CLUSTER SETS OF FUNCTIONS ON AN N-BALL

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1. Introduction. Let D^n be the open unit ball in E^n, and let S^n be the n-sphere. The *cluster set* $C(f, p)$ of a function $f : D^n \to S^n$ at a point p on the boundary of D^n is the set of points y in S^n such that there exists a sequence of points $x_m \to p$, x_m in D^n, with $f(x_m) \to y$. Given an arc γ in $\bar{D}^n$, meeting the boundary bdy(D^n) only in p, the *arc cluster set* $C(f, \gamma)$ is the set of points y in S^n such that there exists a sequence $x_m \to p$, x_m in γ, with $f(x_m) \to y$.

If f is continuous, then $C(f, p)$ and $C(f, \gamma)$ are continua; this result follows easily from a theorem of Whyburn **(8**, p. 14**)**. Conversely, W. Gross **(4)** and, later, Weigand **(7)** proved that, given any closed, connected set C in the plane and a point p on the boundary of the open unit disk D^2, there is a meromorphic function $f : D^2 \to E^2$ for which $C(f, p)$ is C.

A differentiable map f, mapping an open set U of E^n into E^n, will be called *quasi-conformal* if there exists a real number $B > 1$ such that, given any point q in U and the directional derivatives d_1 and d_2 in any two directions at q, then $d_1/d_2 \leqslant B$. This definition is equivalent to that of Callender **(2**, p. 1**)**, who generalizes a standard definition in the plane **(1)**.

In Sections 2 and 3 we shall prove the following results for $n \geqslant 2$; S^n will be considered as E^n plus a point at infinity, denoted by ∞.

1.1. THEOREM. *Given any continuum $C \subset S^n$ and $p \in$ bdy(D^n), there exists a C^∞ quasi-conformal local homeomorphism $f : D^n \to E^n$ such that $C(f, p) = C$. Indeed, each arc cluster set $C(f, \gamma)$ at p is C.*

1.2. THEOREM. *The function f may be chosen to be a homeomorphism if and only if $S^n - C$ has a component whose boundary is C.*

In each case the constant of quasi-conformality may be chosen independent of C. For $n > 2$, the f of (1.2) may not be chosen to be a conformal homeomorphism in general, since any such map is the composition of rotations, translations, transformations of similitude, and inversions.

Some remarks on arc cluster sets of bounded analytic functions are made in Section 3. In the Remark of (2.1) is a Riemann Mapping Theorem for quasi-conformal homeomorphisms.

The interior of a space X is denoted by int X, the closure by $\bar{X}$ or Cl$[X]$, and the restriction of the function f to X by $f \mid X$. A "map" is a continuous function, and the distance from x to y is denoted by $d(x, y)$.

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2. The main theorems. First, we state three results which are well known and easy to prove.

(I) *If f and g are quasi-conformal with constants A and B, and if their composition gf is defined, then gf is also quasi-conformal with constant less than or equal to AB.*

(II) *If f is a quasi-conformal diffeomorphism with constant A, then f^{-1} is also quasi-conformal with constant A.*

(III) *If f is differentiable, then the directional derivative of f at $(x_1, x_2, \ldots, x_n)$, in the direction whose direction cosines are $(a_1, a_2, \ldots, a_n)$ $(a_1{}^2 + \ldots + a_n{}^2 = 1)$ is*

$$\left[\sum_{j=1}^{n}\left(\sum_{i=1}^{n} a_i \partial f_j/\partial x_i\right)^2\right]^{\frac{1}{2}}.$$

2.1. Proposition. *Suppose that f is a C' function mapping the open subset U of E^n into E^n and having non-zero Jacobian everywhere. If X is an open set with closure compact and contained in U, then the restriction of f to X is quasi-conformal.*

The proof is immediate.

Remark. It is a folk theorem **(**cf. **6)** that, if U is any 3-cell in E^3 whose boundary is a (C^2) regular embedding f of S^2 in E^3, then there exists a C' map g of a region V containing D^3 such that the restriction of g to bdy(D^3) is f and the Jacobian of g is never zero. Thus, by the above lemma, there exists a homeomorphism h of $\bar{D}^3$ onto $\bar{U}$, such that the restriction of h to D^3 is quasi-conformal.

