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SPLITTING INVARIANT SUBSPACES IN THE HARDY SPACE OVER THE BIDISK

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Abstract

Let H^2 be the Hardy space over the bidisk. It is known that Hilbert–Schmidt invariant subspaces of H^2 have nice properties. An invariant subspace which is unitarily equivalent to some invariant subspace whose continuous spectrum does not coincide with $\overline{\mathbb{D}}$ is Hilbert–Schmidt. We shall introduce the concept of splittingness for invariant subspaces and prove that they are Hilbert–Schmidt.

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1. Introduction

Let $H^2 = H^2(\mathbb{D}^2)$ be the Hardy space over the bidisk $\mathbb{D}^2$ with variables z and w. Then $H^2 = H^2(z) \otimes H^2(w)$, where $H^2(z)$ is the z-variable Hardy space. A nonzero closed subspace M of H^2 is said to be invariant if $zM \subset M$ and $wM \subset M$. For an invariant subspace L of $H^2(z)$, by the Beurling theorem, $L = \varphi(z)H^2(z)$ for some inner function $\varphi(z)$. The structure of invariant subspaces of $H^2 = H^2(\mathbb{D}^2)$ is extremely complicated (see [3, 14]). For a function ϕ in $H^\infty(\mathbb{D}^2)$, we denote by T_ϕ the multiplication operator on H^2 by ϕ. For an invariant subspace M of H^2, we write $R_z^M = T_z|_M$ and $R_w^M = T_w|_M$. We will simply write R_z, R_w when no confusion occurs. Then (R_z, R_w) is a pair of commuting isometries on M. In the study of invariant subspaces of H^2, the operators R_z, R_w play important roles in the study of operator theory and function theory. Since

$$M = \bigoplus_{n=0}^{\infty} w^n (M \ominus wM),$$

the space $M \ominus wM$ contains much information about the properties of M.

$$[R_w^*, R_w] := R_w^* R_w - R_w R_w^* = I_M - P_{wM} = P_{M\ominus wM},$$

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where P_E is the orthogonal projection from H^2 onto the closed subspace E of H^2, $[R_w^*, R_z] = 0$ on wM and $[R_w^*, R_z] = R_w^* R_z$ on $M \ominus wM$. So $[R_z^*, R_z][R_w^*, R_w]$ and $[R_w^*, R_z]$ are key operators in the study of invariant subspaces of H^2 (see [4, 5, 7–9, 11, 12, 15, 16, 18–23]).

In [19], Yang defined two numerical invariants for M,

$$\Sigma_0(M) = \big\|[R_z^*, R_z][R_w^*, R_w]\big\|_{\mathrm{HS}}^2, \quad \Sigma_1(M) = \big\|[R_w^*, R_z]\big\|_{\mathrm{HS}}^2,$$

where $\|\cdot\|_{\mathrm{HS}}$ is the Hilbert–Schmidt norm, and showed that

$$\big\|[R_z^*, R_z][R_w^*, R_w]\big\|_{\mathrm{HS}}^2 = \big\|[R_w^*, R_w][R_z^*, R_z]\big\|_{\mathrm{HS}}^2$$

and

$$\big\|[R_w^*, R_z]\big\|_{\mathrm{HS}}^2 = \big\|[R_z^*, R_w]\big\|_{\mathrm{HS}}^2.$$

In [19, Proposition 3.3], he showed also that if M is unitarily equivalent to M_1, then $\Sigma_0(M) = \Sigma_0(M_1)$ and $\Sigma_1(M) = \Sigma_1(M_1)$. In [22], Yang introduced the concept of Hilbert–Schmidtness for M. It is equivalent to the fact that $P_M - R_z R_z^* - R_w R_w^* - R_z T_w R_z^* R_w^*$ is Hilbert–Schmidt (see [5, Proposition 1.1]). By [5, Corollary 3.3], M is Hilbert–Schmidt if and only if $\Sigma_0(M) + \Sigma_1(M) < \infty$. For a given M, it is generally difficult to compute the exact values of $\Sigma_0(M)$ and $\Sigma_1(M)$.

Hilbert–Schmidt invariant subspaces have many nice properties (see [5, 15, 16, 19–23]). Let F_z^M be the compression operator of T_z on $M \ominus wM$. In [19], Yang called F_z^M the fringe operator and studied properties of F_z^M. If M is Hilbert–Schmidt, then, by [21], F_z^M is Fredholm. Hence, by [19, Corollary 4.3], $zM + wM$ is closed and $\dim(M \ominus (zM + wM)) < \infty$.

Let $N = H^2 \ominus M$. Let S_z^N and S_w^N be the compression operators of T_z and T_w on N, that is, $S_z^N f = P_N T_z f$ for $f \in N$. We have $(S_z^N)^* = T_z^*|_N$ and $(S_w^N)^* = T_w^*|_N$. We will simply write S_z, S_w when no confusion occurs. We denote by $\sigma_c(S_z)$ and $\sigma_c(S_w)$ the continuous spectra of S_z and S_w, that is, $\lambda \in \sigma_c(S_z)$ if and only if either $\dim(S_z - \lambda I_N) = \infty$ or $S_z - \lambda I_N$ does not have closed range. Set $\sigma_c(M) = \sigma_c(S_z) \cap \sigma_c(S_w)$. In [19, Theorem 2.3], Yang showed that if $\sigma_c(M) \neq \overline{\mathbb{D}}$, then $\Sigma_0(M) + \Sigma_1(M) < \infty$, so M is Hilbert–Schmidt. If $\varphi(z)H^2 \subset M$ for some inner function $\varphi(z)$, then, by the model theory of Sz.-Nagy and Foiaş [13, 17], $\sigma_c(M) \neq \overline{\mathbb{D}}$, so there are a lot of Hilbert–Schmidt invariant subspaces. If M is a unitarily equivalent to an invariant subspace M_1 such that $\sigma_c(M_1) \neq \overline{\mathbb{D}}$, then M is Hilbert–Schmidt. In this paper, we shall study a Hilbert–Schmidt invariant subspace M satisfying that $\sigma_c(M_1) = \overline{\mathbb{D}}$ for every M_1 that is unitarily equivalent to M.

In Section 2, we shall define splitting invariant subspaces of H^2 and prove that they are Hilbert–Schmidt. In Section 3, we shall study a Rudin-type invariant subspace $\mathcal{M}$ which was first studied in [14, page 72]. We shall show that $\mathcal{M}$ is splitting, and that $\sigma_c(M_1) = \overline{\mathbb{D}}$ for every M_1 that is unitarily equivalent to $\mathcal{M}$.

Let $\mathcal{M}_0 = z\mathcal{M} + w\mathcal{M}$. Then $\mathcal{M}_0$ is an invariant subspace. We shall show that, under some additional assumptions, $\mathcal{M}_0$ is Hilbert–Schmidt, $\mathcal{M}_0$ is not splitting and $\sigma_c(M_2) = \overline{\mathbb{D}}$ for every M_2 that is unitarily equivalent to $\mathcal{M}_0$.

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2. Splitting invariant subspaces

Let $\varphi(z)$ be a nonconstant inner function. An invariant subspace M of H^2 is said to be *splitting* for $\varphi(z)$ if

$$(\#1)\quad M = (M \cap \varphi(z)H^2) \oplus (M \cap (H^2 \ominus \varphi(z)H^2))$$

and

$$(\#2)\quad M \cap (H^2 \ominus \varphi(z)H^2) \neq \{0\}.$$

Similarly, we may define a splitting invariant subspace for a nonconstant inner function $\psi(w)$. We say simply that M is splitting if M is splitting either for $\varphi(z)$ or for $\psi(w)$. In this section, we shall study splitting invariant subspaces M for $\varphi(z)$. We set

$$A = A(\varphi) = M \cap (H^2 \ominus \varphi(z)H^2). \tag{2.1}$$

We write

$$K_\varphi(z) = H^2(z) \ominus \varphi(z)H^2(z) \quad \text{and} \quad K_\psi(w) = H^2(w) \ominus \psi(w)H^2(w).$$

Lemma 2.1. *Let M be a splitting invariant subspace for $\varphi(z)$. Then $wA \subset A$ and there is an inner function $\psi(w)$ (may be constant) such that $M \cap \psi(w)H^2 = A \oplus \varphi(z)\psi(w)H^2$, $A \subset \psi(w)K_\varphi(z) \otimes H^2(w)$, $K_\varphi(z) \otimes K_\psi(w) \perp M$ and $T_z^*\varphi(z)\psi(w) \not\perp A$. Moreover, if $\eta(w)$ is an inner function satisfying $A \subset \eta(w)H^2$, then $\psi(w)H^2 \subset \eta(w)H^2$.*

