A REMARK ON A RESULT OF MARVIN MARCUS

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(received September 24, 1962)

Marcus [2] has proved the following theorem.

Suppose A is a non-negative normal matrix satisfying $p(A) = 0$ in which $p(\lambda)$ is a monic polynomial no two of whose non-zero roots have the same modulus. Then there exists a permutation matrix P such that PAP^* is a direct sum, $PAP^* = A_1 \oplus A_2 \oplus \dots \oplus A_m$, in which each A_i is either O or primitive.

This note gives a generalisation of this result, dropping the non-negative assumption and weakening the normality assumption.

Remark 1. If A is an n by n matrix whose elements are real and if AA^T and A^TA have the same diagonal elements then there exists a permutation matrix P such that $P^{-1}AP =$ diag $(A_1, A_2, \dots, A_m)$ in which the A_i are irreducible.

Proof: There exists a permutation matrix P such that

$$P^{-1}AP = \begin{bmatrix} A_{11} & 0 & \dots & 0 \\ A_{21} & A_{22} & \dots & 0 \\ . & . & & . \\ A_{m1} & & & A_{mm} \end{bmatrix}$$

* This research was supported by the United States Air Force Office of Scientific Research.

Canad. Math. Bull. vol. 6, no. 1, January 1963.

https://doi.org/10.4153/CMB-1963-003-2 Published online by Cambridge University Press

where $A_{11}, A_{22}, \ldots, A_{mm}$ are irreducible.

Let A_{11} be a t by t matrix. Since AA^T and A^TA have the same diagonal elements it follows that the sum of the squares of the elements in row i of A is equal to the sum of the squares of the elements in column i of A, for $i = 1, 2, \ldots, n$. The same is true of $P^{-1}AP$. Also, the sum of the squares of the elements in the first t rows of $P^{-1}AP$ is equal to the sum of the squares of the elements of A_{11} which in turn is equal to the sum of the squares of the elements of the first t columns of $P^{-1}AP$. It follows that $A_{21}, A_{31}, \ldots, A_{m1}$ have all their elements equal to 0.

Similar remarks relative to A_{22} show that $A_{32}, A_{42}, \ldots, A_{m2}$ have all their entries equal to 0. Repeating the argument we have $A_{uv} = 0$ for $u \neq v$. Put $A_i = A_{ii}$ and Remark 1 is proved.

The <u>directed graph</u> D_A of an n-square matrix A is defined as follows. It has vertex set $(1, 2, \ldots, n)$, and the ordered pair (i, j) is an edge of D_A if and only if $A_{ij} \neq 0$. A directed graph D with vertex set V is <u>cyclically k-partite</u> $(k \geq 2)$, if and only if V can be partitioned, $V = V_1 + V_2 + \ldots + V_k$ such that (i, j) is an edge of D only if $i \in V_1$ and $j \in V_2$, or $i \in V_2$ and $j \in V_3$, or ... or $i \in V_k$ and $j \in V_1$.

It has been remarked [1] that if D_A is cyclically k-partite then the characteristic polynomial of A has the form $f(\lambda^k)\lambda^p$. Remark 2 and Remark 3 follow from this observation.

Remark 2. If D_A is cyclically k-partite $(k \geq 2)$, then for every non-zero characteristic root λ of A there exist at least $k-1$ distinct roots which are distinct from λ and have the same modulus as λ.

Remark 3. If the minimal polynomial of a matrix A has a root $\lambda \neq 0$ and has no root μ such that $\mu \neq \lambda$ and $|\lambda| = |\mu|$, then there is no integer $k \geq 2$ for which D_A is cyclically k-partite.

Remark 1 and Remark 3 give us the following theorem.

THEOREM 1. Let A be an n by n matrix with real elements such that AA^T and A^TA have the same diagonal elements. Suppose $p(\lambda)$ is a monic polynomial such that (i) $p(A) = 0$, and (ii) $p(\lambda)$ has no pair of roots λ and μ with $\lambda \neq \mu$ and $|\lambda| = |\mu|$. Then there exists a permutation matrix P such that $P^{-1}AP = \text{diag}\,[A_1, A_2, \ldots, A_m]$ in which the matrices A_i are irreducible. Moreover, for each A_i, either every root is zero or there exists no integer $k \geq 2$ such that the directed graph D_{A_i} is cyclically k-partite.

In the result of Marcus [2] the assumption that A is non-negative implies that an A_i which is not zero has a non-zero characteristic root λ and thus the first alternative that every root of A_i should be zero is not possible. If A_i is imprimitive then D_A is cyclically d-partite where d is the index of imprimitivity. Thus the second alternative, that there is no $k \geq 2$ such that D_{A_i} is cyclically k-partite, implies that A_i is primitive. Thus Theorem 1 generalizes Marcus' Theorem.

REFERENCES

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https://doi.org/10.4153/CMB-1963-003-2 Published online by Cambridge University Press