We do not need the next lemma in such generality, but the result has independent interest. Given a positive real-valued map f on (a, b) or $[a, b)$ (both a and b or one of them may be infinite), S_f denotes

$$\left\{(x_1, x_2, \ldots, x_n)|a < x_1 < b \quad \text{and} \left[\sum_{i=2}^{n} x_i^2\right]^{\frac{1}{2}} < f(x_1)\right\},$$

the interior of a surface of revolution; $S_f[c, d]$ denotes the set of points of S_f which satisfy $c \leqslant x_1 \leqslant d$.

2.2. Lemma. *Let $a > 0$, $b > 0$, and let $g : (-a, b) \to (0, \infty)$ be a differentiable (C^k, C^∞) map such that dg/dx is non-positive and bounded below and $g(x) \to 0$ as $x \to b$. Let $h : (-a, \infty) \to (0, \infty)$ be a differentiable (C^k, C^∞) map with dh/du bounded above and below. Then there exists a differentiable (C^k, C^∞) quasi-conformal homeomorphism Q of $S_g[0, b)$ onto $S_h[0, \infty)$.*

Proof. There exists **(5**, pp. 62–63**)** a unique real-valued function $u(x)$ defined on a neighbourhood of 0 satisfying:

(1) $$du/dx = h(u(x))/g(x) \qquad \text{and} \qquad u(0) = 0.$$

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(Since dh/du and $1/g(x)$ are bounded—the latter for x bounded away from b—the Lipschitz Condition follows from the Mean Value Theorem.) Suppose that u is defined on $[0, \epsilon)$, where $0 < \epsilon < b$ and ϵ is maximal. Observe that

$$\int_0^{u(x)} (h(t))^{-1} dt = \int_0^x (g(y))^{-1} dy,$$

where the right-hand integral has a finite limit as $x \to \epsilon$. Since $h(u) \leqslant Bu + h(0)$, where B is a (positive) upper bound on dh/du, the left-hand integral is greater than $B^{-1} \ln(Bu(x) + h(0))$. Thus $u(x)$ and $(dh/du)(x)$ have finite limits as $x \to \epsilon$. In a neighbourhood of $x = \epsilon$ the differential equation has a unique solution (call it u_1) such that $u_1(\epsilon)$ is the limit of $u(x)$ as $x \to \epsilon$. Then u may be extended past ϵ as u_1, and the maximality of ϵ is thus contradicted, unless $\epsilon = b$.

There exists $A > 0$ such that $-A < dg/dx \leqslant 0$, so that the left-hand integral approaches ∞ as $x \to b$; thus $u(x) \to \infty$ as $x \to b$. Since $du/dx > 0$, u is a C' (C^{k+1}, C^∞) homeomorphism of $[0, b)$ onto $[0, \infty)$.

Let Q,

$$Q : (x_1, x_2, \ldots, x_n) \to (u_1, u_2, \ldots, u_n),$$

be defined by: $u_1 = u_1(x_1)$, given by (1), and

$$u_j = h(u_1(x_1)) \cdot x_j / g(x_1)$$

$(j = 2, 3, \ldots, n)$. Clearly Q is a differentiable (C', C^∞) homeomorphism mapping S_g onto S_h. The directional derivative of Q in the direction with direction cosines $(a_1, a_2, \ldots, a_n)$ is $r \cdot du_1/dx_1$, where

$$r = \left[a_1^2 + \sum_{j=2}^{n} \left(\frac{a_1 x_j}{g(x_1)}\left[\frac{dh}{du_1} - \frac{dg}{dx_1}\right] + a_j\right)^2\right]^{\frac{1}{2}}.$$

Since r is bounded above and below by positive numbers, Q is quasi-conformal.

Remarks. Let $h^{(i)}$ denote the ith derivative of h, where $h^{(0)} = h$. If g and h are C^∞, if $g^{(i)}(0) = h^{(i)}(0)$, and if Q is extended to the half-space $x_1 < 1$ as the identity, then Q is C^∞ and quasi-conformal.

Let the interior S of a closed topological 3-ball be

$$\{(x_1, x_2, x_3) : 0 < x_1 < 1, \quad |x_2| < 1 - x_1, \quad |x_3| < (1 - x_1)^2\}.$$

Then Q also defines a quasi-conformal homeomorphism of S onto

$$\{(u_1, u_2, u_3) : 0 < u_1 < \infty, \quad |u_2| < u_1, \quad |u_3| < 1\}$$

(use $g(x) = (1 - x)^2$, $h(u) \equiv 1$).