Proof. By (#2) and (2.1), $A \neq \{0\}$, $wA \subset A$ and $zA \not\subset A$. For $f \in A$, we write

$$zf = f_1 \oplus f_2 \in \varphi(z)H^2 \oplus (H^2 \ominus \varphi(z)H^2).$$

Since $f \in H^2 \ominus \varphi(z)H^2$, $f_1 \in \varphi(z)H^2(w)$,

$$zw^n f = w^n f_1 \oplus w^n f_2 \in \varphi(z)H^2(w) \oplus (H^2 \ominus \varphi(z)H^2)$$

for every $n \geq 0$ and $\{f_1 : f \in A\} \neq \{0\}$. Then, by the Beurling theorem, there is an inner function $\psi(w)$ such that

$$\bigvee_{n\geq 0} w^n\{f_1 : f \in A\} = \varphi(z)\psi(w)H^2(w),$$

where $\bigvee_{n\geq 0} E_n$ is the closed linear span of $E_0, E_1, \ldots$. This shows that $T_z^*\varphi(z)\psi(w) \not\perp A$. By (#1), $f_1 \in M \cap \varphi(z)H^2$. Hence $\varphi(z)\psi(w)H^2(w) \subset M$, so $\varphi(z)\psi(w)H^2 \subset M$. One easily sees that $A \subset \psi(w)H^2$, and $\psi(w)H^2 \subset \eta(w)H^2$ for every inner function $\eta(w)$ satisfying $A \subset \eta(w)H^2$. By (#1) and (2.1), $M \cap \psi(w)H^2 = A \oplus \varphi(z)\psi(w)H^2$, $A \subset \psi(w)K_\varphi(z) \otimes H^2(w)$ and $K_\varphi(z) \otimes K_\psi(w) \perp M$. □

An inner function $\psi(w)$ given in Lemma 2.1 is unique except for constant multiplication and depends on $\varphi(z)$. So $\psi(w)$ is said to be the associated inner function of $\varphi(z)$ for M.

Let M be a splitting invariant subspace for $\varphi(z)$ and $\psi(w)$ be the associated inner function of $\varphi(z)$. By Lemma 2.1,

$$L_1 := A \oplus \varphi(z)\psi(w)H^2 = M \cap \psi(w)H^2. \tag{2.2}$$

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Since $\varphi(z)\psi(w)H^2 \subset M \cap \varphi(z)H^2$, let

$$B = B(\varphi) = (M \cap \varphi(z)H^2) \ominus \varphi(z)\psi(w)H^2.$$

Then $zB \subset B$ and $B \subset \varphi(z)H^2(z) \otimes K_\psi(w)$. By Lemma 2.1, again, $A \subset \psi(w)K_\varphi(z) \otimes H^2(w)$, so $A \perp B$. By (#1) and (2.1),

$$M = A \oplus B \oplus \varphi(z)\psi(w)H^2. \tag{2.3}$$

We set

$$L_2 := B \oplus \varphi(z)\psi(w)H^2 = M \cap \varphi(z)H^2. \tag{2.4}$$

Then L_1 and L_2 are invariant subspaces and $L_1 \cap L_2 = \varphi(z)\psi(w)H^2$. Since $\psi(w)$ is the associated inner function of $\varphi(z)$,

$$\bigvee\{f(0,w) : f \in A\} = \psi(w)H^2(w).$$

When $B \neq \{0\}$, $\psi(w)$ is nonconstant and M is splitting for $\psi(w)$. Let $\varphi_1(z)$ be the associated inner function of $\psi(w)$ for M. Then $\varphi_1(z)H^2(z) \subset \varphi(z)H^2(z)$. We shall show the following theorem.

Theorem 2.2. *If M is a splitting invariant subspace of H^2, then M is Hilbert–Schmidt.*

To show Theorem 2.2, we use several known facts, as mentioned in the introduction. We will list them as lemmas.

Lemma 2.3.

(i) $\Sigma_0(M) = \|P_{M\ominus zM}P_{M\ominus wM}\|^2_{\mathrm{HS}}$.
(ii) *If $\{\psi_n\}_{n\geq 1}$ is an orthonormal basis of $M \ominus wM$, then $\Sigma_1(M) = \sum_{n=1}^\infty \|R_w^* R_z \psi_n\|^2$.*

Lemma 2.4. *Let M be an invariant subspace of H^2. Then M is Hilbert–Schmidt if and only if $\Sigma_0(M) + \Sigma_1(M) < \infty$.*

Lemma 2.5. *Let M be an invariant subspace of H^2. If $\sigma_c(M) \neq \overline{\mathbb{D}}$, then $\Sigma_0(M) + \Sigma_1(M) < \infty$.*

Let M_1 and M_2 be invariant subspaces of H^2. A unitary operator $T : M_1 \to M_2$ is called a unitary module map if $T_zT = TT_z$ and $T_wT = TT_w$ on M_1. We say that M_1 is unitarily equivalent to M_2 if there is a unitary module map $T : M_1 \to M_2$.

Lemma 2.6. *Let M_1 and M_2 be invariant subspaces of H^2. If M_1 is unitarily equivalent to M_2, then $\Sigma_0(M_1) = \Sigma_0(M_2)$ and $\Sigma_1(M_1) = \Sigma_1(M_2)$.*

Proof of Theorem 2.2. We may assume that M is splitting for $\varphi(z)$. Let $\psi(w)$ be the associated inner function of $\varphi(z)$. By (2.2), $L_1 \subset \psi(w)H^2$. Then $T^*_{\psi(w)}L_1$ is an invariant subspace and $T^*_{\psi(w)} : L_1 \to T^*_{\psi(w)}L_1$ is a unitary module map. By Lemma 2.6, $\Sigma_0(L_1) = \Sigma_0(T^*_{\psi(w)}L_1)$ and $\Sigma_1(L_1) = \Sigma_1(T^*_{\psi(w)}L_1)$. By (2.2), again, $T^*_{\psi(w)}L_1 = T^*_{\psi(w)}A \oplus \varphi(z)H^2$.

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Let $N_1 = H^2 \ominus T^*_{\psi(w)} L_1$. Then $N_1 \subset H^2 \ominus \varphi(z)H^2$. Hence $\varphi(S_z^{N_1}) = 0$, so, by the model theory of Sz.-Nagy and Foiaş [13, 17]

$$\sigma_c(S_z^{N_1}) \subset \sigma(S_z^{N_1}) \subset \{z \in \mathbb{D} : \varphi(z) = 0\} \cup \partial\mathbb{D} \neq \overline{\mathbb{D}}.$$

Hence $\sigma_c(T^*_{\psi(w)} L_1) \neq \overline{\mathbb{D}}$. By Lemma 2.5, $\Sigma_0(T^*_{\psi(w)} L_1) + \Sigma_1(T^*_{\psi(w)} L_1) < \infty$, so

$$\Sigma_0(L_1) + \Sigma_1(L_1) < \infty. \tag{2.5}$$

Similarly,

$$\Sigma_0(L_2) + \Sigma_1(L_2) < \infty. \tag{2.6}$$

To show that M is Hilbert–Schmidt, we shall compute the values $\Sigma_0(M)$ and $\Sigma_1(M)$, respectively. First, we shall show that $\Sigma_0(M) < \infty$. By (2.3) and (2.4), $M = A \oplus L_2$. Since $wA \subset A$ and $wL_2 \subset L_2$,

$$M \ominus wM = (A \ominus wA) \oplus (L_2 \ominus wL_2).$$

Let $\{g_n\}_{n\geq 1}$ and $\{f_n\}_{n\geq 1}$ be orthonormal bases of $A \ominus wA$ and $L_2 \ominus wL_2$, respectively. By Lemma 2.3(i),

$$\Sigma_0(M) = \sum_{n=1}^{\infty} (\|P_{M\ominus zM} g_n\|^2 + \|P_{M\ominus zM} f_n\|^2). \tag{2.7}$$

Since $M = B \oplus L_1$ and $zB \subset B$,

$$M \ominus zM = (B \ominus zB) \oplus (L_1 \ominus zL_1).$$

By (2.3), $A \perp B$, so $g_n \perp B \ominus zB$. Since $L_1 = A \oplus \varphi(z)\psi(w)H^2$ and

$$L_1 \ominus wL_1 = (A \ominus wA) \oplus \varphi(z)\psi(w)H^2(z),$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \|P_{M\ominus zM} g_n\|^2 &= \sum_{n=1}^{\infty} \|P_{L_1\ominus zL_1} g_n\|^2 \\
&= \sum_{n=1}^{\infty} \|P_{L_1\ominus zL_1} P_{L_1\ominus wL_1} g_n\|^2 \\
&\leq \|P_{L_1\ominus zL_1} P_{L_1\ominus wL_1}\|^2_{\mathrm{HS}} \\
&= \Sigma_0(L_1) \quad \text{by Lemma 2.3.}
\end{aligned}$$

By (2.7),

$$\Sigma_0(M) \leq \Sigma_0(L_1) + \sum_{n=1}^{\infty} \|P_{M\ominus zM} f_n\|^2. \tag{2.8}$$

Also

$$\sum_{n=1}^{\infty} \|P_{M\ominus zM} f_n\|^2 = \sum_{n=1}^{\infty} (\|P_{B\ominus zB} f_n\|^2 + \|P_{L_1\ominus zL_1} f_n\|^2).$$

