on $\mathcal{S}$. However, $\mathcal{S}$ is irreducible. Here the generating set of projections has infinite cardinality. In the following example the generating set has cardinality 3. This cardinality is minimal as every pair of projections on an infinite-dimensional Hilbert space has a non-trivial common invariant subspace [2].

EXAMPLE. Let $H = l^2(\mathbb{Z}^+)$ and let $A = \operatorname{diag}(a_1, a_2, a_3, \ldots)$ where the sequence $e = (a_i) \in l^2$ and $a_i \neq a_j$ whenever $i \neq j$, and $a_i > 0$, for every $i \geq 1$. On $H \oplus H$ put $K = H \oplus (0)$, $L = G(A)$, $M = G(A) + \big((0) \oplus \langle e \rangle\big)$, where $G(A)$ denotes the graph of A. Let P, Q and R denote the projections with ranges K, L and M, respectively.

Now $L \subseteq M$ so $QR = RQ = Q$. Thus $PQR = PQ$ so $r(PQR) = r(PQ)$. But $r(PQ) = 1$ by Corollary 3.3.2, so by our first remark following Theorem 3.3, $r \equiv 1$ on the semigroup $\mathcal{S}$ generated by P, Q and R. Now P, Q and R have no common non-trivial invariant subspace. For, every projection commuting with each of P, Q, R has the form $E \oplus F$ where E and F are projections on H satisfying $FA = AE$ and $Fe = 0$ or e. Now $EA = AF$ gives $FA^2 = AEA = A^2F$, so F is diagonal. If $Fe = 0$, then $F = 0$ and $E = 0$. If $Fe = e$, then $F = 1$ and $E = 1$. Thus $\mathcal{S}$ is irreducible.

PROPOSITION 3.6. *Let $\mathcal{S}$ be a semigroup generated by projections on which r is submultiplicative. If $\mathcal{S} \setminus \{0\}$ is not a semigroup then $\mathcal{S}$ is reducible.*

PROOF. Suppose that $\mathcal{S} \setminus \{0\}$ is not a semigroup. Then, there exist elements $S, T \in \mathcal{S} \setminus \{0\}$ such that $ST = 0$. Let $\mathcal{J}$ be the ideal $\mathcal{S}$ generated by S. Let $J \in \mathcal{J}$. Since J contains S as a factor and, by Theorem 3.4, r is permutable on $\mathcal{S}$, $r(JT) = 0$. By Theorem 3.4, $JT = 0$. Thus $\mathcal{R}(T)$, and so $\overline{\mathcal{R}(T)}$, is invariant under $\mathcal{J}$. Since $\overline{\mathcal{R}(T)}$ is non-trivial, $\mathcal{J}$ is reducible. By Lemma 2, $\mathcal{S}$ is also reducible. ∎

PROPOSITION 3.7. *Let $\mathcal{S}$ be a semigroup of compact operators on a Hilbert space H such that $r(K) = \|K\|$ for every $K \in \mathcal{S}$. Then there exists a non-zero, finite-dimensional subspace M of H, with both M and $M^\perp$ invariant under $\mathcal{S}$, such that $\mathcal{S}|_M$ consists of multiples of unitary operators.*

PROOF. We can assume that $\mathcal{S} = \mathbb{C}\mathcal{S} = \bar{\mathcal{S}} \neq \{0\}$. By Lemma 3, $\mathcal{S}$ contains non-zero idempotent operators of finite rank. Note that all such idempotents are orthogonal projections since $r(K) = \|K\|$, for every $K \in \mathcal{S}$. Let P be a projection in $\mathcal{S}$ of minimal positive rank. Let S be any element of $\mathcal{S}$. Relative to the decomposition $H = \mathcal{R}(P) \oplus \mathcal{R}(1-P)$ we have

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}.$$

Observe that $X = PSP|_{\mathcal{R}(P)}$ is a multiple of a unitary operator; otherwise, by Lemma 3 (applied to $PSP/\|PSP\|$), we would obtain a projection in $\mathcal{S}$ with positive rank strictly less than the rank of P.

We show that $Z = 0$, as follows. By hypothesis, $r(SP) = \|SP\|$ and $r(PSP) = \|PSP\|$. But SP and PSP have the same spectral radius. This implies that

$$\left\| \begin{pmatrix} X & 0 \\ Z & 0 \end{pmatrix}^* \begin{pmatrix} X & 0 \\ Z & 0 \end{pmatrix} \right\| = \|X^*X\|$$

or $\|X^*X+Z^*Z\| = \|X^*X\|$. Since X is a multiple of a unitary operator, $X^*X = p$, for some $p \geq 0$. Then $\|p + Z^*Z\| = p$ implies that $Z^*Z = 0$, so $Z = 0$.

Similarly, considering PS instead of SP, we find that $Y = 0$. ∎

Above, we have exhibited three classes of operator semigroups (those consisting of normal, respectively compact, operators together with those generated by projections) each having the property that the submultiplicativity of r on any member implies its permutability. Other classes with this property can be obtained by taking tensor products. In the following we use the facts that $r(A\otimes B) = r(A)r(B)$ and $(A\otimes B)(C\otimes D) = AC\otimes BD$, whenever $A, C \in \mathcal{B}(H_1)$ and $B, D \in \mathcal{B}(H_2)$ for Hilbert spaces H_1 and H_2.

PROPOSITION 3.8. *Let $\mathcal{C}$ and $\mathcal{D}$ be classes of semigroups of operators, on Hilbert spaces H and K, respectively, each with the property that the submultiplicativity of r on any one of its members implies its permutability. Then the class of semigroups of the form $\mathcal{S} \otimes \mathcal{T}$, with $\mathcal{S} \in \mathcal{C}$, $\mathcal{T} \in \mathcal{D}$ also has this property.*

PROOF. Let $\mathcal{S} \in \mathcal{C}$, $\mathcal{T} \in \mathcal{D}$ and suppose that r is submultiplicative on $\mathcal{S} \otimes \mathcal{T}$. We can suppose that r is not identically zero on $\mathcal{S} \otimes \mathcal{T}$, so $r(A) \neq 0$ and $r(B) \neq 0$ for some elements $A \in \mathcal{S}$, $B \in \mathcal{T}$. Let $X, Y \in \mathcal{S}$. Then $r\big((X \otimes B)(Y \otimes B)\big) \leq r(X \otimes B)r(Y \otimes B)$ gives $r(XY) \leq r(X)r(Y)$. Hence r is permutable on $\mathcal{S}$. Similarly, r is permutable on $\mathcal{T}$. It follows, almost immediately, that r is permutable on $\mathcal{S} \otimes \mathcal{T}$. ∎

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Department of Mathematics
University of Western Australia
Nedlands, WA 6009
Australia

Department of Mathematics, Statistics and Computing Science
Dalhousie University
Halifax, Nova Scotia
B3H 3J5