2.3. Let $g : (-1, 1) \to (0, 1)$ be a C^∞ function such that $g(x) = [1 - x^2]^{\frac{1}{2}}$ for $x \leqslant 0.9$, $g(x) = 4 - 4x$ in some neighbourhood of 1,

$$4 - 4x \leqslant g(x) \leqslant [1 - x^2]^{\frac{1}{2}}$$

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for $x \geqslant 0.9$, and $g'(x) < 0$ for $x > 0$. We shall now construct a homeomorphism F of $\mathrm{Cl}[S_g(-1, 1)]$ onto $\bar{D}^n$, $F|S_g(-1, 1)$ C^∞ and quasi-conformal.

Let

$$u = \left[\sum_{i=2}^{n} x_i^2\right]^{\frac{1}{2}};$$

let $\zeta(u)$ be the inverse of $g(x_1)$ for $x_1 > 0$ (thus $\zeta(u) = 1 - \frac{1}{4}u$ in some neighbourhood of u). There is a C^∞ function $\lambda : (-\infty, \infty) \rightarrow [0, 1]$ such that $\lambda(s) = 0$ for $s \leqslant 0$, $\lambda(s) = 1$ for $s \geqslant 1$, and $\lambda'(s) > 0$ for $0 < s < 1$. Let

$$s(x_1, u) = \frac{x_1 - 1 + [\frac{1}{2} + \epsilon]u}{[\frac{1}{6} + \epsilon]u},$$

where ϵ is to be specified later, $\frac{1}{2} \geqslant \epsilon > 0$; and let

$$\psi(x_1, u) = 1 + \lambda(s(x_1, u)) \cdot \frac{[1 - u^2]^{\frac{1}{2}} - \zeta(u)}{\zeta(u) - 1 + \frac{1}{2}u}.$$

The homeomorphism F is given by $F_i = x_i$ $(i = 2, 3, \ldots, n)$, $F_1 = x_1$ for $u \geqslant 0.5$, and $u = 0$, and

$$F_1 = 1 - \tfrac{1}{2}u + \psi(x_1, u) \cdot (x_1 - 1 + \tfrac{1}{2}u)$$

for $0 < u < 0.6$. (The function F results from smoothing the homeomorphism S defined by $S_i = x_i$ $(i = 2, 3, \ldots, n)$, $S_1 = x_1$ for $x_1 \leqslant 1 - \frac{1}{2}u$, and S is linear for $x_1 \geqslant 1 - \frac{1}{2}u$.)

Observe that the expression for F_1 yields x_1 for $0.5 \leqslant u < 0.6$ or $x_1 \leqslant 1 - (\frac{1}{2} + \epsilon)u$ (in particular, for $u = 0$). For $0 < u < 0.6$, F_1 is defined by the composition, sum, product, and reciprocal of C^∞ maps, and thus is itself C^∞. Now

$$\frac{\partial F_1}{\partial x_1} = \psi + (x_1 - 1 + \tfrac{1}{2}u) \cdot \frac{\partial \psi}{\partial x_1}$$

for $u < 0.6$. Since $\psi \geqslant 1$, and since, for $x_1 \geqslant 1 - \frac{1}{2}u$, the other term is positive, $\partial F_1/\partial x_1 \geqslant 1$ for such x_1. For $x_1 < 1 - (\frac{1}{2} + \epsilon)u$, F_1 is x_1 and $\partial F_1/\partial x_1 = 1$. For

$$1 - (\tfrac{1}{2} + \epsilon)u < x_1 < 1 - \tfrac{1}{2}u,$$

$$|(x_1 - 1 + \tfrac{1}{2}u) \cdot \partial x/\partial x_1| \leqslant \frac{1 - \frac{1}{2}u - x_1}{(\frac{1}{6} + \epsilon)u} \cdot \frac{d\lambda}{ds} \cdot \frac{[1 - u^2]^{\frac{1}{2}} - \zeta(u)}{\zeta(u) - 1 + \frac{1}{2}u}.$$

Since the product of the last two factors is bounded for $0 < u \leqslant 0.5$, say by M, the expression is bounded by $6M\epsilon$. Thus, for suitable ϵ $(0 < \epsilon \leqslant \frac{1}{2})$, $\partial F_1/\partial x_1 \geqslant \frac{1}{2}$, and F is a homeomorphism.