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Since

$$L_2 \ominus zL_2 = (B \ominus zB) \oplus \varphi(z)\psi(w)H^2(w),$$

$$\begin{aligned}\sum_{n=1}^{\infty} \|P_{M\ominus zM} f_n\|^2 &\le \sum_{n=1}^{\infty} \|P_{L_2\ominus zL_2} f_n\|^2 + \sum_{n=1}^{\infty} \|P_{L_1\ominus zL_1} f_n\|^2 \\ &= \sum_{n=1}^{\infty} \|P_{L_2\ominus zL_2} P_{L_2\ominus wL_2} f_n\|^2 + \sum_{n=1}^{\infty} \|P_{L_1\ominus zL_1} f_n\|^2 \\ &= \|P_{L_2\ominus zL_2} P_{L_2\ominus wL_2}\|_{\mathrm{HS}}^2 + \sum_{n=1}^{\infty} \|P_{L_1\ominus zL_1} f_n\|^2 \\ &= \Sigma_0(L_2) + \sum_{n=1}^{\infty} \|P_{L_1\ominus zL_1} f_n\|^2.\end{aligned}$$

Hence, by (2.8),

$$\Sigma_0(M) \le \Sigma_0(L_1) + \Sigma_0(L_2) + \sum_{n=1}^{\infty} \|P_{L_1\ominus zL_1} f_n\|^2. \tag{2.9}$$

Let $\{h_k\}_{k\ge1}$ be an orthonormal basis of $L_1 \ominus zL_1$. Then, for each $n \ge 1$,

$$\|P_{L_1\ominus zL_1} f_n\|^2 = \sum_{k=1}^{\infty} |\langle P_{L_1\ominus zL_1} f_n, h_k\rangle|^2 = \sum_{k=1}^{\infty} |\langle f_n, h_k\rangle|^2.$$

Since $L_1 \ominus zL_1 \perp z\varphi(z)\psi(w)H^2$, we may write

$$h_k = h_{k,1} \oplus \varphi(z)\psi(w)\eta_k(w) \in A \oplus \varphi(z)\psi(w)H^2(w).$$

Since $f_n \in L_2$ and $L_2 \perp A$,

$$\|P_{L_1\ominus zL_1} f_n\|^2 = \sum_{k=1}^{\infty} |\langle f_n, h_k\rangle|^2 = \sum_{k=1}^{\infty} |\langle f_n, \varphi(z)\psi(w)\eta_k(w)\rangle|^2.$$

Since $f_n \in L_2 \ominus wL_2$, $f_n \perp w\varphi(z)\psi(w)H^2$. Hence we may write

$$f_n = f_{n,1} \oplus \varphi(z)\psi(w)\sigma_n(z) \in B \oplus \varphi(z)\psi(w)H^2(z).$$

Since $B \perp \varphi(z)\psi(w)H^2$,

$$\begin{aligned}\|P_{L_1\ominus zL_1} f_n\|^2 &= \sum_{k=1}^{\infty} |\langle \varphi(z)\psi(w)\sigma_n(z), \varphi(z)\psi(w)\eta_k(w)\rangle|^2 \\ &= |\sigma_n(0)|^2 \sum_{k=1}^{\infty} |\eta_k(0)|^2.\end{aligned}$$

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Here

$$\begin{aligned}\sum_{k=1}^{\infty}|\eta_k(0)|^2 &= \sum_{k=1}^{\infty}|\langle h_k, \varphi(z)\psi(w)\rangle|^2 = \sum_{k=1}^{\infty}\|P_{\mathbb{C}\cdot\varphi\psi}h_k\|^2 \\ &\le \sum_{k=1}^{\infty}\|P_{L_1\ominus wL_1}h_k\|^2 = \sum_{k=1}^{\infty}\|P_{L_1\ominus wL_1}P_{L_1\ominus zL_1}h_k\|^2 \\ &= \|P_{L_1\ominus wL_1}P_{L_1\ominus zL_1}\|_{\mathrm{HS}}^2 = \Sigma_0(L_1) \quad \text{by Lemma 2.3.}\end{aligned}$$

Similarly, $\sum_{n=1}^{\infty}|\sigma_n(0)|^2 \le \Sigma_0(L_2)$. Hence

$$\sum_{n=1}^{\infty}\|P_{L_1\ominus zL_1}f_n\|^2 \le \Sigma_0(L_1)\sum_{n=1}^{\infty}|\sigma_n(0)|^2 \le \Sigma_0(L_1)\Sigma_0(L_2).$$

By (2.5), (2.6) and (2.9),

$$\Sigma_0(M) \le \Sigma_0(L_1) + \Sigma_0(L_2) + \Sigma_0(L_1)\Sigma_0(L_2) < \infty.$$

Next, we shall prove that $\Sigma_1(M) < \infty$. Since $\{g_n, f_n : n \ge 1\}$ is an orthonormal basis of $M \ominus wM$, by Lemma 2.3(ii),

$$\Sigma_1(M) = \sum_{n=1}^{\infty}(\|R_w^*R_zg_n\|^2 + \|R_w^*R_zf_n\|^2).$$

Since $M \ominus wM = (A \ominus wA) \oplus (L_2 \ominus wL_2)$ and $wA \perp L_2$,

$$R_w^*R_z = R_w^{M*}R_z^M = R_w^{L_2*}R_z^{L_2} \quad \text{on } L_2.$$

Since $\{f_n\}_{n\ge1}$ is an orthonormal basis of $L_2 \ominus wL_2$, by Lemma 2.3, again,

$$\sum_{n=1}^{\infty}\|R_w^*R_zf_n\|^2 = \Sigma_1(L_2).$$

Hence

$$\Sigma_1(M) = \Sigma_1(L_2) + \sum_{n=1}^{\infty}\|R_w^*R_zg_n\|^2. \tag{2.10}$$

Since $\{g_n\}_{n\ge1}$ is an orthonormal basis of $A \ominus wA$ and $A \ominus wA \subset L_1 \ominus wL_1$,

$$\sum_{n=1}^{\infty}\|R_w^{L_1*}R_z^{L_1}g_n\|^2 \le \Sigma_1(L_1). \tag{2.11}$$

By Lemma 2.1, $P_AT_z^*\varphi(z)\psi(w) \ne 0$. Since $zwA \perp \varphi(z)\psi(w)$, $P_AT_z^*\varphi(z)\psi(w) \in A \ominus wA$, so we may assume that

$$g_1 = \frac{P_AT_z^*\varphi(z)\psi(w)}{\|P_AT_z^*\varphi(z)\psi(w)\|}.$$

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For each $n \geq 2$,

$$0 = \langle g_n, g_1 \rangle = \frac{1}{\|P_A T_z^* \varphi(z)\psi(w)\|} \langle g_n, P_A T_z^* \varphi(z)\psi(w) \rangle,$$

so $zg_n \perp \varphi(z)\psi(w)$. Hence

$$zg_n \in A \oplus \varphi(z)\psi(w)wH^2(w) \subset A \oplus \varphi(z)\psi(w)H^2 = L_1.$$

This shows that $R_w^* R_z g_n = R_w^{L_1 *} R_z^{L_1} g_n$ for every $n \geq 2$. Therefore, by (2.11),

$$\begin{aligned} \sum_{n=1}^{\infty} \|R_w^* R_z g_n\|^2 &= \|R_w^* R_z g_1\|^2 + \sum_{n \geq 2}^{\infty} \|R_w^{L_1 *} R_z^{L_1} g_n\|^2 \\ &\leq \|R_w^* R_z g_1\|^2 + \Sigma_1(L_1). \end{aligned}$$

Thus, by (2.5), (2.6) and (2.10), $\Sigma_1(M) < \infty$. By Lemma 2.4, M is Hilbert–Schmidt. □

As mentioned in the introduction, by Yang's works we have the following corollary.

Corollary 2.7. *Let M be a splitting invariant subspace of H^2. Then $zM + wM$ is closed, $1 \leq \dim(M \ominus (zM + wM)) < \infty$ and F_z^M on $M \ominus wM$ is Fredholm.*

Example 2.8. Let $\varphi(z), \psi(w)$ be nonconstant inner functions and $M = \varphi(z)H^2 + \psi(w)H^2$. Then M is a splitting invariant subspace for $\varphi(z)$. □

Example 2.9. Let μ, ν be bounded positive singular measures on $\partial\mathbb{D}$. Let

$$\psi_\mu(z) = \exp\bigg(-\int_{\partial\mathbb{D}} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(e^{i\theta})\bigg), \quad z \in \mathbb{D}.$$

Then $\psi_\mu(z)$ is an inner function (see [6]). Let

$$M = \bigvee_{0<t<\infty} \psi_\mu(z)^t \psi_\nu(w)^{1/t} H^2.$$

Then it is clear that M is a splitting invariant subspace for $\psi_\mu(z)$. □

Proposition 2.10. *Let η be an inner function on $\mathbb{D}^2$. If ηH^2 is splitting, then $\eta = \varphi_1(z)\psi_1(w)$ for some inner functions $\varphi_1(z)$ and $\psi_1(w)$.*

Proof. We may assume that ηH^2 is splitting for a nonconstant inner function $\varphi(z)$. Let $\psi(w)$ be the associated inner function of $\varphi(z)$ for ηH^2. Then, by Lemma 2.1, $\varphi(z)\psi(w)H^2 \subset \eta H^2$ and $K_\varphi(z) \otimes K_\psi(w) \perp \eta H^2$. There is $\sigma \in H^2$ satisfying $\eta\sigma = \varphi(z)\psi(w)$. We note that σ is an inner function on $\mathbb{D}^2$. We have $T_z^* \varphi(z) T_w^* \psi(w) \perp \eta H^2$, so $\varphi(z)\psi(w) \perp zw\eta H^2$. Hence $\sigma \perp zwH^2$.

Suppose that σ is not a one variable function. Then we may write $\sigma = f(z) \oplus g(w)$, where $f(z) \in H^2(z)$, $g(w) \in H^2(w)$ and $g(0) = 0$. Also $g(w) \neq 0$ and $f(z)$ is not constant. For every $n \geq 1$, $\langle f(z), z^n f(z) \rangle = \langle \sigma, z^n \sigma \rangle = 0$, so $f(z) = c\varphi_1(z)$ for some nonconstant inner function $\varphi_1(z)$ and nonzero constant c. Since

$$1 = \|\sigma\|^2 = \|f(z)\|^2 + \|g(w)\|^2 = |c|^2 + \|g(w)\|^2$$

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and $\|g(w)\|^2 \neq 0$, $0 < |c| < 1$. Since σ is inner, $\sigma(z, \lambda)$ is inner for almost every $\lambda \in \partial\mathbb{D}$. We have $\sigma(z, \lambda) = c\varphi_1(z) + g(\lambda)$. Since $g(w) \neq 0$, this leads to a contradiction. Then σ is a one variable inner function.

Suppose that $\sigma = \sigma(z)$. Since $\eta\sigma = \varphi(z)\psi(w)$, $\varphi(z)/\sigma \in H^2(z)$. Put $\varphi_1(z) = \varphi(z)/\sigma$ and $\psi_1(w) = \psi(w)$. Then $\eta = \varphi_1(z)\psi_1(w)$. Similarly, we get the assertion for the case $\sigma = \sigma(w)$. □

3. Rudin-type invariant subspaces

Let $\{\varphi(z)\}_{n=-\infty}^{\infty}$ and $\{\psi(w)\}_{n=-\infty}^{\infty}$ be sequences of nonconstant one variable inner functions satisfying the following conditions:

(α1) $\zeta_n(z) := \varphi_n(z)/\varphi_{n+1}(z)$ is a nonconstant inner function for every $-\infty < n < \infty$;

(α2) $\varphi_n(z) \to 1$ as $n \to \infty$ for every $z \in \mathbb{D}$;

(α3) $\varphi_n(z) \to 0$ as $n \to -\infty$ for every $z \in \mathbb{D}$;

(α4) $\xi_n(w) := \psi_{n+1}(w)/\psi_n(w)$ is a nonconstant inner function for every $-\infty < n < \infty$;

(α5) $\psi_n(w) \to 1$ as $n \to -\infty$ for every $w \in \mathbb{D}$; and

(α6) $\psi_n(w) \to 0$ as $n \to \infty$ for every $w \in \mathbb{D}$.

Moreover, we assume that

(α7) $\varphi_n(0) \geq 0, \quad \psi_n(0) \geq 0, \quad \zeta_n(0) \geq 0$ and $\xi_n(0) \geq 0$ for every $-\infty < n < \infty$.

Let

$$\mathcal{M} = \bigvee_{n=-\infty}^{\infty} \varphi_{n+1}(z)\psi_n(w)H^2. \tag{3.1}$$

Then $\mathcal{M}$ is an invariant subspace. This type of invariant subspace was first studied by Rudin [14, page 72] (see also [8–10, 15, 16]), so $\mathcal{M}$ is called a Rudin-type invariant subspace. By (α2) and (α3), $\mathcal{M} \not\subset \varphi(z)H^2$ and $\varphi(z)H^2 \not\subset \mathcal{M}$ for every nonconstant inner function $\varphi(z)$. By (α5) and (α6), $\mathcal{M} \not\subset \psi(w)H^2$ and $\psi(w)H^2 \not\subset \mathcal{M}$ for every nonconstant inner function $\psi(w)$.

By (α1), (α2), (α4), (α5) and (α7), we may assume that

$$\varphi_n(z) = \prod_{k=n}^{\infty} \zeta_k(z), \quad \psi_n(w) = \prod_{k=-\infty}^{n-1} \xi_k(w)$$

and

$$\begin{aligned}\mathcal{M} &= \bigoplus_{n=-\infty}^{\infty} \varphi_{n+1}(z)\psi_n(w)H^2(z) \otimes K_{\xi_n}(w) \\ &= \bigoplus_{n=-\infty}^{\infty} \varphi_{n+1}(z)\psi_n(w)K_{\zeta_n}(z) \otimes H^2(w).\end{aligned}$$

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Now it is clear that $\mathcal{M}$ is splitting for $\varphi_1(z)$ and $\psi_1(w)$ is the associated inner function of $\varphi_1(z)$, so, by Theorem 2.2 and Corollary 2.7, we have the following corollary.

Corollary 3.1. *$\mathcal{M}$ is Hilbert–Schmidt, $z\mathcal{M} + w\mathcal{M}$ is closed,* $\dim(\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M})) < \infty$ *and $F_z^{\mathcal{M}}$ on $\mathcal{M} \ominus w\mathcal{M}$ is Fredholm.*

Let $\mathcal{N} = H^2 \ominus \mathcal{M}$. Then

$$\mathcal{N} = \bigoplus_{n=-\infty}^{\infty} \psi_n(w) K_{\varphi_{n+1}}(z) \otimes K_{\xi_n}(w)$$
$$= \bigoplus_{n=-\infty}^{\infty} \varphi_{n+1}(z) K_{\zeta_n}(z) \otimes K_{\psi_n}(w).$$

We shall prove the following theorem.

Theorem 3.2. $\sigma_c(\mathcal{M}) = \overline{\mathbb{D}}$.

To prove Theorem 3.2, we use the following lemma freely (see [2, 13, 17]). For a one variable inner function $\varphi(z)$, we define the operator $S_z^{K_\varphi}$ on $K_\varphi(z)$ by $S_z^{K_\varphi} = P_{K_\varphi(z)} T_z|_{K_\varphi(z)}$. We write S_z when no confusion occurs. We have $S_z^* = T_z^*|_{K_\varphi(z)}$.

Lemma 3.3. *Let $\varphi(z)$ be a nonconstant inner function. Then:*

(i) $T_z^*\varphi(z) \in K_\varphi(z)$;
(ii) $S_z = T_z$ *on* $K_\varphi(z) \ominus \mathbb{C} \cdot T_z^*\varphi(z)$; *and*
(iii) $S_z T_z^*\varphi(z) = -\varphi(0)(1 - \overline{\varphi(0)}\varphi(z))$.

For $\alpha \in \mathbb{D}$, let $k_\alpha(z) = 1/(1 - \overline{\alpha} z)$. We have $k_\alpha(z) \in H^2(z)$ and $T_z^* k_\alpha(z) = \overline{\alpha} k_\alpha(z)$.

Lemma 3.4. *Let $\varphi(z)$ be a nonconstant inner function. Then, for every $\alpha \in \mathbb{D}$, there exists $f(z) \in K_\varphi(z)$ with $\|f(z)\| = 1$ satisfying $\|(S_z^* - \overline{\alpha} I) f(z)\| \le 4|\varphi(\alpha)|/\sqrt{1 - |\varphi(\alpha)|^2}$.*

Proof. We note that $S_z^* = T_z^*|_{K_\varphi(z)}$. Let

$$\eta(z) = (1 - \overline{\varphi(\alpha)}\varphi(z)) k_\alpha(z) \in K_\varphi(z).$$

Then $\eta(z)$ is the reproducing kernel of $K_\varphi(z)$ for the point $z = \alpha$. We have $\|\eta(z)\|^2 = (1 - |\varphi(a)|^2)/(1 - |\alpha|^2)$. For $h(z), g(z) \in H^2(z)$ satisfying $(hg)(z) \in H^2(z)$, $T_z^* h(z) = (h(z) - h(0))/z$ and

$$T_z^*(hg)(z) = T_z^*(h(z))g(z) + h(0) T_z^* g(z).$$

Hence

$$(S_z^* - \overline{\alpha} I)\eta(z) = -\overline{\varphi(\alpha)} \frac{\varphi(z) - \varphi(0)}{z} k_\alpha(z) + \overline{\alpha}(1 - \overline{\varphi(\alpha)}\varphi(0)) k_\alpha(z)$$
$$- \overline{\alpha}(1 - \overline{\varphi(\alpha)}\varphi(z)) k_\alpha(z)$$
$$= \overline{\varphi(\alpha)} \frac{\varphi(z) - \varphi(0)}{z} (\overline{\alpha} z - 1) k_\alpha(z).$$

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Therefore

$$\frac{\|(S_z^* - \overline{\alpha}I)\eta(z)\|}{\|\eta(z)\|} \le \frac{4\sqrt{1-|\alpha|^2}|\varphi(\alpha)|\|k_\alpha(z)\|}{\sqrt{1-|\varphi(\alpha)|^2}} = \frac{4|\varphi(\alpha)|}{\sqrt{1-|\varphi(\alpha)|^2}}.$$

Set $f = \eta(z)/\|\eta(z)\|$. Then we get the assertion. □

For an invariant subspace M of H^2, let $N = H^2 \ominus M$. Then $T_z^* N \subset N$ and $T_w^* N \subset N$, so N is called a backward shift invariant subspace. We may define the compression operators S_z^N, S_w^N of T_z, T_w on N. We have $(S_z^N)^* = T_z^*|_N$ and $(S_w^N)^* = T_w^*|_N$.