Similarly, one can prove that $|\partial F_1/\partial x_1|$ $(i = 1, 2, \ldots, n)$ are bounded above. Since the directional derivative in the direction with direction cosines $(a_1, a_2, \ldots, a_n)$ is

$$\left[\left(\sum_{i=1}^{n} a_i \frac{\partial F_1}{\partial x_i}\right)^2 + \sum_{j=2}^{n} (a_j)^2\right]^{\frac{1}{2}},$$

it follows that F is quasi-conformal.

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Let h be any C^∞ homeomorphism such that $h^{(i)}(0) = g^{(i)}(0)$, and h satisfies the hypotheses of (2.2). If h is extended to $[-1, 1]$ by $h \equiv g$ on $[-1, 0]$, then there is a quasi-conformal homeomorphism Q of $S_g(-1, 1)$ onto $S_h(-1, \infty)$; let H be the (natural) extension of QF^{-1} to $\bar{D}^n - \{(1, 0, \ldots, 0)\}$. *Thus, there is a homeomorphism H of $\bar{D}^n - \{(1, 0, \ldots, 0)\}$ onto* $\mathrm{Cl}[S_h(-1, \infty)]$ *(closure in E^n), $H|D^n$ quasi-conformal and C^∞.*

2.4. LEMMA. *Given any continuum C on S^n, there exists a C^∞ map $\alpha : [0, \infty) \to E^n$ such that*

(1) *on some neighbourhood of* 0 $\alpha_1 = x$ *and α_i is constant* $(i = 2, 3, \ldots, n)$.

(2) $\sum_{i=1}^{n} (d\alpha_i/dx)^2 > 0$, *and*

(3) *as $x \to \infty$, the set of limit points of $\alpha(x)$ is C.*

Proof. First, suppose that $C \subset E^n$. Let U_m be the connected open set $\{y \in E^n : d(y, C) < 1/m\}$, and let $y_{m,j}$ $(j = 1, 2, \ldots, j_m; m = 1, 2, \ldots)$ be a set of points $1/m$-dense in U_m. Define β on some interval $[0, a_1]$ so that $\beta([0, a_1)]$ meets each point $y_{1,j}$ $(j = 1, 2, \ldots, j_1)$, $\beta([0, a_1]) \subset U_1$, $\beta(a_1) \in U_2$, and $\beta|[0, a_1]$, satisfies (1) and (2). Extend β to $[a_1, a_2]$ so that $\beta([a_1, a_2])$ meets each point $y_{2,j}$ $(j = 1, 2, \ldots, j_2)$, $\beta([a_1, a_2]) \subset U_2$, $\beta(a_2) \in U_3$, and $\beta|[0, a_2]$ satisfies (1) and (2); the rest of the definition of β is clear. If the domain of β is $[0, b)$, b finite, let $\alpha = \beta\gamma$, where $\gamma(u) = bu(1 + u)^{-1}$; otherwise, let $\alpha = \beta$.

If $C \not\subset E^n$ and $C \neq \{\infty\}$, replace U_m by

$$[S(q, m) - S(q, m - 1)] \cup [S(q, m) \cap U_m]$$

$(m = 1, 2, \ldots)$, where $q \in C - \{\infty\}$ and $S(q, m)$ is $\{r : d(r, q) < m\}$, and proceed as above. If $C = \{\infty\}$, let $\alpha_1 = x$ and $\alpha_i = 0$ $(i = 2, 3, \ldots, n)$.

2.5. LEMMA. *If $S^n - C$ has a component K whose boundary is C, then α in* 2.4 *may be chosen so that, in addition,*

(4) $\alpha([0, \infty)) \subset K$, *and*

(5) *α is one-to-one.*

Proof. We may as well suppose that $0 \in K$ and that $d(0, C) > 1$. We consider three cases.

(a) $n = 2$. Let $\gamma : [0, \infty) \to D^2$ be the spiral map given in polar co-ordinates by: $r(t) = t(\pi + t)^{-1}$, $\theta(t) = t$ $(0 \leqslant t < \infty)$. Let g be a conformal map of D^2 onto K. If $g(\gamma([0, \infty)))$ meets ∞, it meets it for at most one t, so that there exists $\delta > 0$ such that $\infty \notin g(\gamma([\delta, \infty)))$; with suitable reparametrization and a change on a neighbourhood of δ (now 0) to satisfy (1), $g\gamma|[\delta, \infty)$ is the desired map α. (A proof not involving the Riemann Mapping Theorem and analogous to that given for $n \geqslant 3$ could be given.)

(b) $n \geqslant 3$ and $\infty \notin C$. Let $\mathfrak{X}_m$ be the family of all closed n-cubes which have edges parallel to the co-ordinate axes (of E^n), side 2^{-m}, and vertices

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$(e_1 \cdot 2^{-m}, e_2 \cdot 2^{-m}, \ldots, e_n \cdot 2^{-m})$, where e_j is an integer $(j = 1, 2, \ldots, n;$ $m = 1, 2, \ldots)$. Let X_m be the union of those n-cubes of $\mathfrak{X}_m$ which are contained in K. If $\infty \in K$, let Y_m be the unbounded component of X_m, together with ∞; otherwise, let Y_m be the component of X_m containing 0. Then $Y_{m+1} \subset Y_m$, $S^n - Y_m$ is connected, and

$$\bigcup_{m=1}^{\infty} \operatorname{int}(Y_m) = K.$$

The Mayer-Vietoris sequence

$$\ldots \to H_1(S^n) \to H_0\,(\operatorname{bdy}(Y_m)) \to H_0(S^n \;-\operatorname{int}(Y_m)) + H_0(Y_m) \to H_0(S^n),$$

using homology (mod 2), is exact. It follows that $\operatorname{bdy}(Y_m)$ is connected. For any component Z_m of $K - Y_m$, $\operatorname{bdy}(Z_m)$ meets $\operatorname{bdy}(Y_m)$; thus $K - Y_m$ is connected (and open).

For each m, choose a set of points $y_{m,j}$ $(j = 1, 2, \ldots, j_m)$ $1/m$-dense in $K - Y_m$ and distinct for distinct indices. As in the previous lemma, define a one-to-one function $\alpha : [0, \infty) \to K$ such that (1) and (2) are satisfied, $\alpha([0, \infty))$ joins the points $y_{m,j}$ in lexicographic order, and there exist $\delta_m > 0$, $\delta_{m+1} > \delta_m$, with $\alpha([\delta_m, \infty)) \subset K - Y_m$. (The assumption that $n \geqslant 3$ is used here.) Since

$$\bigcap_{m=1}^{\infty} \operatorname{Cl}[K - Y_{m]} = C,$$

the set of limit points of $\alpha(x)$ is C as $x \to \infty$.

(c) $n \geqslant 3$ and $\infty \in C$. Let X_m be the n-cubes of $\mathfrak{X}_m$ which are contained in $K \cap S(0, m + 1)$, and let Y_m be the component containing 0 of X_m. The argument proceeds as above, except that the points $y_{m,j}$ are defined to be $1/m$-dense in $(K - Y_m) \cap \mathrm{S}(0, m + 2)$.

2.6. *Proof of* (1.1) *and* (1.2). If C is a single point in E^n, f is a rigid motion of $\bar{D}^n$ into E^n, sending p into that point. If $C = \{\infty\}$, f is the H of (2.3).

Otherwise, let α be the map given by (2.4) in general, and by (2.5) in case C satisfies the hypotheses of (1.2). We may suppose that α is reparametrized so that x (call it x_1 now) is arc length;

$$\sum_{i=1}^{n} (d\alpha_i/dx_1)^2 = 1.$$

Since C has more than one point, the domain of α will still be $[0, \infty)$, rather than a finite interval. For each x_1, choose a normal $(n - 1)$-frame v^j $(j = 2, 3, \ldots, n)$ to $\alpha([0, \infty))$ at $\alpha(x_1)$, so that each co-ordinate function v_i^j is C^∞ on $[0, \infty)$ and (by conclusion (1) of (2.4)) $v_j^j = 1$ and $v_i^j = 0$ for $i \neq j$ on a neighbourhood of 0. Define $A : \{(x_1, x_2, \ldots, x_n) : x_1 \geqslant 0\} \to E_n$ by

$$A_i(x_1, x_2, \ldots, x_n) = \alpha_i(x_1) + \sum_{j=2}^{n} x_j v_i^j(x_1)$$

$(i = 1, 2, \ldots, n)$. Let $v_i^1(x_1) = (d\alpha_i/dx_1)(x_1)$.