Lemma 3.5. *Let N be a backward shift invariant subspace of H^2. If there are sequences of nonconstant inner functions $\{\varphi_n(z)\}_{n\ge 0}$ and $\{\psi_n(w)\}_{n\ge 0}$ such that $K_{\varphi_n}(z) \otimes K_{\psi_n}(w) \subset N$ for every $n \ge 0$ and $\varphi_n(\alpha) \to 0$ for every $\alpha \in \mathbb{D}$, then $\sigma(S_z^N) = \overline{\mathbb{D}}$.*

Proof. Let $\alpha \in \mathbb{D}$. By Lemma 3.4, for each $n \ge 0$ there exists $f_n(z) \in K_{\varphi_n}(z)$ with $\|f_n(z)\| = 1$ satisfying

$$\|((S_z^{K_{\varphi_n}})^* - \overline{\alpha}I)f_n(z)\| \le \frac{4|\varphi_n(\alpha)|}{\sqrt{1-|\varphi_n(\alpha)|^2}}.$$

Let $g_n(w) \in K_{\psi_n}(w)$ with $\|g_n(w)\| = 1$. Then $f_n(z)g_n(w) \in K_{\varphi_n}(z) \otimes K_{\psi_n}(w) \subset N$ and $\|f_n(z)g_n(w)\| = 1$. By the assumption,

$$\begin{aligned}\|((S_z^N)^* - \overline{\alpha}I)f_n(z)g_n(w)\| &= \|g_n(w)\|\,\|((S_z^{K_{\varphi_n}})^* - \overline{\alpha}I)f_n(z)\| \\ &\le \frac{4|\varphi_n(\alpha)|}{\sqrt{1-|\varphi_n(\alpha)|^2}} \to 0\end{aligned}$$

as $n \to \infty$. Hence $(S_z^N)^* - \overline{\alpha}I$ is not invertible, so $\overline{\alpha} \in \sigma((S_z^N)^*) = \overline{\sigma(S_z^N)}$. Thus we get $\sigma(S_z^N) = \overline{\mathbb{D}}$. □

Recall that $S_z^{\mathcal{N}} = P_{\mathcal{N}} T_z|_{\mathcal{N}}$ and $S_w^{\mathcal{N}} = P_{\mathcal{N}} T_w|_{\mathcal{N}}$. We have $K_{\varphi_n}(z) \otimes K_{\psi_n}(w) \subset \mathcal{N}$ for every $-\infty < n < \infty$. By (α3) and (α6) and by applying Lemma 3.5 we obtain the following corollary, which is proved by Yang [24, Theorem 4.3].

Corollary 3.6. $\sigma(S_z^{\mathcal{N}}) = \sigma(S_w^{\mathcal{N}}) = \overline{\mathbb{D}}$.

Proof of Theorem 3.2. Recall that $\sigma_c(\mathcal{M}) = \sigma_c(S_z^{\mathcal{N}}) \cap \sigma_c(S_w^{\mathcal{N}})$. Since we are working on $\mathcal{N}$, we write S_z and S_w, for short. It is sufficient to prove that $\sigma_c(S_z) = \overline{\mathbb{D}}$. Let $Z(\varphi_n) = \{z \in \mathbb{D} : \varphi_n(z) = 0\}$. By ($\alpha$1), $Z(\varphi_n) \subset Z(\varphi_k)$ for $-\infty < k < n < \infty$ and $\bigcup_{n=-\infty}^{\infty} Z(\varphi_n)$ is at most a countable set. Let $\lambda \in \mathbb{D}$. We shall show that $\lambda \in \sigma_c(S_z)$. We study two cases separately.

Case 1. Suppose that $\lambda \in \bigcup_{n=-\infty}^{\infty} Z(\varphi_n)$. Then there is an integer n_0 such that $\varphi_n(\lambda) = 0$ for every $-\infty < n \le n_0$. We write

$$b_\lambda(z) = \frac{z-\lambda}{1-\overline{\lambda}z}, \quad z \in \mathbb{D}.$$

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For each $n \le n_0$, there is an inner function $\sigma_n(z)$ satisfying $\varphi_n(z) = b_\lambda(z)\sigma_n(z)$. Then $\sigma_n(z)/(1-\overline{\lambda}z) \in K_{\varphi_n}(z)$. Let

$$f_n = \frac{\sigma_n(z)}{1-\overline{\lambda}z}\psi_{n-1}(w)T_w^*\xi_{n-1}(w).$$

We have $f_n \in \mathcal{N}$ and

$$(z-\lambda)f_n = \varphi_n(z)\psi_{n-1}(w)T_w^*\xi_{n-1}(w) \in \mathcal{M}.$$

Hence

$$\{f_n : -\infty < n \le n_0\} \subset \ker(S_z - \lambda I_{\mathcal{N}}).$$

Since $f_n \perp f_k$ for $k < n \le n_0$, $\ker(S_z - \lambda I_{\mathcal{N}}) = \infty$. Hence $\lambda \in \sigma_c(S_z)$.

Case 2. Suppose that $\lambda \notin \bigcup_{n=-\infty}^{\infty} Z(\varphi_n)$. Let $g \in \ker(S_z - \lambda I_{\mathcal{N}})^*$. Then $(S_z - \lambda I_{\mathcal{N}})^* g = 0$, so $g \perp (z-\lambda)H^2$. Hence there is $h(w) \in H^2(w)$ such that $g = h(w)/(1-\overline{\lambda}z)$. Since $g \in \mathcal{N}$, $g \perp \varphi_{n+1}(z)\psi_n(w)K_{\xi_n}(w)$ for every $-\infty < n < \infty$. Since $\varphi_{n+1}(\lambda) \neq 0$, $h(w) \perp \psi_n(w)K_{\xi_n}(w)$. By condition ($\alpha$6),

$$\psi_k(w)H^2(w) = \bigoplus_{n=k}^{\infty} \psi_n(w)K_{\xi_n}(w)$$

for every $-\infty < k < \infty$. Hence $h(w) \perp \psi_k(w)H^2(w)$. By condition ($\alpha$5),

$$H^2(w) = \bigvee_{k=-\infty}^{\infty} \psi_k(w)H^2(w),$$

so $h(w) \perp H^2(w)$. This shows that $h(w) = 0$ and $g = 0$. Thus we get $\ker(S_z - \lambda I_{\mathcal{N}})^* = \{0\}$.

Next, we shall show that $\ker(S_z - \lambda I_{\mathcal{N}}) = \{0\}$. Let $f \in \mathcal{N}$ and $(S_z - \lambda I_{\mathcal{N}})f = 0$. For each integer j, let

$$\mathcal{M}_j = T^*_{\psi_j(w)}\mathcal{M} \quad \text{and} \quad \mathcal{N}_j = T^*_{\psi_j(w)}\mathcal{N}.$$

Then $\mathcal{M}_j$ is an invariant subspace and $\mathcal{N}_j = H^2 \ominus \mathcal{M}_j$.

$$\mathcal{M}_j = \bigvee_{n=j}^{\infty} \varphi_{n+1}(z)\frac{\psi_n(w)}{\psi_j(w)}H^2.$$

Hence $\varphi_{j+1}(S_z^{\mathcal{N}_j}) = 0$.

Set $\mathcal{N}_{j,1} = \psi_j(w)\mathcal{N}_j$. Then $\mathcal{N}_{j,1} \subset \mathcal{N}$. Let $\mathcal{N}_{j,2} = \mathcal{N} \ominus \mathcal{N}_{j,1}$. We have $S_z\mathcal{N}_{j,1} \subset \mathcal{N}_{j,1}$ and $S_z\mathcal{N}_{j,2} \subset \mathcal{N}_{j,2}$. Hence $S_zP_{\mathcal{N}_{j,1}} = P_{\mathcal{N}_{j,1}}S_z$. It is not difficult to show that $S_z|_{\mathcal{N}_{j,1}}$ is unitarily equivalent to $S_z^{\mathcal{N}_j}$, that is, $T^*_{\psi_j(w)}S_z|_{\mathcal{N}_{j,1}} = S_z^{\mathcal{N}_j}T^*_{\psi_j(w)}|_{\mathcal{N}_{j,1}}$. Hence

$$\sigma(S_z|_{\mathcal{N}_{j,1}}) = \sigma(S_z^{\mathcal{N}_j}) \subset Z(\varphi_{j+1}) \cup \partial\mathbb{D}.$$

Since $\varphi_{j+1}(\lambda) \neq 0$, $\lambda \notin \sigma(S_z|_{\mathcal{N}_{j,1}})$. Since $(S_z - \lambda I_{\mathcal{N}})f = 0$,

$$0 = P_{\mathcal{N}_{j,1}}(S_z - \lambda I_{\mathcal{N}})f = (S_z|_{\mathcal{N}_{j,1}} - \lambda I_{\mathcal{N}_{j,1}})P_{\mathcal{N}_{j,1}}f.$$

Hence $P_{\mathcal{N}_{j,1}} f = 0$, so $f \perp \mathcal{N}_{j,1}$ for every $-\infty < j < \infty$. We have $\mathcal{N}_{j,1} \subset \mathcal{N}_{k,1}$ for $k < j$ and $\mathcal{N} = \bigvee_{j=-\infty}^{\infty} \mathcal{N}_{j,1}$. Therefore $f \perp \mathcal{N}$. Since $f \in \mathcal{N}$, $f = 0$. Thus $\ker(S_z - \lambda I_{\mathcal{N}}) = \{0\}$.

To show that $\lambda \in \sigma_c(S_z)$, suppose that $\lambda \notin \sigma_c(S_z)$. Then $S_z - \lambda I_{\mathcal{N}}$ has closed range. Since $\ker(S_z - \lambda I_{\mathcal{N}}) = \ker(S_z - \lambda I_{\mathcal{N}})^* = \{0\}$, $\lambda \notin \sigma(S_z)$. This contradicts the fact given in Corollary 3.6. Hence $\lambda \in \sigma_c(S_z)$.

By Cases 1 and 2, $\mathbb{D} \subset \sigma_c(S_z) \subset \overline{\mathbb{D}}$. To show that $\sigma_c(S_z) = \overline{\mathbb{D}}$, let $\lambda \in \partial\mathbb{D}$ satisfy $\lambda \notin \sigma_c(S_z)$. Then $S_z - \lambda I_{\mathcal{N}}$ has closed range. Let $g \in \mathcal{N}$ satisfy $(S_z - \lambda I_{\mathcal{N}})^* g = 0$. Then $g \perp (z - \lambda)H^2$, so $g \perp H^2$. Hence $g = 0$ and $\ker(S_z - \lambda I_{\mathcal{N}})^* = \{0\}$. Let $h \in \mathcal{N}$ satisfy $S_z h = \lambda h$. Then $\|S_z h\| = \|h\|$. Hence $zh \in \mathcal{N}$ and $(z - \lambda)h = 0$. This shows that $h = 0$ and $\ker(S_z - \lambda I_{\mathcal{N}}) = \{0\}$. Therefore $\lambda \notin \sigma(S_z)$. This also contradicts the fact given in Corollary 3.6. Thus we get $\sigma_c(S_z) = \overline{\mathbb{D}}$. □

As mentioned in the introduction, if M is a unitarily equivalent to an invariant subspace M_1 such that $\sigma_c(M_1) \neq \overline{\mathbb{D}}$, then M is Hilbert–Schmidt. We shall show the following theorem.

Theorem 3.7. *Let M_1 be an invariant subspace of H^2 which is unitarily equivalent to $\mathcal{M}$. Then $\sigma_c(M_1) = \overline{\mathbb{D}}$.*

To prove Theorem 3.7, we first show the following lemma.

Lemma 3.8. *Let M be an invariant subspace of H^2 and η be an inner function on $\mathbb{D}^2$. If $\sigma_c(M) = \overline{\mathbb{D}}$, then $\sigma_c(\eta M) = \overline{\mathbb{D}}$.*

Proof. Let $N = H^2 \ominus M$, $M_1 = \eta M$ and $N_1 = H^2 \ominus M_1$. To show that $\sigma_c(M_1) = \overline{\mathbb{D}}$, we suppose the contrary. We may assume that there is $\lambda \in \mathbb{D}$ such that $\lambda \notin \sigma_c(S_z^{N_1})$ (see the proof of Theorem 3.2). Then $S_z^{N_1} - \lambda I_{N_1}$ has closed range and $\dim\ker(S_z^{N_1} - \lambda I_{N_1}) < \infty$.

First, we shall show that $\dim\ker(S_z^N - \lambda I_N) < \infty$. Since η is inner,

$$\eta H^2 = \eta(M \oplus N) = M_1 \oplus \eta N \subset H^2.$$

Then $\eta N \subset N_1$ and $T_\eta|_N : N \to \eta N$ is a unitary operator. Let $f \in N$ and write $zf = f_1 \oplus f_2 \in M \oplus N$. Then $S_z^N f = f_2$ and $T_\eta S_z^N f = \eta f_2$. Since $z\eta f = \eta f_1 \oplus \eta f_2 \in M_1 \oplus \eta N$, $S_z^{N_1} T_\eta f = \eta f_2 = T_\eta S_z^N f$. Hence $S_z^{N_1} T_\eta = T_\eta S_z^N$ on N, so

$$(S_z^{N_1} - \lambda I_{N_1})T_\eta = T_\eta(S_z^N - \lambda I_N) \quad \text{on } N.$$

Therefore

$$\dim\ker(S_z^N - \lambda I_N) \leq \dim\ker(S_z^{N_1} - \lambda I_{N_1}) < \infty.$$

Next, we shall show that $S_z^N - \lambda I_N$ has closed range. Since $S_z^{N_1} - \lambda I_{N_1}$ has closed range, there exists $\delta > 0$ such that $\delta\|g\| \leq \|(S_z^{N_1} - \lambda I_{N_1})g\|$ for every $g \in N_1 \ominus \ker(S_z^{N_1} - \lambda I_{N_1})$. Since $\dim\ \ker(S_z^{N_1} - \lambda I_{N_1}) < \infty$,

$$\dim P_{\eta N} \ker(S_z^{N_1} - \lambda I_{N_1}) < \infty.$$

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Let E be a closed subspace of N such that

$$P_{\eta N}\ker(S_z^{N_1}-\lambda I_{N_1})=\eta E.$$

Then $\dim E<\infty$ and $\eta N=\eta(N\ominus E)\oplus\eta E$. Let $f\in N\ominus E$ and $h\in\ker(S_z^{N_1}-\lambda I_{N_1})$. Then $\eta f\perp M_1$, $P_{\eta N}h=\eta\sigma$ for some $\sigma\in E$ and

$$\langle\eta f,h\rangle=\langle\eta f,P_{\eta N}h\rangle=\langle\eta f,\eta\sigma\rangle=\langle f,\sigma\rangle=0.$$

Hence

$$\eta(N\ominus E)\subset H^2\ominus(M_1+\ker(S_z^{N_1}-\lambda I_{N_1})).$$

Therefore $\eta(N\ominus E)\subset N_1\ominus\ker(S_z^{N_1}-\lambda I_{N_1})$ and

$$\delta\|f\|=\delta\|\eta f\|\le\|(S_z^{N_1}-\lambda I_{N_1})\eta f\|,\quad f\in N\ominus E.$$

Thus we get

$$\delta\|f\|\le\|T_\eta(S_z^{N}-\lambda I_{N})f\|=\|(S_z^{N}-\lambda I_{N})f\|,\quad f\in N\ominus E.$$

This shows that $(S_z^N-\lambda I_N)(N\ominus E)$ is closed. Since $\dim E<\infty$, $(S_z^N-\lambda I_N)N$ is closed. By the last paragraph, $\lambda\notin\sigma_c(S_z^N)$, so $\lambda\notin\sigma_c(M)$. This contradicts the assumption. Thus we get $\mathbb{D}\subset\sigma_c(M_1)$ and $\sigma_c(M_1)=\overline{\mathbb{D}}$. □

Proof of Theorem 3.7. By [1], there is a unimodular function u on $\partial\mathbb{D}\times\partial\mathbb{D}$ such that $M_1=u\mathcal{M}$. We write $H_z^2=H^2(z)\otimes L^2(w)$, where $L^2(w)$ is the w-variable Lebesgue space on $\partial\mathbb{D}$. Since $\varphi_{n+1}(z)\psi_n(w)\in\mathcal{M}$,

$$u\varphi_{n+1}(z)\psi_n(w)\in M_1\subset H^2\subset H_z^2.$$

Hence $u\varphi_{n+1}(z)\in H_z^2$ for every $-\infty<n<\infty$. We have $\varphi_{n+1}(z)=\varphi_{n+1}(0)+zT_z^*\varphi_{n+1}(z)$. Then $1=|\varphi_{n+1}(0)|^2+\|T_z^*\varphi_{n+1}(z)\|^2$. By ($\alpha$2), $\|T_z^*\varphi_{n+1}(z)\|\to0$ as $n\to\infty$.

$$zT_z^*\varphi_{n+1}(z)=\varphi_{n+1}(z)-\varphi_{n+1}(0)=\varphi_{n+1}(z)-1+1-\varphi_{n+1}(0),$$

so

$$\|T_z^*\varphi_{n+1}(z)\|=\|zT_z^*\varphi_{n+1}(z)\|\ge\|\varphi_{n+1}(z)-1\|-|1-\varphi_{n+1}(0)|.$$

By (α2), again, $\|\varphi_{n+1}(z)-1\|\to0$ as $n\to\infty$. Hence $u\in H_z^2$. Similarly, $u\in H_w^2$. Then $u\in H_z^2\cap H_w^2=H^2$. Therefore u is an inner function. By Theorem 3.2, $\sigma_c(\mathcal{M})=\overline{\mathbb{D}}$. By Lemma 3.8, $\sigma_c(M_1)=\sigma_c(u\mathcal{M})=\overline{\mathbb{D}}$. □

Let M be an invariant subspace of H^2. Let

$$\Omega=\Omega(M)=M\ominus(zM+wM)$$

and

$$M_0=\overline{zM+wM}=M\ominus\Omega.$$

Then M_0 is an invariant subspace, $z\Omega\subset M_0$ and $w\Omega\subset M_0$. We write $N=H^2\ominus M$ and $N_0=H^2\ominus M_0$. Then $N_0=N\oplus\Omega$. In the last part of this paper, we shall show the following theorem.