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On the line L defined by $x_i = 0$ $(i = 2, 3, \ldots, n)$ the Jacobian determinant has entries v_i^j, and thus equals ± 1 there. Thus there exists a decreasing function $\lambda : [0, \infty) \to (0, \infty)$ such that $A|S_\lambda[0, \infty)$ has non-zero Jacobian, and hence is a local homeomorphism.

In the case that C satisfies (1.2), we may suppose that $A(S_\lambda[0, \infty)) \subset K$. We shall show that, for some natural number m, $A|(L \cup R_m)$ is one-to-one, where R_m is $\mathrm{Cl}(S_{1/m}[0, 1])$. If not, then there exist distinct points q_m and r_m in $L \cup R_m$ such that $A(q_m) = A(r_m)$ $(m = 1, 2, \ldots)$. We may suppose that $A(R_m) \subset K$ (and thus is at positive distance from C). Since $A|L$ is one-to-one, we may suppose that $q_m \in R_m$; thus $\{q_m\}$ has a limit point q in L. If $\{r_m\}$ has no limit point in L, then the set of limit points of $\{A(r_m)\}$ is contained in C, contradicting the fact that $A(r_m) = A(q_m)$. Therefore, we may suppose that r_m has a limit point r in L. Since $A|L$ is one-to-one, and since $A(q) = A(r)$, $q = r$. Since A is locally one-to-one at each point of L, a contradiction follows.

Similarly, there exists a natural number k such that

$$A|(L \cup \mathrm{Cl}(S_{1/m}[0, 1]) \cup \mathrm{Cl}(S_{1/k}[1, 2]))$$

is one-to-one. By an induction argument there exists a decreasing function $\mu : [0, \infty) \to (0, \infty)$ such that $A|S_\mu[0, \infty)$ is one-to-one, $\mu \leqslant \lambda$.

On L the directional derivative d of A in the direction with direction cosines $(a_1, a_2, \ldots, a_n)$ is

$$\left[\sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_j V_i^j \right)^2 \right]^{\frac{1}{2}}.$$

Thus d is the length of $V(a)$, the image of the vector $a = (a_1, a_2, \ldots, a_n)$ under the linear transformation given by the matrix $V = (v_i^j)$. Since a has unit length, and V is an orthogonal matrix (the v^j constitute an n-frame), $d \equiv 1$. Thus, given any $\epsilon > 0$, there exists a decreasing function $\nu : [0, \infty) \to (0, \infty)$ such that $B - 1 < \epsilon$, where B is the constant of quasi-conformality of $A|S_\nu[0, \infty)$.

Let ξ be a C^∞ map of $[0, \infty)$ into $(0, \infty)$ such that, for C satisfying the hypotheses of (1.1) (respectively, (1.2)), $\xi < \min(\lambda, \nu)$ (resp., $\xi < \min(\mu, \nu)$ and $S(0, \xi(0)) \subset K$). We may suppose, in addition, that in some neighbourhood of 0 the map ξ agrees with the map σ given by $\sigma(x_1) = [\xi^2(0) - x_1^2]^{\frac{1}{2}}$. Extend ξ to $[-\xi(0), 0]$ as σ. Thus $S_\xi[-\xi(0), \infty) = T(S_h[-1, \infty))$, where h satisfies the conditions of the last paragraph of (2.3), T is the transformation of similitude given by $T_i(x) = \omega \cdot x_i$, and ω is some positive number. If $\delta > 0$ is in the neighbourhood of conclusion (1) of (2.4) and (2.5), since A is the identity on $S_\xi[0, \delta]$, extend A to $S_\xi[-\xi(0), 0]$ as the identity map.

The desired C^∞ map f is

$$(A|S_\xi[-\xi(0), \infty))T(H|D^n),$$

where H is the map given by (2.3), and ξ depends on whether C satisfies the

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hypotheses of (1.2) or only of (1.1). To prove that, for each arc γ in $D^n \cup \{p\}$, γ ending at p, $C(f, \gamma) = C$, it suffices to prove the analogous result for $A, S_\xi = S_\xi[-\xi(0), \infty)$, and paths to ∞. In $A(S_\xi)$ the distance from each point $A(x_1, x_2, \ldots, x_n)$ to $A(x_1, 0, \ldots, 0)$ is bounded by $n^{\frac{1}{2}} \cdot \xi(x_1)$. As $x_1 \to \infty$, $\xi(x_1) \to 0$ and the set of limit points of $A(x_1, 0, \ldots, 0)$ is C; the result follows.