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Theorem 3.9.

(i) *$\mathcal{M}_0$ is Hilbert–Schmidt.*
(ii) *Let M_1 be an invariant subspace of H^2. If M_1 is unitarily equivalent to $\mathcal{M}_0$, then $\sigma_c(M_1) = \overline{\mathbb{D}}$.*
(iii) *$\mathcal{M}_0$ is splitting if and only if $\varphi_n(0)\psi_n(0) = 0$ for some $-\infty < n < \infty$.*

To prove this theorem, we need two lemmas.

Lemma 3.10. *$(S_z^N - \lambda I_N)N = N \cap (S_z^{N_0} - \lambda I_{N_0})N_0$ for every $\lambda \in \overline{\mathbb{D}} \setminus \{0\}$.*

Proof. Let $h \in N \cap (S_z^{N_0} - \lambda I_{N_0})N_0$. Then there is $f_1 \oplus f_2 \in N \oplus \Omega$ such that $h = (S_z^{N_0} - \lambda I_{N_0})(f_1 \oplus f_2)$.

$$\begin{aligned} h &= (S_z^{N_0} - \lambda I_{N_0})f_1 - \lambda f_2 \\ &= (S_z^N - \lambda I_N)f_1 + P_\Omega S_z^{N_0} f_1 - \lambda f_2 \\ &= (S_z^N - \lambda I_N)f_1 \quad \text{because } h \in N \in (S_z^N - \lambda I_N)N. \end{aligned}$$

Hence $N \cap (S_z^{N_0} - \lambda I_{N_0})N_0 \subset (S_z^N - \lambda I_N)N$.

Let $g_1 \in N$ and $\lambda \in \overline{\mathbb{D}} \setminus \{0\}$. Set $g_2 = P_\Omega S_z^{N_0} g_1/\lambda$.

$$\begin{aligned} (S_z^N - \lambda I_N)g_1 &= (S_z^N - \lambda I_N)g_1 + P_\Omega S_z^{N_0} g_1 - P_\Omega S_z^{N_0} g_1 \\ &= (S_z^{N_0} - \lambda I_{N_0})g_1 - \lambda g_2 \\ &= (S_z^{N_0} - \lambda I_{N_0})(g_1 \oplus g_2). \end{aligned}$$

Since $g_1 \oplus g_2 \in N_0$, we get $(S_z^N - \lambda I_N)N \subset N \cap (S_z^{N_0} - \lambda I_{N_0})N_0$. □

Lemma 3.11. *If $\sigma_c(M) = \overline{\mathbb{D}}$ and $\dim \Omega < \infty$, then $\sigma_c(M_0) = \overline{\mathbb{D}}$.*

Proof. Suppose that $\sigma_c(M_0) \neq \overline{\mathbb{D}}$. We may assume that $\sigma_c(S_z^{N_0}) \neq \overline{\mathbb{D}}$. Then there is $\lambda \in \mathbb{D}$ such that $\dim \ker(S_z^{N_0} - \lambda I_{N_0}) < \infty$ and $S_z^{N_0} - \lambda I_{N_0}$ has closed range (see the proof of Theorem 3.2).

Since

$$(S_z^{N_0} - \lambda I_{N_0}) \ker(S_z^N - \lambda I_N) \subset \Omega$$

and $\dim \Omega < \infty$, there are $f_1, \dots, f_n \in \ker(S_z^N - \lambda I_N)$ such that

$$\ker(S_z^N - \lambda I_N) \ominus (\mathbb{C} \cdot f_1 + \cdots + \mathbb{C} \cdot f_n) \subset \ker(S_z^{N_0} - \lambda I_{N_0}).$$

Since $\dim \ker(S_z^{N_0} - \lambda I_{N_0}) < \infty$, $\dim \ker(S_z^N - \lambda I_N) < \infty$.

Suppose that $\lambda \neq 0$. By Lemma 3.10, $S_z^N - \lambda I_N$ has closed range. Hence $\lambda \notin \sigma_c(S_z^N)$, so $\lambda \notin \sigma_c(M)$. This contradicts that $\sigma_c(M) = \overline{\mathbb{D}}$.

Next, suppose that $\lambda = 0$. Then $S_z^{N_0} N_0$ is closed. Since $S_z^{N_0} N_0 = S_z^{N_0} N$, $S_z^{N_0} N$ is closed. Since $\dim \Omega < \infty$, there are $g_1, \dots, g_m \in S_z^{N_0} N$ such that

$$S_z^{N_0} N \ominus (\mathbb{C} \cdot g_1 + \cdots + \mathbb{C} \cdot g_m) = N \cap S_z^{N_0} N.$$

Hence

$$S_z^N N = P_N S_z^{N_0} N = (N \cap S_z^{N_0} N) + P_N(\mathbb{C} \cdot g_1 + \cdots + \mathbb{C} \cdot g_m),$$

so $S_z^N N$ is closed. Therefore $0 \notin \sigma_c(S_z^N)$, so $0 \notin \sigma_c(M)$. This contradicts that $\sigma_c(M) = \overline{\mathbb{D}}$. Thus we get the assertion. □

Now we shall study $\mathcal{M}$ given in (3.1) and $\mathcal{M}_0$. By Corollary 3.1, $\mathcal{M}$ is Hilbert–Schmidt, $z\mathcal{M} + w\mathcal{M}$ is closed and $\dim(\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M})) < \infty$. We note that $\mathcal{M}_0 = z\mathcal{M} + w\mathcal{M}$, $\Omega(\mathcal{M}) = \mathcal{M} \ominus \mathcal{M}_0$ and $\mathcal{N}_0 = \mathcal{N} \oplus \Omega(\mathcal{M})$.

Proof of Theorem 3.9. (i) We have $\mathcal{M}_0 \subset \mathcal{M}$ and $\dim(\mathcal{M} \ominus \mathcal{M}_0) = \dim \Omega(\mathcal{M}) < \infty$. Since $\mathcal{M}$ is Hilbert–Schmidt, it is not difficult to see that $\mathcal{M}_0$ is Hilbert–Schmidt.

(ii) By Theorem 3.2 and Lemma 3.11, $\sigma_c(\mathcal{M}_0) = \overline{\mathbb{D}}$. By [1], there is a unimodular function u on $\partial\mathbb{D} \times \partial\mathbb{D}$ such that $M_1 = u\mathcal{M}_0$. Since $\mathcal{M}_0 = z\mathcal{M} + w\mathcal{M}$, $zu\mathcal{M} \subset u\mathcal{M}_0 = M_1 \subset H^2$. By the proof of Theorem 3.7, zu is inner. Similarly, wu is inner. Then one easily sees that u is inner. By Lemma 3.8, $\sigma_c(M_1) = \sigma_c(u\mathcal{M}_0) = \overline{\mathbb{D}}$.

(iii) Suppose that $\psi_n(0) = 0$ for some $-\infty < n < \infty$. We shall show that $\mathcal{M}_0$ is splitting. By (α5), there is an integer n_0 such that

$$\psi_{n_0+1}(0) = 0 \quad \text{and} \quad \psi_{n_0}(0) \neq 0. \tag{3.2}$$

Since $\mathcal{M}$ is splitting for $\varphi_{n_0+1}(z)$ and $\psi_{n_0+1}(w)$ is the associated inner function of $\varphi_{n_0+1}(z)$,

$$\mathcal{M} = (\mathcal{M} \cap \varphi_{n_0+1}(z)H^2) \oplus (\mathcal{M} \cap (H^2 \ominus \varphi_{n_0+1}(z)H^2))$$

and

$$\mathcal{M} \cap (H^2 \ominus \varphi_{n_0+1}(z)H^2) \subset \psi_{n_0+1}(w)K_{\varphi_{n_0+1}}(z) \otimes H^2(w). \tag{3.3}$$

We shall show that

$$\Omega(\mathcal{M}) \perp \varphi_{n_0+1}(z)\psi_{n_0+1}(w)H^2. \tag{3.4}$$

Let $f \in \Omega(\mathcal{M})$. Since $f \perp w\mathcal{M}$, we may write

$$f = \bigoplus_{n=-\infty}^{\infty} \varphi_{n+1}(z)\psi_n(w)f_n(z) \in \bigoplus_{n=-\infty}^{\infty} \varphi_{n+1}(z)\psi_n(w)K_{\zeta_n}(z).$$

By (3.2),

$$\bigoplus_{n=-\infty}^{n_0} \varphi_{n+1}(z)\psi_n(w)f_n(z) \perp \varphi_{n_0+1}(z)\psi_{n_0+1}(w)H^2.$$

Also

$$\begin{aligned}\bigoplus_{n=n_0+1}^{\infty} \varphi_{n+1}(z)\psi_n(w)f_n(z) \in \bigoplus_{n=n_0+1}^{\infty} \varphi_{n+1}(z)K_{\zeta_n}(z) \otimes H^2(w) \\ = K_{\varphi_{n_0+1}}(z) \otimes H^2(w).\end{aligned}$$

Hence

$$\bigoplus_{n=n_0+1}^{\infty} \varphi_{n+1}(z)\psi_n(w)f_n(z) \perp \varphi_{n_0+1}(z)\psi_{n_0+1}(w)H^2.$$

Thus we get (3.4).