Now suppose that $f: D^n \to S^n$ is a homeomorphism such that $C(f, p) = C$ ($p \in \text{bdy } D$). To complete the proof of (1.2), it suffices to prove that $S^n - C$ has a component whose boundary is C. We shall show that the range of f is disjoint from the cluster set $C(f, p)$ of f at p. If y is in the range of f, then there is an n-ball D_y about $f^{-1}(y)$, $\bar{D}_y$ in D^n. Thus y is in $f(D_y)$; suppose that y is also in $C(f, p)$. Then there exists a sequence $\{x_k\}$, x_k in D^n, $x_k \to p$, with $f(x_k) \to y$. The sequence $\{x_k\}$ has a subsequence $\{x_{k(m)}\}$ in $D^n - \bar{D}_y$. Thus $f(D_y)$ meets $f(D^n - D_y)$. Since f is a homeomorphism, we have a contradiction.

Since the range of f is connected, it must be in one component of $S^n - C(f, p)$. Thus $C(f, p)$ is the boundary of this component.

Remark. An upper bound for the constant of quasi-conformality of the map Q given in (2.2) depends only on the bounds of $|dg/dx|$ and $|dh/du|$. Thus, for all h with $|dh/du| < 1$ (on $[0, \infty)$), the H of (2.3) have a common bound on their constants of quasi-conformality. Choose $\epsilon < 1$ (say), and choose ξ so that, in addition to its other properties, $|d\xi/dx_1| < 1$; then $|dh/dx_1| < 1$. (Note that, for $x_1 \leqslant 0$, f is a rigid motion.) Thus, in both (1.1) and (1.2) there is some $k > 1$ such that, independent of the continuum C, the constant of quasi-conformality of the map f constructed is less than k.

Given any $\delta > 0$, is it possible to give a method of construction, in either theorem, so that the constant of quasi-conformality of f is less than $1 + \delta$, independent of the continuum C?

The author is grateful to the referee for suggesting a substantial simplification of this section.

3. Arc cluster sets of bounded analytic functions

3.1. COROLLARY. *Given any continuum C in the plane, there is a function $g(z)$, bounded and analytic on D^2 and a local homeomorphism* (*into*) *on the closed disk $\bar{D}^2$ except at $z = 1$, such that, for every arc γ in D^2 approaching $z = 1$, the arc cluster set $C(g, \gamma)$ is C.*

Proof. Let f be the quasi-conformal C^∞ local homeomorphism given by (1.1) for $p = 1$. From its construction it follows that $\text{Cl}[f(D^2)] \subset E^2$ and that f is continuous on $\bar{D}^2 - \{1\}$. Then g is the analytic function given by **(3**; § 3**)**.

Let L_0 be the line segment joining 0 and 1. Let $L_{n,k}$ be a line segment such that it meets the real axis at $z = 1$ in an angle of $\pi(1 + j \cdot 2^{-n-1})$ ($n = 1$,

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$2, \ldots; j = \pm 1, \pm 3, \ldots, \pm(2^n - 1))$; its length is at most $1/n$; and it is short enough so that it does not separate D^2. Remove the segments L_0 and $L_{n,k}$ from D^2, and let U be the resulting region. If k is a conformal homeomorphism of D^2 onto U, then we have the following result for gk, which we again call g.

3.2. COROLLARY. *Given any continuum C in the plane, there is a function $g(z)$, bounded and analytic on D^2 and a local homeomorphism (into) on $\bar{D}^2$ except at the points of a Cantor set on the circle* bdy(D^2), *such that each arc cluster set at a point of the Cantor set is C.*

These results generalize some of G. S. Young (Notices Amer. Math. Soc. Abstract 564-237), and the use of U was suggested by him. From (1.2) we have the following statement.

3.3. COROLLARY. *The map g can be chosen to be a homeomorphism (into) if and only if C is the boundary of a simply connected region of the complex plane.*

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