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For $f \in \Omega(\mathcal{M})$, we may write

$$f = f_1 \oplus f_2 \in (\mathcal{M} \cap \varphi_{n_0+1}(z)H^2) \oplus (\mathcal{M} \cap (H^2 \ominus \varphi_{n_0+1}(z)H^2)).$$

By (3.4), $f_1 \in \varphi_{n_0+1}(z)H^2(z) \otimes K_{\psi_{n_0+1}}(w)$ and, by (3.3),

$$f_2 \in \psi_{n_0+1}(w)K_{\varphi_{n_0+1}}(z) \otimes H^2(w).$$

Then it is not difficult to show that $f_1, f_2 \in \Omega(\mathcal{M})$. Hence

$$\Omega(\mathcal{M}) = (\Omega(\mathcal{M}) \cap \varphi_{n_0+1}(z)H^2) \oplus (\Omega(\mathcal{M}) \cap (H^2 \ominus \varphi_{n_0+1}(z)H^2)).$$

Thus we get

$$\begin{aligned}\mathcal{M}_0 &= \mathcal{M} \ominus \Omega(\mathcal{M}) \\ &= (\mathcal{M}_0 \cap \varphi_{n_0+1}(z)H^2) \oplus (\mathcal{M}_0 \cap (H^2 \ominus \varphi_{n_0+1}(z)H^2)).\end{aligned}$$

This shows that $\mathcal{M}_0$ is splitting.

Similarly, if $\varphi_n(0) = 0$ for some $-\infty < n < \infty$, then we may prove that $\mathcal{M}_0$ is splitting.

To show the converse assertion, suppose that $\varphi_n(0) \neq 0$ and $\psi_n(0) \neq 0$ for every $-\infty < n < \infty$. By [19, pages 532–533], $\Omega(\mathcal{M}) = \mathbb{C} \cdot P_{\mathcal{M}}1$. By ($\alpha$7), one can easily check that

$$\begin{aligned}P_{\mathcal{M}}1 &= \bigoplus_{n=-\infty}^{\infty} \varphi_{n+1}(0)\psi_n(0)\varphi_{n+1}(z)\psi_n(w)(1 - \zeta_n(0)\zeta_n(z)) \\ &= \bigoplus_{n=-\infty}^{\infty} \varphi_{n+1}(0)\psi_n(0)\varphi_{n+1}(z)\psi_n(w)(1 - \xi_n(0)\xi_n(w)).\end{aligned} \tag{3.5}$$

To prove that $\mathcal{M}_0$ is not splitting, we assume that $\mathcal{M}_0$ is splitting. We may assume that $\mathcal{M}_0$ is splitting for $\varphi(z)$. Let $\psi(w)$ be the associated inner function of $\varphi(z)$ for $\mathcal{M}_0$. We have $K_{\varphi_n}(z) \otimes K_{\psi_n}(w) \perp \mathcal{M}$, so $K_{\varphi_n}(z) \otimes K_{\psi_n}(w) \perp \varphi(z)\psi(w)H^2$ for every $-\infty < n < \infty$. Hence either $T_z^*\varphi_n(z) \perp \varphi(z)H^2(z)$ or $T_w^*\psi_n(w) \perp \psi(w)H^2(w)$. This shows that either $\varphi(z)/\varphi_n(z) \in H^2(z)$ or $\psi(w)/\psi_n(w) \in H^2(w)$. By ($\alpha$6), $\psi(w)/\psi_n(w) \notin H^2(w)$ for a large n, so $\varphi(z)/\varphi_n(z) \in H^2(z)$ for a large n. By (α3), $\varphi(z)/\varphi_n(z) \notin H^2(z)$ for a sufficiently small n. Then there is an integer n_0 such that $\varphi(z)/\varphi_{n_0+1}(z) \in H^2(z)$ and $\varphi(z)/\varphi_{n_0}(z) \notin H^2(z)$. We have $\psi(w)/\psi_{n_0}(w) \in H^2(w)$. Hence

$$\varphi(z)\psi(w)H^2 \subset \varphi_{n_0+1}(z)\psi_{n_0}(w)H^2.$$

Since $\psi(w)$ is the associated inner function of $\varphi(z)$ for $\mathcal{M}_0$, $K_\varphi(z) \otimes K_\psi(w) \perp \mathcal{M}_0$, so $K_\varphi(z) \otimes K_\psi(w) \subset \mathcal{N}_0$. Let

$$\sigma(z) = \varphi(z)/\varphi_{n_0+1}(z) \quad \text{and} \quad \eta(w) = \psi(w)/\psi_{n_0}(w).$$

Then $\varphi_{n_0+1}(z)K_\sigma(z) \subset K_\varphi(z)$, $\psi_{n_0}(w)K_\eta(w) \subset K_\psi(w)$ and

$$\begin{aligned}&\varphi_{n_0+1}(z)\psi_{n_0}(w)K_\sigma(z) \otimes K_\eta(w) \\ &\subset (K_\varphi(z) \otimes K_\psi(w)) \cap \varphi_{n_0+1}(z)\psi_{n_0}(w)H^2 \\ &\subset \mathcal{N}_0 \cap \varphi_{n_0+1}(z)\psi_{n_0}(w)H^2 \\ &\subset \mathcal{N}_0 \cap \mathcal{M} = \Omega(\mathcal{M}) = \mathbb{C} \cdot P_{\mathcal{M}}1.\end{aligned}$$

If $K_\sigma(z) \otimes K_\eta(w) \neq \{0\}$, then

$$\varphi_{n_0+1}(z)\psi_{n_0}(w)K_\sigma(z) \otimes K_\eta(w) = \mathbb{C} \cdot P_{\mathcal{M}}1$$

and this contradicts (3.5). Thus $K_\sigma(z) \otimes K_\eta(w) = \{0\}$. Hence we may assume that either $\varphi(z) = \varphi_{n_0+1}(z)$ or $\psi(w) = \psi_{n_0}(w)$. Suppose that $\varphi(z) = \varphi_{n_0+1}(z)$. Since $\mathcal{M}_0$ is splitting for $\varphi_{n_0+1}(z)$,

$$\mathcal{M}_0 = (\mathcal{M}_0 \cap \varphi_{n_0+1}(z)H^2) \oplus (\mathcal{M}_0 \cap (H^2 \ominus \varphi_{n_0+1}(z)H^2)). \tag{3.6}$$

Let

$$\begin{aligned} f = \varphi_{n_0+2}(z)\psi_{n_0+1}(w)&(1 - \zeta_{n_0+1}(0)\zeta_{n_0+1}(z)) \\ &\oplus c\varphi_{n_0+1}(z)\psi_{n_0}(w)(1 - \zeta_{n_0}(0)\zeta_{n_0}(z)) \end{aligned}$$

for some $c \in \mathbb{C}$. Then $f \in \mathcal{M}$. We may take $c \in \mathbb{C}$ such that $\langle f, P_{\mathcal{M}}1\rangle = 0$. Since $\mathcal{M} = \mathcal{M}_0 \oplus \mathbb{C} \cdot P_{\mathcal{M}}1$, $f \in \mathcal{M}_0$. By (3.6),

$$\begin{aligned} f_1 &:= \varphi_{n_0+2}(z)\psi_{n_0+1}(w)(1 - \zeta_{n_0+1}(0)\zeta_{n_0+1}(z)) \in \mathcal{M}_0 \\ \langle f_1, P_{\mathcal{M}}1\rangle &= \varphi_{n_0+2}(0)\psi_{n_0+1}(0)(1 - \zeta_{n_0+1}(0)^2) \neq 0. \end{aligned}$$

This shows $f_1 \notin \mathcal{M}_0$ and this is a contradiction.

Suppose that $\psi(w) = \psi_{n_0}(w)$. Since

$$(\mathcal{M}_0 \cap \varphi_{n_0+1}(z)H^2) \ominus \varphi_{n_0+1}(z)\psi_{n_0}(w)H^2 \neq \{0\},$$

$\mathcal{M}_0$ is splitting for $\psi_{n_0}(w)$ (see above Theorem 2.2). In the same way as the last paragraph, we have a contradiction. As a result, $\mathcal{M}_0$ is not splitting. □

Corollary 3.12. *The splittingness is not stable under the finite dimensional perturbations.*

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